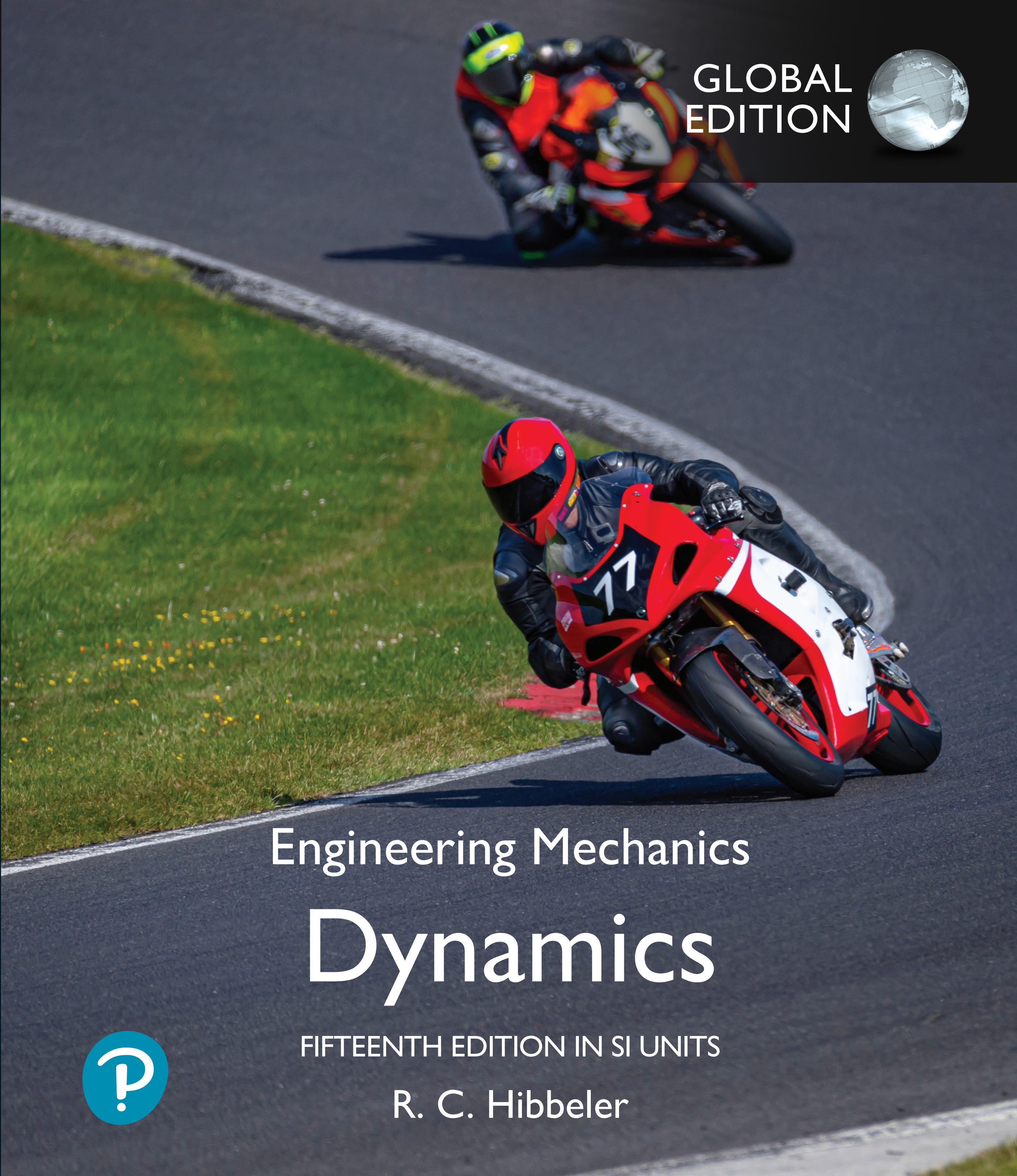

GLOBAL EDITION

Engineering Mechanics

Dynamics

FIFTEENTH EDITION IN SI UNITS

R. C. Hibbeler

ENGINEERING MECHANICS

DYNAMICS

FIFTEENTH EDITION IN SI UNITS

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R. C. HIBBELER

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To the Student

With the hope that this work will stimulate
an interest in Engineering Mechanics
and provide an acceptable guide to its understanding.

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PREFACE

The main purpose of this book is to provide the student with a clear and thorough presentation of the theory and application of engineering mechanics. To achieve this objective, this work has been shaped by the comments and suggestions of hundreds of reviewers in the teaching profession, as well as many of the author's students.

New to this Edition

Expanded Answer Section. The answer section in the back of the book now includes additional information related to the solution of select Fundamental Problems in order to offer the student some guidance in solving the problems.

Re-writing of Text Material. Some concepts have been clarified further in this edition, and throughout the book the accuracy has been enhanced, and important definitions are now in boldface throughout the text to highlight their importance.

New Photos. The relevance of knowing the subject matter is reflected by the real-world applications depicted in the over 14 new or updated photos placed throughout the book. These photos generally are used to explain how the relevant principles apply to real-world situations and how materials behave under load.

New Problems. There are approximately 30% new problems that have been added to this edition, which involve applications to many different fields of engineering.

New Videos. Three types of videos are available that are designed to enhance the most important material in the book. Lecture Videos serve to test the student's ability to understand the concepts, Example Problem Videos are intended to review these problems, and Fundamental Problem Videos guide the student in solving these problems that are in the book. They are available for selected sections in the chapters and marked with this icon. The videos appear on a companion website available for separate purchase at www.pearsonglobaleditions.com.

Hallmark Features

Besides the new features mentioned, other outstanding features that define the contents of the book include the following:

Organization and Approach. Each chapter is organized into well-defined sections that contain an explanation of specific topics, illustrative example problems, and a set of homework problems. The topics within each section are placed into subgroups defined by boldface titles. The purpose of this is to present a structured method for introducing each new definition or concept and to make the book convenient for later reference and review.

Chapter Contents. Each chapter begins with an illustration demonstrating a broad-range application of the material within the chapter. A bulleted list of the chapter contents is provided to give a general overview of the material that will be covered.

Emphasis on Free-Body Diagrams. Drawing a free-body diagram is particularly important when solving problems, and for this reason this step is strongly emphasized throughout the book. In particular, special sections and examples are devoted to show how to draw free-body diagrams. Specific homework problems have also been added to develop this practice.

Procedures for Analysis. A general procedure for analyzing any mechanics problem is presented at the end of the first chapter. Then this procedure is customized to relate to specific types of problems that are covered throughout the book. This unique feature provides the student with a logical and orderly method to follow when applying the theory. The example problems are solved using this outlined method in order to clarify its numerical application. Realize, however, that once the relevant principles have been mastered and enough confidence and judgment have been obtained, the student can then develop his or her own procedures for solving problems.

Important Points. This feature provides a review or summary of the most important concepts in a section and highlights the most significant points that should be known when applying the theory to solve problems.

Fundamental Problems. These problem sets are selectively located just after most of the example problems. They provide students with simple applications of the concepts, and therefore, the chance to develop their problem-solving skills before attempting to solve any of the standard problems that follow. In addition, they can be used for preparing for exams, and they can be used at a later time when preparing for the Fundamentals in Engineering Exam. The partial solutions are given in the back of the book.

Conceptual Understanding. Through the use of photographs placed throughout the book, the theory is applied in a simplified way in order to illustrate some of its more important conceptual features and instill the physical meaning of many of the terms used in the equations.

Homework Problems. Apart from the Fundamental and Conceptual type problems mentioned previously, other types of problems contained in the book include the following:

- **Free-Body Diagram Problems.** Some sections of the book contain introductory problems that only require drawing the free-body diagram for the specific problems within a problem set. These assignments will impress upon the student the importance of mastering this skill as a requirement for a complete solution of any equilibrium problem.

- **General Analysis and Design Problems.** The majority of problems in the book depict realistic situations encountered in engineering practice. Some of these problems come from actual products used in industry. It is hoped that this realism will both stimulate the student's interest in engineering mechanics and provide a means for developing the skill to reduce any such problem from its physical description to a model or symbolic representation to which the principles of mechanics may be applied.

  Throughout the book, in any set of problems, an attempt has been made to arrange them in order of increasing difficulty except for the end of chapter review problems, which are presented in random order.

- **Computer Problems.** An effort has been made to include a few problems that may be solved using a numerical procedure executed on either a desktop computer or a programmable pocket calculator. The intent here is to broaden the student's capacity for using other forms of mathematical analysis without sacrificing the time needed to focus on the application of the principles of mechanics. Problems of this type, which either can or must be solved using numerical procedures, are identified by a "square" symbol (■) preceding the problem number.

The many homework problems in this edition, have been placed into two different categories. Problems that are simply indicated by a problem number have an answer and in some cases an additional numerical result given in the back of the book. An asterisk (*) before every fourth problem number indicates a problem without an answer.

Accuracy. As with the previous editions, apart from the author, the accuracy of the text and problem solutions has been thoroughly checked by Kai Beng Yap and Jun Hwa Lee, along with a team of specialists at EPAM, including Georgii Kolobov, Ekaterina Radchenko, and Artur Akberov.

Contents

The book is divided into 11 chapters, in which the principles are first applied to simple, then to more complicated situations.

The kinematics of a particle is discussed in Chapter 12, followed by a discussion of particle kinetics in Chapter 13 (Equation of Motion), Chapter 14 (Work and Energy), and Chapter 15 (Impulse and Momentum). The concepts of particle dynamics contained in these four chapters are then summarized in a "review" section, and the student is given the chance to identify and solve a variety of problems. A similar sequence of presentation is given for the planar motion of a rigid body: Chapter 16 (Planar Kinematics), Chapter 17 (Equations of Motion), Chapter 18 (Work and Energy), and Chapter 19 (Impulse and Momentum), followed by a summary and review set of problems for these chapters.

If time permits, some of the material involving three-dimensional rigid-body motion may be included in the course. The kinematics and kinetics of this motion are discussed in Chapters 20 and 21, respectively. Chapter 22 (Vibrations) may be included if the student has the necessary mathematical background. Sections of the book that are considered to be beyond the scope of the basic dynamics course are indicated by a star (★) and may be omitted. Note that this material also provides a suitable reference for basic principles when it is discussed in more advanced courses. Finally, Appendix A provides a list of mathematical formulas needed to solve the problems in the book, Appendix B provides a brief review of vector analysis, and Appendix C reviews application of the chain rule.

Alternative Coverage. At the discretion of the instructor, it is possible to cover Chapters 12 through 19 in the following order with no loss in continuity: Chapters 12 and 16 (Kinematics), Chapters 13 and 17 (Equations of Motion), Chapter 14 and 18 (Work and Energy), and Chapters 15 and 19 (Impulse and Momentum).

Acknowledgments

I have endeavored to write this book so that it will appeal to both the student and instructor. Through the years, many people have helped in its development, and I will always be grateful for their valued suggestions and comments. Specifically, I wish to thank all the individuals who have sent comments to me. These include J. Aurand, J. Ari-Gur, R. Boyd, O. Byer, E. Erisman, C. Heinke, H. Kuhlman, E. Most, S. Moustafa, H. Nazeri, D. Pox, J. Ross, D. Rowlison, R. Scott, K. Steurer.

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During the production process I am thankful for the assistance of Rose Kernan, my production editor, and Marta Samsel, who worked on the cover of the book. And finally, to my wife, Conny, who helped in the proofreading of the manuscript for publication.

Lastly, many thanks are extended to all my students and to members of the teaching profession who have freely taken the time to offer their suggestions and comments. Since this list is too long to mention, it is hoped that those who have given help in this manner will accept this anonymous recognition.

I would greatly appreciate hearing from you if at any time you have any comments, suggestions, or issues related to any matters regarding this edition.

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Resources for Instructors

Instructor's Solutions Manual This supplement provides complete solutions supported by problem statements and problem figures. The Instructor's Solutions Manual is available in the Instructor Resource Center.

PowerPoint Slides A complete set of all the figures and tables from the textbook are available in PowerPoint format.

Resources for Students

Videos Developed by the author, three different types of videos are now available to reinforce learning the basic theory and applying the principles. The first set provides a lecture review and a self-test of the material related to the theory and concepts presented in the book. The second set provides a self-test of the example problems and the basic procedures used for their solution. The third set provides an engagement for solving the Fundamental Problems throughout the book. They are available for selected sections in the chapters and marked with a video icon. The videos can be accessed in the Pearson eText or from a website available for purchase separately at www.pearsonglobaleditions.com.

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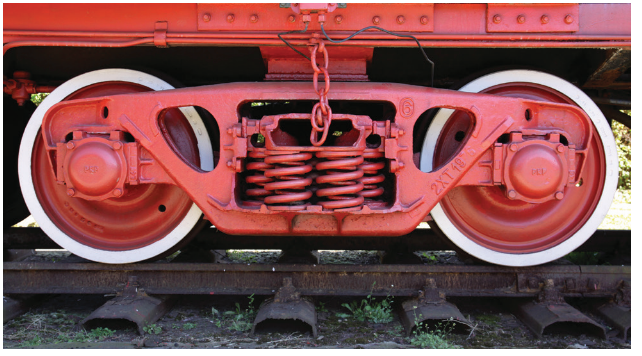

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ENGINEERING MECHANICS

DYNAMICS

FIFTEENTH EDITION IN SI UNITS

CHAPTER 12

Although these jet planes are rather large, from a distance their motion can be analyzed as if each were a particle.

KINEMATICS OF A PARTICLE

Lecture Summary and Quiz, Example, and Problem-solving videos are available where this icon appears.

CHAPTER OBJECTIVES

- To introduce the concepts of position, displacement, velocity, and acceleration.
- To study particle motion along a straight line and represent this motion graphically.
- To investigate particle motion along a curved path using different coordinate systems.
- To present an analysis of dependent motion of two particles.
- To examine the principles of relative motion of two particles using translating axes.

12.1 INTRODUCTION

Engineering mechanics is the study of the state of rest or motion of bodies subjected to the action of forces. It is divided into two areas, namely, statics and dynamics. ***Statics*** is concerned with the equilibrium of a body that is either at rest or moves with constant velocity. Here we will consider ***dynamics***, which deals with the accelerated motion of a body. This subject will be presented in two parts: *kinematics*, which treats only the geometric aspects of the motion, and *kinetics*, which is the analysis of the forces causing the motion. To develop these principles, the dynamics of a particle will be discussed first, followed by topics in rigid-body dynamics in two and then three dimensions.

Historically, the principles of dynamics developed when it was possible to make an accurate measurement of time. Galileo Galilei (1564–1642) was one of the first major contributors to this field. His work consisted of experiments using pendulums and falling bodies. The most significant contributions to dynamics, however, were made by Isaac Newton (1642–1727), who is noted for his formulation of the three fundamental laws of motion and the law of universal gravitational attraction. Shortly after these laws were postulated, important techniques for their application were developed by Euler, D'Alembert, Lagrange, and others.

There are many problems in engineering whose solutions require application of the principles of dynamics. For example, bridges and frames are subjected to moving loads and natural forces caused by wind and earthquakes. The structural design of any vehicle, such as an automobile or airplane, requires consideration of the motion to which it is subjected. This is also true for many mechanical devices, such as motors, pumps, movable tools, industrial manipulators, and machinery. Furthermore, predictions of the motions of artificial satellites, projectiles, and spacecraft are based on the theory of dynamics. With further advances in technology, there will be an even greater need for knowing how to apply the principles of this subject.

Problem Solving. Dynamics is considered to be more involved than statics since both the forces applied to a body and its motion must be taken into account. Also, many applications require using calculus, rather than just algebra and trigonometry. In any case, the most effective way of learning the principles of dynamics is ***to solve problems***. To be successful at this, it is necessary to present the work in a logical and orderly manner as suggested by the following sequence of steps:

1. Read the problem carefully and try to correlate the actual physical situation with the theory you have studied.
2. Draw any necessary diagrams and tabulate the problem data.
3. Establish a coordinate system and apply the relevant principles, generally in mathematical form.
4. Solve the necessary equations using a consistent set of units, and report the answer with no more than three significant figures, which is generally the accuracy of the given data.
5. Study the answer using technical judgment and common sense to determine whether or not it seems reasonable.

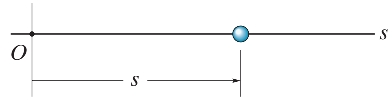
In applying this general procedure, do the work as neatly as possible. Being neat generally stimulates clear and orderly thinking, and vice versa. If you are having trouble developing your problem-solving skills, consider watching the videos available at www.pearson.com/hibbeler.

12.2 RECTILINEAR KINEMATICS: CONTINUOUS MOTION

We will begin our study of dynamics by discussing the kinematics of a particle that moves along a straight path. Recall that a ***particle*** has a mass but negligible size and shape, so we will limit application to those objects that have dimensions that are of no consequence in the analysis of the motion. For example, a rocket, projectile, or a vehicle can be considered as a particle, as long as its motion is characterized by the motion of its mass center, and any rotation of the body is neglected.

Rectilinear Kinematics. The kinematics of a particle is characterized by specifying, at any given instant, the particle's position, velocity, and acceleration.

Position. The rectilinear or straight-line path of a particle will be defined using a single coordinate axis s, Fig. 12–1a. The origin O on the path is a fixed point, and from this point the ***position coordinate*** s is used to specify the location of the particle at any given instant. The magnitude of s is the distance from O to the particle, usually measured in meters (m), and the sense of direction is defined by the algebraic sign of s. Although the choice is arbitrary, here s will be positive when the particle is located to the right of the origin, and it will be negative if the particle is located to the left of O. Position is actually a vector quantity since it has both magnitude and direction; however, it is being represented by the algebraic scalar s, rather than in boldface $\mathbf{s}$, since the direction always remains along the coordinate axis.

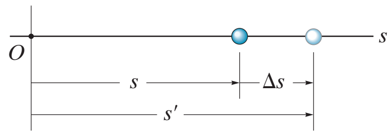


Position

(a)

Displacement. The ***displacement*** of the particle is defined as the *change in its position*. For example, if the particle moves from one point to another, Fig. 12–1b, the displacement is

$$\Delta s = s' - s$$



Displacement

(b)

Fig. 12–1

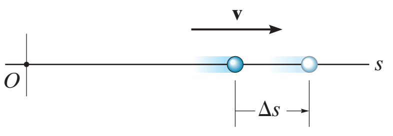
In this case Δs is *positive* since the particle's final position is to the *right* of its initial position, i.e., $s' > s$. Displacement is also a *vector quantity*, and it should be distinguished from the distance the particle travels. Specifically, the *distance traveled* is a *positive scalar* that represents the total length of path over which the particle travels.

Velocity. If the particle moves through a displacement Δs during the time interval Δt, the ***average velocity*** of the particle is

$$v_{avg} = \frac{\Delta s}{\Delta t}$$

If we take smaller and smaller values of Δt, the magnitude of Δs becomes smaller and smaller. Consequently, the ***instantaneous velocity*** is a vector defined as $v = \lim_{\Delta t \to 0} (\Delta s/\Delta t)$, or

$$(\overset{+}{\rightarrow}) \qquad v = \frac{ds}{dt} \qquad (12\text{–}1)$$



Velocity

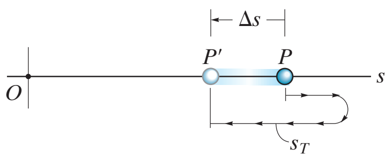
(c)

Since Δt or dt is always positive, the sign used to define the *sense* of the velocity is the same as that of Δs or ds. For example, if the particle is moving to the *right*, Fig. 12–1*c*, the velocity is *positive*; whereas if it is moving to the *left*, the velocity is *negative*. (This is emphasized here by the arrow written at the left of Eq. 12–1.) The *magnitude* of the velocity is known as the ***speed***, and it is generally expressed in units of m/s.

Occasionally, the term "average speed" is used. The ***average speed*** is always a positive scalar and is defined as the total distance traveled by a particle, s_T, divided by the elapsed time Δt; i.e.,

$$(v_{avg})_{sp} = \frac{s_T}{\Delta t}$$

For example, the particle in Fig. 12–1*d* travels along the path of length s_T in time Δt, so its average speed is $(v_{avg})_{sp} = s_T/\Delta t$, but its average velocity is $v_{avg} = -\Delta s/\Delta t$.



Average velocity and
Average speed

(d)

Fig. 12–1 (cont.)

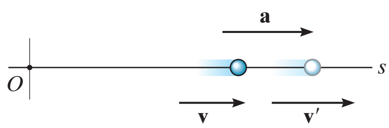
Acceleration. If the velocity of the particle is known at two points, then the ***average acceleration*** of the particle during the time interval Δt is defined as

$$a_{\text{avg}} = \frac{\Delta v}{\Delta t}$$

Here Δv represents the difference in the velocity during the time interval Δt, i.e., $\Delta v = v' - v$, Fig. 12–1*e*.

The ***instantaneous acceleration*** at time t is a *vector* that is found by taking smaller and smaller values of Δt and corresponding smaller and smaller values of Δv, so that $a = \lim_{\Delta t \to 0} (\Delta v / \Delta t)$, or

$$(\xrightarrow{+}) \qquad a = \frac{dv}{dt} \qquad (12\text{–}2)$$



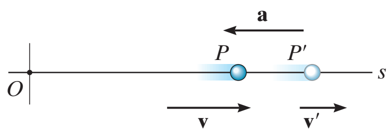
Acceleration

(e)

Substituting Eq. 12–1 into this result, we can also write

$$(\xrightarrow{+}) \qquad a = \frac{d^2s}{dt^2}$$

Both the average and instantaneous acceleration can be either positive or negative. In particular, when the particle is *slowing down*, or its speed is decreasing, the particle is said to be ***decelerating***. In this case, v' in Fig. 12–1*f* is *less* than v, and so $\Delta v = v' - v$ will be negative. Consequently, a will also be negative, and therefore it will act to the *left*, in the *opposite sense* to v. Also, notice that if the particle is originally at rest, then it can have an acceleration if a moment later it has a velocity v'. Units commonly used to express the magnitude of acceleration are m/s^2.



Deceleration

(f)

Fig. 12–1 (cont.)

Finally, an important differential relation involving the displacement, velocity, and acceleration along the path may be obtained by eliminating the time differential dt between Eqs. 12–1 and 12–2. We have

$$dt = \frac{ds}{v} = \frac{dv}{a}$$

or

$$(\xrightarrow{+}) \qquad a\,ds = v\,dv \qquad (12\text{–}3)$$

Although we have now produced three important kinematic equations, realize that the above equation is not independent of Eqs. 12–1 and 12–2.

Constant Acceleration, $a = a_c$. When the acceleration is constant, each of the three kinematic equations $a_c = dv/dt$, $v = ds/dt$, and $a_c\,ds = v\,dv$ can be integrated to obtain formulas that relate a_c, v, s, and t.

Velocity as a Function of Time. Integrating $a_c = dv/dt$, assuming that initially $v = v_0$ when $t = 0$, we get

$$\int_{v_0}^{v} dv = \int_0^t a_c\,dt$$

$$(\overset{+}{\rightarrow}) \qquad v = v_0 + a_c t \qquad (12\text{–}4)$$

Constant Acceleration

Position as a Function of Time. Integrating $v = ds/dt = v_0 + a_c t$, assuming that initially $s = s_0$ when $t = 0$, yields

$$\int_{s_0}^{s} ds = \int_0^t (v_0 + a_c t)\,dt$$

$$(\overset{+}{\rightarrow}) \qquad s = s_0 + v_0 t + \tfrac{1}{2} a_c t^2 \qquad (12\text{–}5)$$

Constant Acceleration

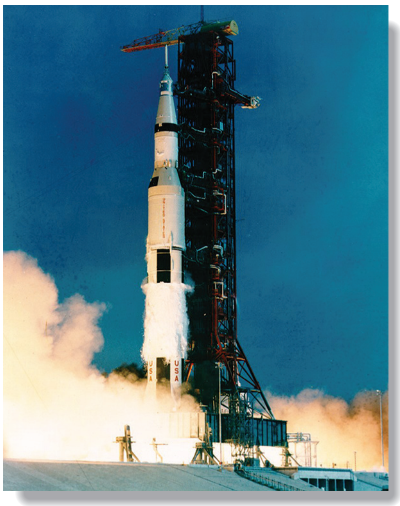
Velocity as a Function of Position. If we solve for t in Eq. 12–4 and substitute it into Eq. 12–5, or integrate $v\,dv = a_c\,ds$, assuming that initially $v = v_0$ at $s = s_0$, we get

$$\int_{v_0}^{v} v\,dv = \int_{s_0}^{s} a_c\,ds$$

$$(\overset{+}{\rightarrow}) \qquad v^2 = v_0^2 + 2a_c(s - s_0) \qquad (12\text{–}6)$$

Constant Acceleration

The algebraic signs of s_0, v_0, and a_c, used in these equations, are determined from the positive direction of the s axis as indicated by the arrow written at the left of each equation. It is important to remember that these equations are useful *only when the acceleration is constant and when $t = 0$, $s = s_0$, $v = v_0$*. A typical example of constant accelerated motion occurs when a body falls freely toward the earth. If air resistance is neglected and the distance of fall is short, then the constant *downward* acceleration of the body when it is close to the earth is approximately $9.81\ \text{m/s}^2$.



During the time this rocket undergoes rectilinear motion, its altitude as a function of time can be measured and expressed as $s = s(t)$. Its velocity can then be found using $v = ds/dt$, and its acceleration can be determined from $a = dv/dt$.

IMPORTANT POINTS

- Dynamics is the study of bodies that have accelerated motion.
- Kinematics is a study of the geometry of the motion.
- Kinetics is a study of the forces that cause the motion.
- Rectilinear kinematics refers to straight-line motion.
- Speed refers to the magnitude of velocity.
- Average speed is the total distance traveled divided by the total time. This is different from the average velocity, which is the displacement divided by the time.
- A particle that is slowing down is decelerating.
- A particle can have an acceleration and yet have zero velocity.
- The relationship $a\,ds = v\,dv$ is derived from $a = dv/dt$ and $v = ds/dt$, by eliminating dt.

PROCEDURE FOR ANALYSIS

Coordinate System.

- Establish a position coordinate s along the path and specify its *fixed origin* and positive direction.
- Since motion is along a straight line, the vector quantities position, velocity, and acceleration can be represented as algebraic scalars. For analytical work the sense of s, v, and a is then defined by their *algebraic signs*.
- The positive sense for each of these scalars can be indicated by an arrow shown alongside each kinematic equation as it is applied.

Kinematic Equations.

- If a relation is known between any *two* of the four variables a, v, s, and t, then a third variable can be obtained by using one of the kinematic equations, $a = dv/dt$, $v = ds/dt$ or $a\,ds = v\,dv$, since each equation relates all three variables.*
- Whenever integration is performed, it is important that the position and velocity be known at a given instant in order to evaluate either the constant of integration if an indefinite integral is used, or the limits of integration if a definite integral is used.
- Remember that Eqs. 12–4 through 12–6 have limited use. These equations apply *only* when the *acceleration is constant* and the initial conditions are $s = s_0$ and $v = v_0$ when $t = 0$.

* Some standard differentiation and integration formulas are given in Appendix A.

Refer to the companion website for Lecture Summary and Quiz videos.

EXAMPLE 12.1

The car in Fig. 12–2 moves in a straight line such that for a short time its velocity is defined by $v = (0.9t^2 + 0.6t)$ m/s, where t is in seconds. Determine its position and acceleration when $t = 3$ s. When $t = 0, s = 0$.

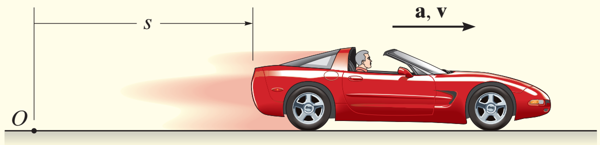


Fig. 12–2

SOLUTION

Coordinate System. The position coordinate extends from the fixed origin O to the car, positive to the right.

Position. Since $v = f(t)$, the car's position can be determined from $v = ds/dt$, since this equation relates v, s, and t. Noting that $s = 0$ when $t = 0$, we have*

$$(\overset{+}{\rightarrow}) \qquad v = \frac{ds}{dt} = (0.9t^2 + 0.6t)$$

$$\int_0^s ds = \int_0^t (0.9t^2 + 0.6t)dt$$

$$s\Big|_0^s = 0.3t^3 + 0.3t^2\Big|_0^t$$

$$s = 0.3t^3 + 0.3t^2$$

When $t = 3$ s,

$$s = 0.3(3)^3 + 0.3(3)^2 = 10.8 \text{ m} \qquad \textit{Ans.}$$

Acceleration. Since $v = f(t)$, the acceleration is determined from $a = dv/dt$, since this equation relates a, v, and t.

$$(\overset{+}{\rightarrow}) \qquad a = \frac{dv}{dt} = \frac{d}{dt}(0.9t^2 + 0.6t) = 1.8t + 0.6$$

When $t = 3$ s,

$$a = 1.8(3) + 0.6 = 6.00 \text{ m/s}^2 \rightarrow \qquad \textit{Ans.}$$

NOTE: The formulas for constant acceleration *cannot* be used to solve this problem, because the acceleration is a function of time.

* The *same result* can be obtained by evaluating a constant of integration C rather than using definite limits on the integral. For example, integrating $ds = (0.9t^2 + 0.6t)dt$ yields $s = 0.3t^3 + 0.3t^2 + C$. Using the condition that at $t = 0$, $s = 0$, then $C = 0$.

EXAMPLE 12.2

A small projectile is fired vertically *downward* into a fluid with an initial velocity of 60 m/s. Due to the drag resistance of the fluid the projectile experiences a deceleration of $a = (-0.4v^3)\ \text{m/s}^2$, where v is in m/s. Determine the projectile's velocity and position 4 s after it is fired.

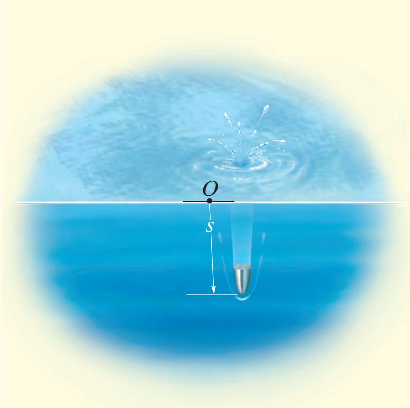


Fig. 12–3

SOLUTION

Coordinate System. Since the motion is downward, the position coordinate is positive downward, with origin located at O, Fig. 12–3.

Velocity. Here $a = f(v)$ and so we must determine the velocity as a function of time using $a = dv/dt$, since this equation relates v, a, and t. (Why not use $v = v_0 + a_c t$?) Separating the variables and integrating, with $v_0 = 60$ m/s when $t = 0$, yields*

$$(+\downarrow) \qquad a = \frac{dv}{dt} = -0.4v^3$$

$$\int_{60\text{ m/s}}^{v} \frac{dv}{-0.4v^3} = \int_0^t dt$$

$$\frac{1}{-0.4}\left(\frac{1}{-2}\right)\frac{1}{v^2}\Bigg|_{60}^{v} = t - 0$$

$$\frac{1}{0.8}\left[\frac{1}{v^2} - \frac{1}{(60)^2}\right] = t$$

$$v = \left\{\left[\frac{1}{(60)^2} + 0.8t\right]^{-1/2}\right\}\text{ m/s}$$

Here the positive root is taken, since the projectile will continue to move downward. When $t = 4$ s,

$$v = 0.559\text{ m/s}\downarrow \qquad \textit{Ans.}$$

Position. Knowing $v = f(t)$, we can obtain the projectile's position from $v = ds/dt$, since this equation relates s, v, and t. Using the initial condition $s = 0$, when $t = 0$, we have

$$(+\downarrow) \qquad v = \frac{ds}{dt} = \left[\frac{1}{(60)^2} + 0.8t\right]^{-1/2}$$

$$\int_0^s ds = \int_0^t \left[\frac{1}{(60)^2} + 0.8t\right]^{-1/2} dt$$

$$s = \frac{2}{0.8}\left[\frac{1}{(60)^2} + 0.8t\right]^{1/2}\Bigg|_0^t$$

$$s = \frac{1}{0.4}\left\{\left[\frac{1}{(60)^2} + 0.8t\right]^{1/2} - \frac{1}{60}\right\}\text{ m}$$

When $t = 4$ s,

$$s = 4.43\text{ m} \qquad \textit{Ans.}$$

* The *same result* can be obtained by evaluating a constant of integration C rather than using definite limits on the integral.

EXAMPLE 12.3

During a test the rocket in Fig. 12–4 travels upward at 75 m/s, and when it is 40 m from the ground its engine fails. Determine the maximum height s_B reached by the rocket and its speed just before it hits the ground. While in motion the rocket is subjected to a constant downward acceleration of 9.81 m/s^2 due to gravity. Neglect the effect of air resistance.

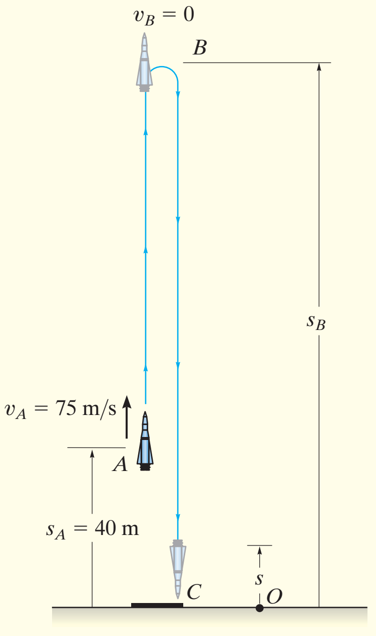


Fig. 12–4

SOLUTION

Coordinate System. The origin O for the position coordinate s is taken at ground level with positive upward, Fig. 12–4.

Maximum Height. Since the rocket is traveling *upward*, $v_A = +75$ m/s when $t = 0$. At the maximum height $s = s_B$ the velocity $v_B = 0$. For the entire motion, the acceleration is $a_c = -9.81\ \text{m/s}^2$ (negative since it acts in the *opposite* sense to positive velocity or positive displacement). Since a_c is *constant* the rocket's position may be related to its velocity at the two points A and B on the path by using Eq. 12–6, namely,

$$(+\uparrow) \qquad v_B^2 = v_A^2 + 2a_c(s_B - s_A)$$

$$0 = (75\ \text{m/s})^2 + 2(-9.81\ \text{m/s}^2)(s_B - 40\ \text{m})$$

$$s_B = 327\ \text{m} \qquad \textit{Ans.}$$

Velocity. To obtain the velocity of the rocket just before it hits the ground, we can apply Eq. 12–6 between points B and C, Fig. 12–4.

$$(+\uparrow) \qquad v_C^2 = v_B^2 + 2a_c(s_C - s_B)$$

$$= 0 + 2(-9.81\ \text{m/s}^2)(0 - 327\ \text{m})$$

$$v_C = -80.1\ \text{m/s} = 80.1\ \text{m/s}\downarrow \qquad \textit{Ans.}$$

The negative root was chosen since the rocket is moving downward.

Similarly, Eq. 12–6 may also be applied between points A and C, i.e.,

$$(+\uparrow) \qquad v_C^2 = v_A^2 + 2a_c(s_C - s_A)$$

$$= (75\ \text{m/s})^2 + 2(-9.81\ \text{m/s}^2)(0 - 40\ \text{m})$$

$$v_C = -80.1\ \text{m/s} = 80.1\ \text{m/s}\downarrow \qquad \textit{Ans.}$$

NOTE: It should be realized that the rocket is subjected to a *deceleration* from A to B of 9.81 m/s^2, and then from B to C it is *accelerated* at this rate. Furthermore, even though the rocket momentarily comes to *rest* at B ($v_B = 0$) the acceleration at B is still 9.81 m/s^2 downward!

EXAMPLE 12.4

A metallic particle is subjected to the influence of a magnetic field as it travels downward from plate A to plate B, Fig. 12–5. If the particle is released from rest at the midpoint C, $s = 100$ mm, and the acceleration is $a = (4s)$ m/s^2, where s is in meters, determine the velocity of the particle when it reaches plate B, $s = 200$ mm, and the time it takes to travel from C to B.

SOLUTION

Coordinate System. As shown in Fig. 12–5, s is positive downward, measured from plate A.

Velocity. Since $a = f(s)$, the velocity as a function of position can be obtained by using $v\,dv = a\,ds$. Realizing that $v = 0$ at $s = 0.1$ m, we have

$$(+\downarrow) \qquad v\,dv = a\,ds$$

$$\int_0^v v\,dv = \int_{0.1\text{ m}}^s 4s\,ds$$

$$\frac{1}{2}v^2\Big|_0^v = \frac{4}{2}s^2\Big|_{0.1\text{ m}}^s$$

$$v = 2(s^2 - 0.01)^{1/2}\text{ m/s} \qquad (1)$$

At $s = 200$ mm $= 0.2$ m,

$$v_B = 0.346\text{ m/s} = 346\text{ mm/s}\downarrow \qquad \textit{Ans.}$$

Time. The time for the particle to travel from C to B can be obtained using $v = ds/dt$ and Eq. 1, where $s = 0.1$ m when $t = 0$. From Appendix A,

$$(+\downarrow) \qquad ds = v\,dt$$

$$= 2(s^2 - 0.01)^{1/2}dt$$

$$\int_{0.1}^s \frac{ds}{(s^2 - 0.01)^{1/2}} = \int_0^t 2\,dt$$

$$\ln\left(\sqrt{s^2 - 0.01} + s\right)\Big|_{0.1}^s = 2t\Big|_0^t$$

$$\ln\left(\sqrt{s^2 - 0.01} + s\right) + 2.303 = 2t$$

At $s = 0.2$ m,

$$t = \frac{\ln\left(\sqrt{(0.2)^2 - 0.01} + 0.2\right) + 2.303}{2} = 0.658\text{ s} \qquad \textit{Ans.}$$

NOTE: The formulas for constant acceleration cannot be used here because the acceleration changes with position, i.e., $a = 4s$.

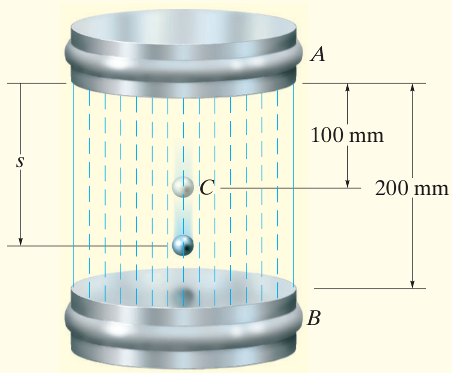
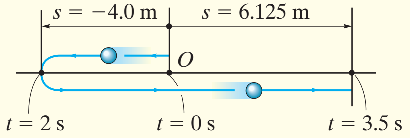


Fig. 12–5

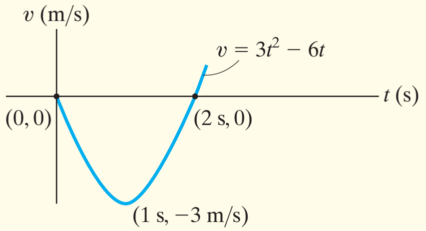
12

EXAMPLE 12.5

A particle moves along a horizontal path with a velocity of $v = (3t^2 - 6t)$ m/s, where t is in seconds. If it is initially located at the origin O, determine the distance traveled in 3.5 s, and the particle's average velocity and average speed during the time interval.



(a)



(b)

Fig. 12–6

SOLUTION

Coordinate System. Here positive motion is to the right, measured from the origin O, Fig. 12–6*a*.

Distance Traveled. Since $v = f(t)$, the position as a function of time may be found by integrating $v = ds/dt$ with $t = 0$, $s = 0$.

$$(\overset{+}{\rightarrow}) \qquad ds = v\,dt$$
$$= (3t^2 - 6t)\,dt$$
$$\int_0^s ds = \int_0^t (3t^2 - 6t)\,dt$$
$$s = (t^3 - 3t^2) \text{ m} \qquad (1)$$

In order to determine the distance traveled in 3.5 s, it is necessary to investigate the path of motion. If we graph the velocity function, Fig. 12–6*b*, then it shows that for $0 < t < 2$ s the velocity is *negative*, which means the particle is traveling to the *left*, and for $t > 2$ s the velocity is *positive*, and hence the particle is traveling to the *right*. Also, note that $v = 0$ when $t = 2$ s. The particle's position when $t = 0$, $t = 2$ s, and $t = 3.5$ s can be determined from Eq. 1. This yields

$$s|_{t=0} = 0 \quad s|_{t=2\text{ s}} = -4.0 \text{ m} \quad s|_{t=3.5\text{ s}} = 6.125 \text{ m}$$

The path is shown in Fig. 12–6*a*. Hence, the distance traveled in 3.5 s is

$$s_T = 4.0 + 4.0 + 6.125 = 14.125 \text{ m} = 14.1 \text{ m} \qquad \textit{Ans.}$$

Velocity. The *displacement* from $t = 0$ to $t = 3.5$ s is

$$\Delta s = s|_{t=3.5\text{ s}} - s|_{t=0} = 6.125 \text{ m} - 0 = 6.125 \text{ m}$$

and so the average velocity is

$$v_{avg} = \frac{\Delta s}{\Delta t} = \frac{6.125 \text{ m}}{3.5 \text{ s} - 0} = 1.75 \text{ m/s} \rightarrow \qquad \textit{Ans.}$$

The average speed is defined in terms of the *total distance traveled* s_T. This positive scalar is

$$(v_{avg})_{sp} = \frac{s_T}{\Delta t} = \frac{14.125 \text{ m}}{3.5 \text{ s} - 0} = 4.04 \text{ m/s} \qquad \textit{Ans.}$$

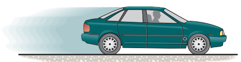
NOTE: In this problem, the acceleration is $a = dv/dt = (6t - 6)$ m/s^2, which is not constant.

Refer to the companion website for a self quiz of these Example problems.

FUNDAMENTAL PROBLEMS

Partial solutions and answers to all Fundamental Problems are given in the back of the book. Video solutions are available for select Fundamental Problems on the companion website.

F12–1. Initially, the car travels along a straight road with a speed of 35 m/s. If the brakes are applied and the speed of the car is reduced to 10 m/s in 15 s, determine the constant deceleration of the car.



Prob. F12–1

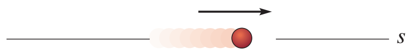
F12–2. A ball is thrown vertically upward with a speed of 15 m/s. Determine the time of flight when it returns to its original position.

Prob. F12–2

F12–3. A particle travels along a straight line with a velocity of $v = (4t - 3t^2)$ m/s, where t is in seconds. Determine the position of the particle when $t = 4$ s. $s = 0$ when $t = 0$.

F12–4. A particle travels along a straight line with a speed $v = (0.5t^3 - 8t)$ m/s, where t is in seconds. Determine the acceleration of the particle when $t = 2$ s.

F12–5. The position of the particle is $s = (2t^2 - 8t + 6)$ m, where t is in seconds. Determine the time when the velocity of the particle is zero, and the total distance traveled by the particle when $t = 3$ s.



Prob. F12–5

F12–6. A particle travels along a straight line with an acceleration of $a = (10 - 0.2s)$ m/s^2, where s is measured in meters. Determine the velocity of the particle when $s = 10$ m if $v = 5$ m/s at $s = 0$.

Prob. F12–6

F12–7. A particle moves along a straight line such that its acceleration is $a = (4t^2 - 2)$ m/s^2, where t is in seconds. When $t = 0$, the particle is located 2 m to the left of the origin, and when $t = 2$ s, it is 20 m to the left of the origin. Determine the position of the particle when $t = 4$ s.

F12–8. A particle travels along a straight line with a velocity of $v = (20 - 0.05s^2)$ m/s, where s is in meters. Determine the acceleration of the particle at $s = 15$ m.

12

PROBLEMS

12–1. A particle is moving along a straight line such that its position is defined by $s = (10t^2 + 20)$ mm, where t is in seconds. Determine (a) the displacement of the particle during the time interval from $t = 1$ s to $t = 5$ s, (b) the average velocity of the particle during this time interval, and (c) the acceleration when $t = 1$ s.

12–2. Starting from rest, a particle moving in a straight line has an acceleration of $a = (2t - 6)$ m/s^2, where t is in seconds. What is the particle's velocity when $t = 6$ s, and what is its position when $t = 11$ s?

12–3. A particle moves along a straight line such that its position is defined by $s = (t^2 - 6t + 5)$ m. Determine the average velocity, the average speed, and the acceleration of the particle when $t = 6$ s.

***12–4.** A particle travels along a straight line with a velocity $v = (12 - 3t^2)$ m/s, where t is in seconds. When $t = 1$ s, the particle is located 10 m to the left of the origin. Determine the acceleration when $t = 4$ s, the displacement from $t = 0$ to $t = 10$ s, and the distance the particle travels during this time period.

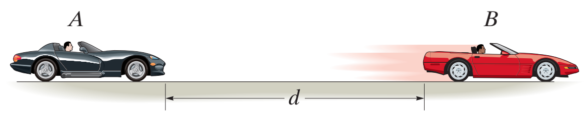
12–5. The acceleration of a particle as it moves along a straight line is given by $a = (2t - 1)$ m/s^2, where t is in seconds. If $s = 1$ m and $v = 2$ m/s when $t = 0$, determine the particle's velocity and position when $t = 6$ s. Also, determine the total distance the particle travels during this time period.

12–6. The velocity of a particle traveling in a straight line is given by $v = (6t - 3t^2)$ m/s, where t is in seconds. If $s = 0$ when $t = 0$, determine the particle's deceleration and position when $t = 3$ s. How far has the particle traveled during the 3-s time interval, and what is its average speed?

12–7. A particle moving along a straight line is subjected to a deceleration $a = (-2v^3)$ m/s^2, where v is in m/s. If it has a velocity $v = 8$ m/s and a position $s = 10$ m when $t = 0$, determine its velocity and position when $t = 4$ s.

***12–8.** A particle moves along a straight line such that its position is defined by $s = (2t^3 + 3t^2 - 12t - 10)$ m. Determine the velocity, average velocity, and the average speed of the particle when $t = 3$ s.

12–9. When two cars A and B are next to one another, they are traveling in the same direction with speeds v_A and v_B, respectively. If B maintains its constant speed, while A begins to decelerate at a_A, determine the distance d between the cars at the instant A stops.



Prob. 12–9

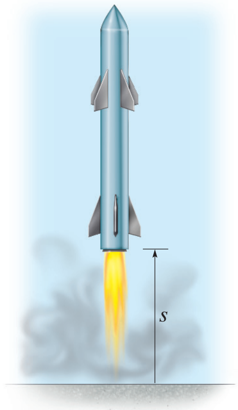
12–10. A particle moves along a straight path with an acceleration of $a = (5/s)$ m/s^2, where s is in meters. Determine the particle's velocity when $s = 2$ m, if it is released from rest when $s = 1$ m.

12–11. A particle moves along a straight line with an acceleration of $a = 5/(3s^{1/3} + s^{5/2})$ m/s^2, where s is in meters. Determine the particle's velocity when $s = 2$ m, if it starts from rest when $s = 1$ m. Use a numerical method to evaluate the integral.

***12–12.** A particle travels along a straight-line path such that in 4 s it moves from an initial position $s_A = -8$ m to a position $s_B = +3$ m. Then in another 5 s it moves from s_B to $s_C = -6$ m. Determine the particle's average velocity and average speed during the 9-s time interval.

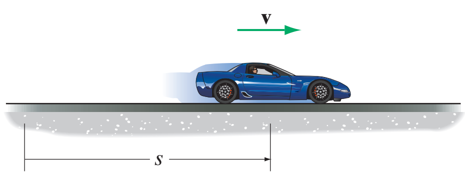
12–13. The speed of a particle traveling along a straight line within a liquid is measured as a function of its position as $v = (100 - s)$ mm/s, where s is in millimeters. Determine (a) the particle's deceleration when it is located at point A, where $s_A = 75$ mm, (b) the distance the particle travels before it stops, and (c) the time needed to stop the particle.

12–14. The acceleration of a rocket traveling upward is given by $a = (6 + 0.02s)\ \text{m/s}^2$, where s is in meters. Determine the rocket's velocity when $s = 2$ km and the time needed to reach this attitude. Initially, $v = 0$ and $s = 0$ when $t = 0$.



Prob. 12–14

12–15. The sports car travels along the straight road such that $v = 3\sqrt{100 - s}$ m/s, where s is in meters. Determine the time for the car to reach $s = 60$ m. How much time does it take to stop?

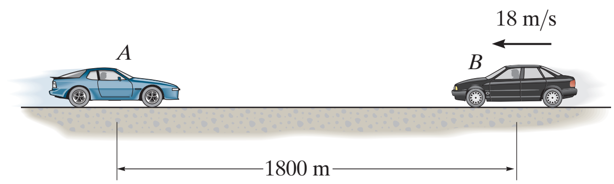


Prob. 12–15

***12–16.** A particle is moving with a velocity of v_0 when $s = 0$ and $t = 0$. If it is subjected to a deceleration of $a = -kv^3$, where k is a constant, determine its velocity and position as functions of time.

12–17. A particle is moving along a straight line with an initial velocity of 6 m/s when it is subjected to a deceleration of $a = (-1.5v^{1/2})\ \text{m/s}^2$, where v is in m/s. Determine how far it travels before it stops. How much time does this take?

12–18. Car A starts from rest at $t = 0$ and travels along a straight road with a constant acceleration of 1.8 m/s^2 until it reaches a speed of 24 m/s. Afterwards it maintains this speed. Also, when $t = 0$, car B located 1800 m down the road is traveling towards A at a constant speed of 18 m/s. Determine the distance traveled by car A when they pass each other.

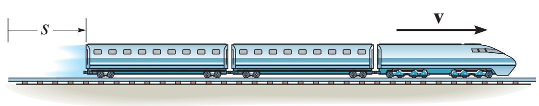


Prob. 12–18

12–19. A train starts from rest at station A and accelerates at 0.5 m/s^2 for 60 s. Afterwards it travels with a constant velocity for 15 min. It then decelerates at 1 m/s^2 until it is brought to rest at station B. Determine the distance between the stations.

***12–20.** A sandbag is dropped from a balloon which is ascending vertically at a constant speed of 6 m/s. If the bag is released with the same upward velocity of 6 m/s when $t = 0$ and hits the ground when $t = 8$ s, determine the speed of the bag as it hits the ground and the altitude of the balloon at this instant.

12–21. When a train is traveling along a straight track at 2 m/s, it begins to accelerate at $a = (60v^{-4})\ \text{m/s}^2$, where v is in m/s. Determine its velocity v and the position 3 s after the acceleration.

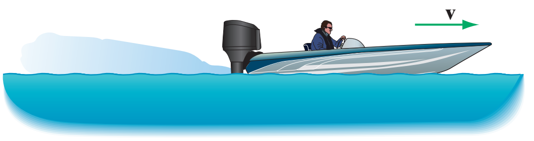


Prob. 12–21

12

12–22. When a particle falls through the air, its initial acceleration $a = g$ diminishes until it is zero, and thereafter it falls at a constant or terminal velocity v_f. If this variation of the acceleration can be expressed as $a = (g/v_f^2)(v_f^2 - v^2)$, determine the time needed for the velocity to become $v = v_f/2$. Initially the particle falls from rest.

12–23. The acceleration of the boat is defined by $a = (1.5\,v^{1/2})$ m/s. Determine its speed when $t = 4$ s if it has a speed of 3 m/s when $t = 0$.



Prob. 12–23

***12–24.** A particle is moving along a straight line such that its acceleration is defined as $a = (-2v)$ m/s^2, where v is in meters per second. If $v = 20$ m/s when $s = 0$ and $t = 0$, determine the particle's position, velocity, and acceleration as functions of time.

12–25. When a particle is projected vertically upward with an initial velocity of v_0, it experiences an acceleration $a = -(g + kv^2)$, where g is the acceleration due to gravity, k is a constant, and v is the velocity of the particle. Determine the maximum height reached by the particle.

12–26. If the effects of atmospheric resistance are accounted for, a freely falling body has an acceleration defined by the equation $a = 9.81[1 - v^2(10^{-4})]$ m/s^2, where v is in m/s and the positive direction is downward. If the body is released from rest at a *very high altitude*, determine (a) the velocity when $t = 5$ s, and (b) the body's terminal or maximum attainable velocity (as $t \to \infty$).

12–27. A ball is thrown with an upward velocity of 5 m/s from the top of a 10-m-high building. One second later another ball is thrown upward from the ground with a velocity of 10 m/s. Determine the height from the ground where the two balls pass each other.

***12–28.** As a body is projected to a high altitude above the earth's surface, the variation of the acceleration of gravity with respect to altitude y must be taken into account. Neglecting air resistance, this acceleration is determined from the formula $a = -g_0[R^2/(R + y)^2]$, where g_0 is the constant gravitational acceleration at sea level, R is the radius of the earth, and the positive direction is measured upward. If $g_0 = 9.81$ m/s^2 and $R = 6356$ km, determine the minimum initial velocity (escape velocity) at which a projectile should be shot vertically from the earth's surface so that it does not fall back to the earth. *Hint:* This requires that $v = 0$ as $y \to \infty$.

12–29. Accounting for the variation of gravitational acceleration a with respect to altitude y (see Prob. 12–28), derive an equation that relates the velocity of a freely falling particle to its altitude. Assume that the particle is released from rest at an altitude y_0 from the earth's surface. With what velocity does the particle strike the earth if it is released from rest at an altitude $y_0 = 500$ km? Use the numerical data in Prob. 12–28.

12–30. A train is initially traveling along a straight track at a speed of 90 km/h. For 6 s it is subjected to a constant deceleration of 0.5 m/s^2, and then for the next 5 s it has a constant deceleration a_c. Determine a_c so that the train stops at the end of the 11-s time period.

12–31. Two cars A and B start from rest at a stop line. Car A has a constant acceleration of $a_A = 8$ m/s^2, while Car B has an acceleration of $a_B = (2t^{3/2})$ m/s^2, where t is in seconds. Determine the distance between the cars when A reaches a velocity of $v_A = 120$ km/h.

***12–32.** A sphere is fired downward into a medium with an initial speed of 27 m/s. If it experiences a deceleration of $a = (-6t)$ m/s^2, where t is in seconds, determine the distance traveled before it stops.

12–33. The velocity of a particle traveling along a straight line is $v = v_0 - ks$, where k is constant. If $s = 0$ when $t = 0$, determine the position and acceleration of the particle as a function of time.

12–34. Ball A is thrown vertically upward from the top of a 30-m-high building with an initial velocity of 5 m/s. At the same instant another ball B is thrown upward from the ground with an initial velocity of 20 m/s. Determine the height from the ground and the time at which they pass.

12.3 RECTILINEAR KINEMATICS: ERRATIC MOTION

When a particle has erratic or changing motion, then its position, velocity, and acceleration *cannot* be described by a single continuous mathematical function along the entire path. Instead, a series of functions will be required to specify the motion at different intervals. For this reason, it is convenient to represent the motion as a graph. If this graph relates any two of the variables s, v, a, t, then it can be used to construct subsequent graphs relating two other variables since the variables are related by the differential relationships $v = ds/dt$, $a = dv/dt$, or $a\,ds = v\,dv$. Several situations are possible.

The s–t, v–t, and a–t Graphs. To construct the v–t graph given the s–t graph, Fig. 12–7*a*, the equation $v = ds/dt$ should be used, since it relates the variables s and t to v. This equation states that

$$\frac{ds}{dt} = v$$

$$\text{slope of } s\text{–}t \text{ graph} = \text{velocity}$$

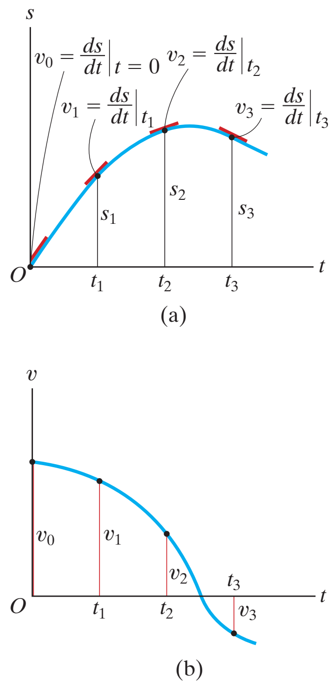


Fig. 12–7

For example, by measuring the slope on the s–t graph when $t = t_1$, the velocity is v_1, Fig. 12–7*a*. The v–t graph can be constructed by plotting this and other values at each instant, Fig. 12–7*b*.

The a–t graph can be constructed from the v–t graph in a similar manner, since

$$\frac{dv}{dt} = a$$

$$\text{slope of } v\text{–}t \text{ graph} = \text{acceleration}$$

Examples of various measurements are shown in Fig. 12–8*a* and plotted in Fig. 12–8*b*.

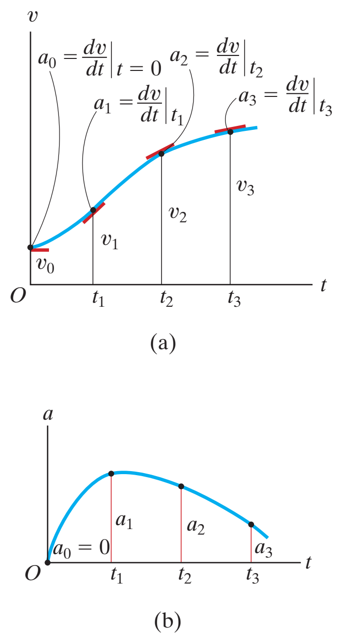


Fig. 12–8

If the s–t curve for each interval of motion can be expressed by a mathematical function $s = s(t)$, then the equation of the v–t and a–t graph for the same interval can be obtained from succesive derivatives of this function with respect to time since $v = ds/dt$ and $a = dv/dt$. Since differentiation reduces a polynomial of degree n to that of degree $n - 1$, then if the s–t graph is parabolic (a second-degree curve), the v–t graph will be a sloping line (a first-degree curve), and the a–t graph will be a constant or a horizontal line (a zero-degree curve).

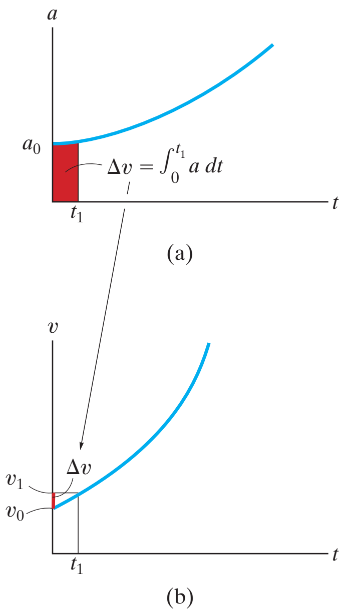


Fig. 12–9

If the a–t graph is given, Fig. 12–9a, the v–t graph may be constructed using $a = dv/dt$, written as

$$\Delta v = \int a\, dt$$

change in velocity = area under a–t graph

Therefore, to construct the v–t graph, we begin with the particle's initial velocity v_0 and then add to this small increments of area (Δv) determined from the a–t graph. In this manner successive points, $v_1 = v_0 + \Delta v$, etc., are determined, Fig. 12–9b. When doing this, an algebraic addition of the area increments of the a–t graph is necessary, since areas lying above the t axis correspond to an increase in v ("positive" area), whereas those lying below the axis indicate a decrease in v ("negative" area).

Similarly, if the v–t graph is given, Fig. 12–10a, it is possible to determine the s–t graph using $v = ds/dt$, written as

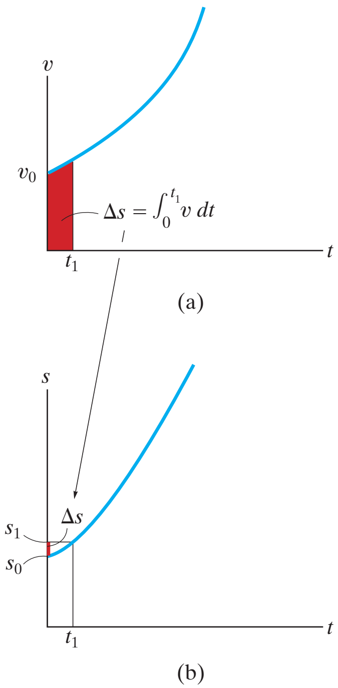


Fig. 12–10

$$\Delta s = \int v\, dt$$

displacement = area under v–t graph

Here we begin with the particle's initial position s_0 and add (algebraically) to this small area increments Δs determined from the v–t graph, Fig. 12–10b.

Due to the integration, if *segments* of the a–t graph can be described by a series of equations, then each of these equations can be successively *integrated* to yield equations describing the corresponding segments of the v–t and s–t graphs. As a result, if the a–t graph is linear (a first-degree curve), integration will yield a v–t graph that is parabolic (a second-degree curve) and an s–t graph that is cubic (third-degree curve).

The v–s and a–s Graphs. If the a–s graph can be constructed, then points on the v–s graph can be determined by using $v\,dv = a\,ds$. Integrating this equation between the limits $v = v_0$ at $s = s_0$ and $v = v_1$ at $s = s_1$, we have,

$$\frac{1}{2}(v_1^2 - v_0^2) = \int_{s_0}^{s_1} a\,ds$$

area under a–s graph

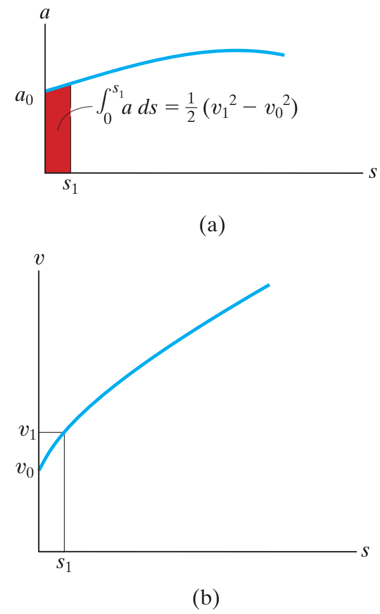


Fig. 12–11

For example, if the red area in Fig. 12–11*a* is determined, and the initial velocity v_0 at $s_0 = 0$ is known, then $v_1 = \left(2\int_0^{s_1} a\,ds + v_0^2\right)^{1/2}$, Fig. 12–11*b*. Other points on the v–s graph can be determined in this same manner.

If the v–s graph is known, the acceleration a at any position s can be determined using $a\,ds = v\,dv$, written as

$$a = v\left(\frac{dv}{ds}\right)$$

acceleration = velocity times slope of v–s graph

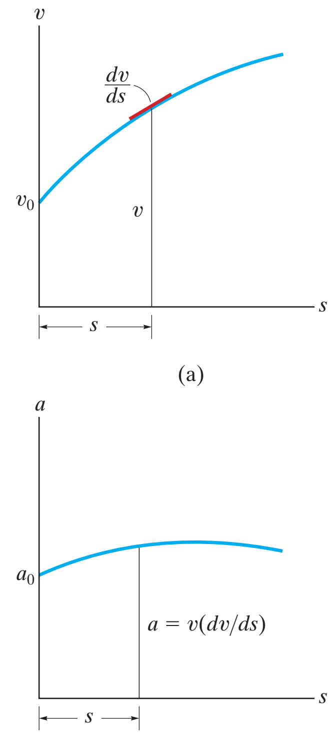


Fig. 12–12

For example, at point (s, v) in Fig. 12–12*a*, the slope dv/ds of the v–s graph is measured. Then with v and dv/ds known, the value of a can be calculated, Fig. 12–12*b*.

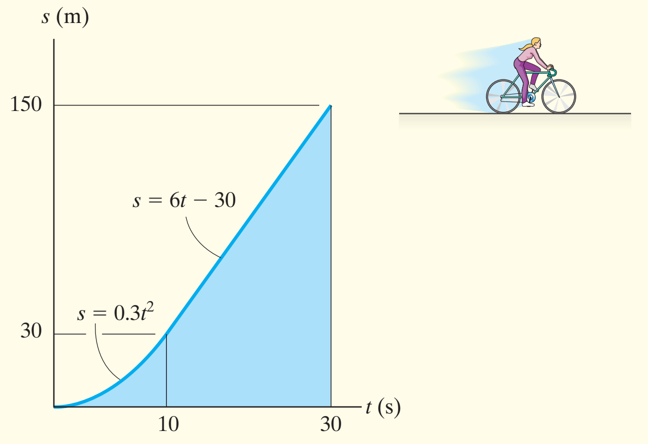
The v–s graph can also be constructed from the a–s graph, or vice versa, by approximating the known graph in various intervals with mathematical functions, $v = f(s)$ or $a = g(s)$, and then using $a\,ds = v\,dv$ to obtain the other graph.

Refer to the companion website for Lecture Summary and Quiz videos.

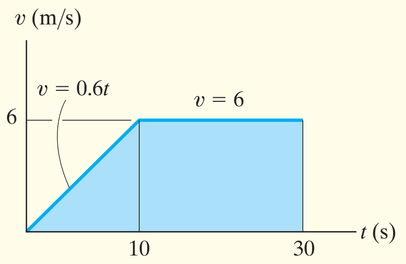
12

EXAMPLE 12.6

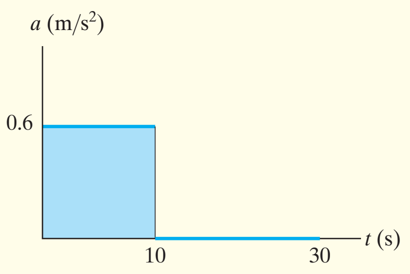
A bicycle moves along a straight road such that its position is described by the graph shown in Fig. 12–13*a*. Construct the v–t and a–t graphs for $0 \le t \le 30$ s.



(a)



(b)



(c)

Fig. 12–13

SOLUTION

v–t Graph. Since $v = ds/dt$, the v–t graph can be determined by differentiating the equations defining the s–t graph, Fig. 12–13*a*. We have

$$0 \le t < 10 \text{ s}; \qquad s = (0.3t^2)\text{ m} \qquad v = \frac{ds}{dt} = (0.6t)\text{ m/s}$$

$$10 \text{ s} < t \le 30 \text{ s}; \qquad s = (6t - 30)\text{ m} \qquad v = \frac{ds}{dt} = 6\text{ m/s}$$

These results are plotted in Fig. 12–13*b*. We can also obtain specific values of v by measuring the *slope* of the s–t graph at a given instant. For example, at $t = 20$ s, the slope of the s–t graph is determined from the straight line from 10 s to 30 s, i.e.,

$$t = 20 \text{ s}; \qquad v = \frac{\Delta s}{\Delta t} = \frac{150\text{ m} - 30\text{ m}}{30\text{ s} - 10\text{ s}} = 6\text{ m/s}$$

a–t Graph. Since $a = dv/dt$, the a–t graph can be determined by differentiating the equations defining the lines of the v–t graph. This yields

$$0 \le t < 10 \text{ s}; \qquad v = (0.6t)\text{ m/s} \qquad a = \frac{dv}{dt} = 0.6\text{ m/s}^2$$

$$10 < t \le 30 \text{ s}; \qquad v = 0.6\text{ m/s} \qquad a = \frac{dv}{dt} = 0$$

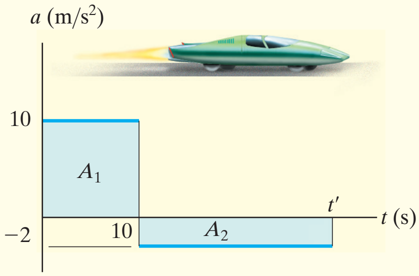
These results are plotted in Fig. 12–13*c*.

NOTE: The sudden change in a at $t = 10$ s represents a discontinuity, but actually this change must occur during a short, but finite time.

EXAMPLE 12.7

The car in Fig. 12–14*a* starts from rest and travels along a straight track such that it accelerates at 10 m/s^2 for 10 s, and then decelerates at 2 m/s^2. Draw the v–t graph and determine the time t' needed to stop the car.

SOLUTION



(a)

***v*–*t* Graph.** Since $dv = a\,dt$, the v–t graph is determined by integrating the straight-line segments of the a–t graph. Using the *initial condition* $v = 0$ when $t = 0$, we have

$$0 \le t < 10 \text{ s}; \qquad a = (10) \text{ m/s}^2; \qquad \int_0^v dv = \int_0^t 10\,dt, \quad v = 10t$$

When $t = 10$ s, $v = 10(10) = 100$ m/s. Using this as the *initial condition* for the next time period, we have

$$10 \text{ s} < t \le t'; \quad a = (-2) \text{ m/s}^2; \int_{100 \text{ m/s}}^v dv = \int_{10 \text{ s}}^t -2\,dt, \quad v = (-2t + 120) \text{ m/s}$$

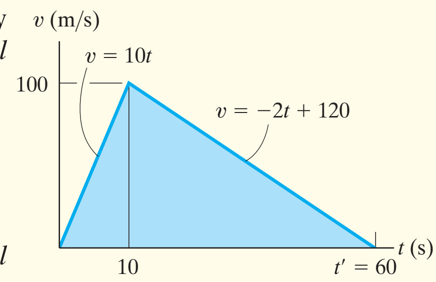
When $t = t'$ we require $v = 0$. This yields, Fig. 12–14*b*,

$$t' = 60 \text{ s} \qquad \textit{Ans.}$$

A direct solution for t' is also possible by realizing that the area under the a–t graph is equal to the change in the car's velocity. We require $\Delta v = 0 = A_1 + A_2$, Fig. 12–14*a*. Thus

$$0 = 10 \text{ m/s}^2(10 \text{ s}) + (-2 \text{ m/s}^2)(t' - 10 \text{ s})$$

$$t' = 60 \text{ s} \qquad \textit{Ans.}$$



(b)

Fig. 12–14

12

EXAMPLE 12.8

The v–s graph describing the motion of a motorcycle is shown in Fig. 12–15a. Construct the a–s graph of the motion and determine the time needed for the motorcycle to reach the position $s = 120$ m.

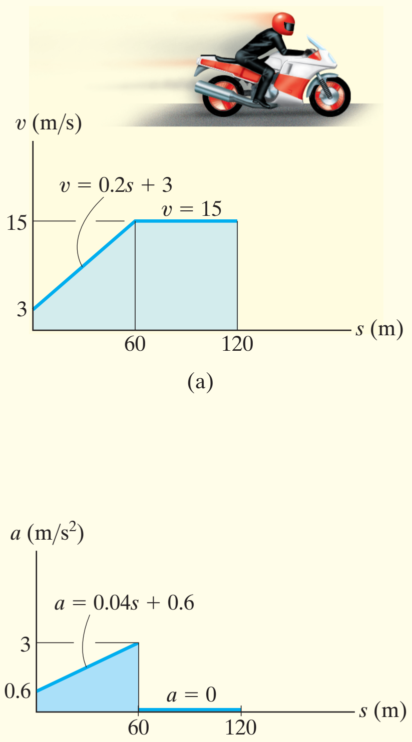


Fig. 12–15

SOLUTION

a–s Graph. Since the equations for segments of the v–s graph are given, the a–s graph can be determined using $a\,ds = v\,dv$.

$0 \leq s < 60$ m; $\quad v = (0.2s + 3)$ m/s

$$a = v\frac{dv}{ds} = (0.2s + 3)\frac{d}{ds}(0.2s + 3) = 0.04s + 0.6$$

$60 \text{ m} < s \leq 120$ m; $\quad v = 15$ m/s

$$a = v\frac{dv}{ds} = (15)\frac{d}{ds}(15) = 0$$

The results are plotted in Fig. 12–15b.

Time. The time can be obtained using the v–s graph and $v = ds/dt$, because this equation relates v, s, and t. For the first segment of motion, $s = 0$ when $t = 0$, so

$0 \leq s < 60$ m; $\quad v = (0.2s + 3)$ m/s; $\quad dt = \dfrac{ds}{v} = \dfrac{ds}{0.2s + 3}$

$$\int_0^t dt = \int_0^s \frac{ds}{0.2s + 3}$$

$$t = (5\ln(0.2s + 3) - 5\ln 3)\text{ s}$$

At $s = 60$ m, $t = 5\ln[0.2(60) + 3] - 5\ln 3 = 8.05$ s. Therefore, using these initial conditions for the second segment of motion,

$60 \text{ m} < s \leq 120$ m; $\quad v = 15$ m/s; $\quad dt = \dfrac{ds}{v} = \dfrac{ds}{15}$

$$\int_{8.05\text{ s}}^t dt = \int_{60\text{ m}}^s \frac{ds}{15}$$

$$t - 8.05 = \frac{s}{15} - 4;$$

$$t = \left(\frac{s}{15} + 4.05\right)\text{s}$$

Therefore, at $s = 120$ m,

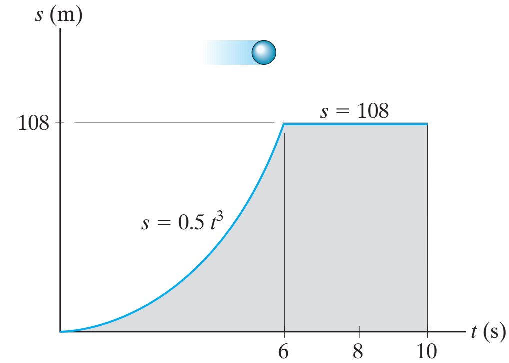
$$t = \frac{120}{15} + 4.05 = 12.0\text{ s} \qquad \textit{Ans.}$$

NOTE: The graphical results can be checked in part by calculating slopes. For example, at $s = 0$, $a = v(dv/ds) = 3(15 - 3)/60 = 0.6\text{ m/s}^2$. Also, the results can be checked in part by inspection. The v–s graph indicates the initial increase in velocity (acceleration) followed by constant velocity ($a = 0$).

Refer to the companion website for a self quiz of these Example problems.

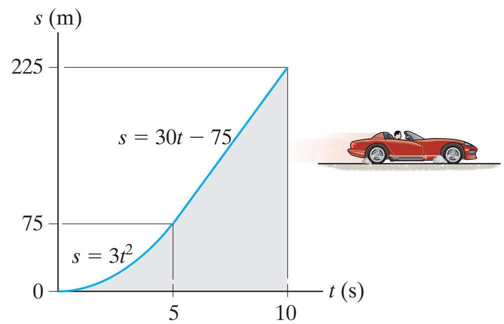
FUNDAMENTAL PROBLEMS

F12–9. Due to an external force, the particle travels along a straight track such that its position is described by the s–t graph. Construct the v–t graph for the same time interval. Take $v = 0$, $a = 0$ when $t = 0$.



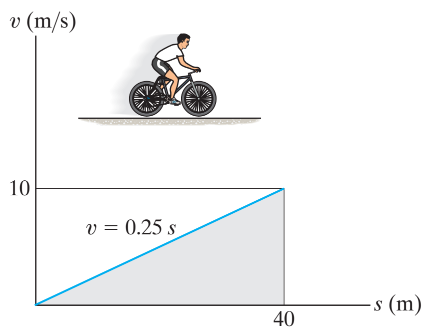
Prob. F12–9

F12–10. The sports car travels along a straight road such that its position is described by the graph. Construct the v–t and a–t graphs for the time interval $0 \le t \le 10$ s.



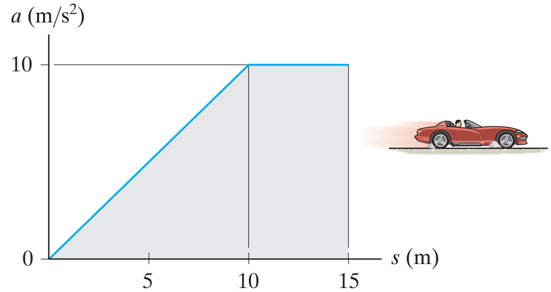
Prob. F12–10

F12–11. The rider begins to apply a force to the rear wheel of his bicycle, thereby initiating an acceleration. If his velocity is described by the v–s graph, construct the a–s graph for the same interval.



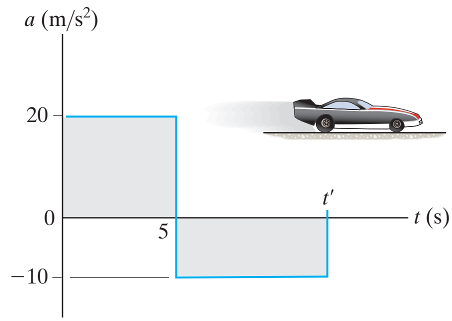
Prob. F12–11

F12–12. The sports car starts from rest and travels along a straight road. Its initial increasing acceleration is caused by the rear wheels of the car as shown on the graph. Construct the v–s graph. What is the velocity of the car when $s = 10$ m and $s = 15$ m?



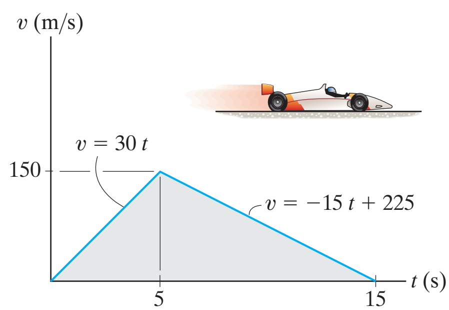
Prob. F12–12

F12–13. The dragster starts from rest and has an acceleration described by the graph. Construct the v–t graph for the time interval $0 \le t \le t'$, where t' is the time for the car to come to rest.



Prob. F12–13

F12–14. The dragster starts from rest and has a velocity described by the graph. Construct the s–t graph during the time interval $0 \le t \le 15$ s. Also, determine the total distance traveled during this time interval.



Prob. F12–14

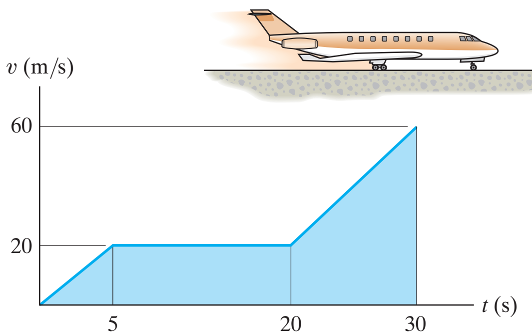
12

PROBLEMS

12–35. If the position of a particle is defined by $s = [3\sin(\pi/4)t + 8]$ m, where t is in seconds, construct the s–t, v–t, and a–t graphs for $0 \leq t \leq 10$ s.

***12–36.** A train starts from station A and for the first kilometer, it travels with a uniform acceleration. Then, for the next two kilometers, it travels with a uniform speed. Finally, the train decelerates uniformly for another kilometer before coming to rest at station B. If the time for the whole journey is six minutes, draw the v–t graph and determine the maximum speed of the train.

12–37. From experimental data, the motion of a jet plane while traveling along a runway is defined by the v–t graph. Construct the s–t and a–t graphs for the motion. When $t = 0, s = 0$.



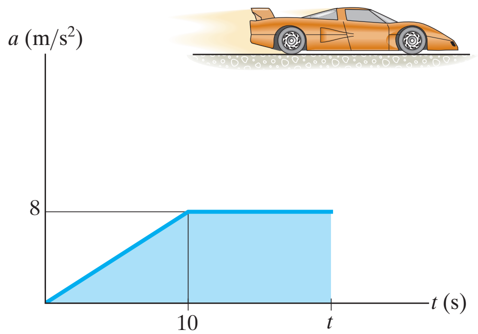
Prob. 12–37

12–38. Two rockets start from rest at the same elevation. Rocket A accelerates vertically at 20 m/s^2 for 12 s and then maintains a constant speed. Rocket B accelerates at 15 m/s^2 until reaching a constant speed of 150 m/s. Construct the a–t, v–t, and s–t graphs for each rocket until $t = 20$ s. What is the distance between the rockets when $t = 20$ s?

12–39. A particle starts from $s = 0$ and travels along a straight line with a velocity $v = (t^2 - 4t + 3)$ m/s, where t is in seconds. Construct the v–t and a–t graphs for the time interval $0 \leq t \leq 4$ s.

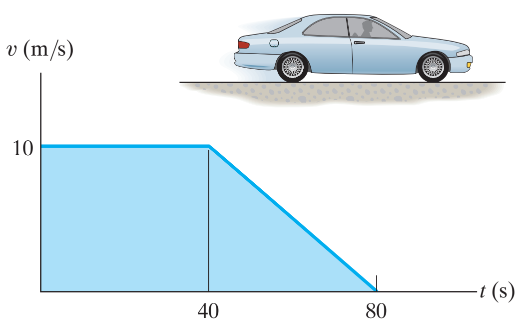
***12–40.** If the position of a particle is defined by $s = [2\sin(\pi/5)t + 4]$ m, where t is in seconds, construct the s–t, v–t, and a–t graphs for $0 \leq t \leq 10$ s.

12–41. A car starting from rest moves along a straight track with an acceleration as shown. Determine the time t for the car to reach a speed of 50 m/s and construct the v–t graph that describes the motion until the time t.



Prob. 12–41

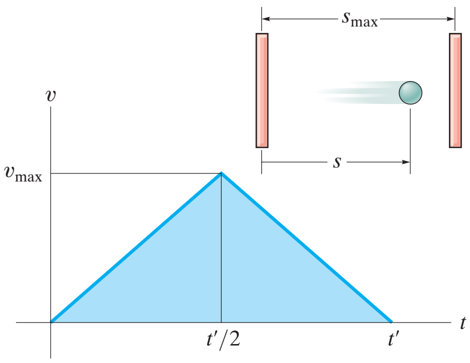
12–42. The velocity of a car is plotted as shown. Determine the total distance the car moves until it stops ($t = 80$ s). Construct the a–t graph.



Prob. 12–42

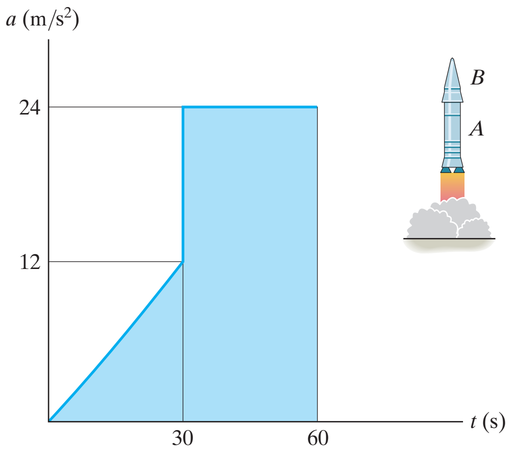
12–43. The v–t graph for a particle moving through an electric field from one plate to another has the shape shown in the figure. The acceleration and deceleration that occur are constant and both have a magnitude of $4\ \text{m/s}^2$. If the plates are spaced 200 mm apart, determine the maximum velocity v_{max} and the time t' for the particle to travel from one plate to the other. Also draw the s–t graph. When $t = t'/2$ the particle is at $s = 100$ mm.

***12–44.** The v–t graph for a particle moving through an electric field from one plate to another has the shape shown in the figure, where $t' = 0.2$ s and $v_{max} = 10$ m/s. Draw the s–t and a–t graphs for the particle. When $t = t'/2$ the particle is at $s = 0.5$ m.



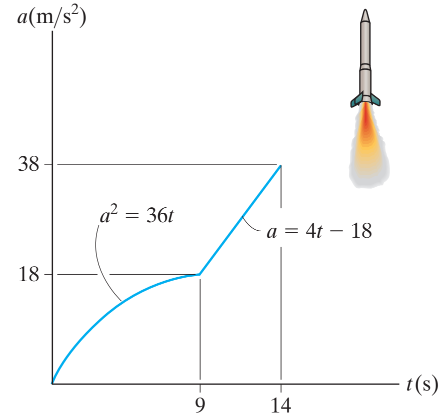
Probs. 12–43/44

12–45. A two-stage rocket is fired vertically from rest at $s = 0$ with the acceleration as shown. After 30 s the first stage, A, burns out and the second stage, B, ignites. Plot the v–t and s–t graphs which describe the motion of the second stage for $0 \leq t \leq 60$ s.



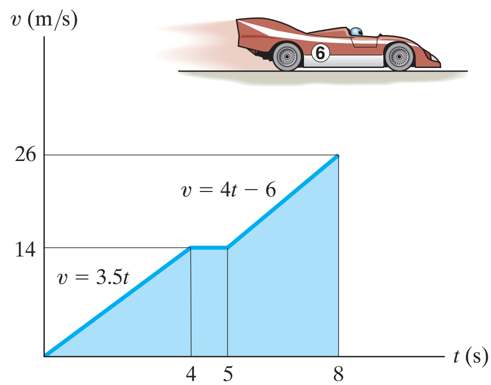
Prob. 12–45

12–46. The rocket has an acceleration described by the graph. If it starts from rest, construct the v–t and s–t graphs for the motion for the time interval $0 \leq t \leq 14$ s.



Prob. 12–46

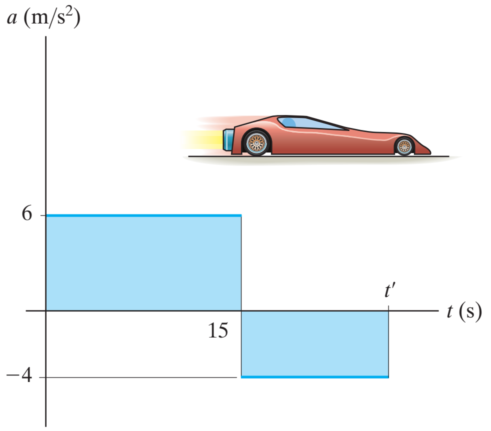
12–47. The race car starts from rest and travels along a straight road until it reaches a speed of 26 m/s in 8 s as shown on the v–t graph. The flat part of the graph is caused by shifting gears. Draw the a–t graph and determine the maximum acceleration of the car.



Prob. 12–47

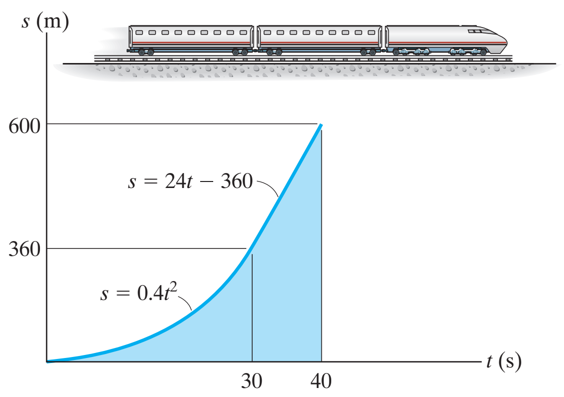
12

***12–48.** The jet car is originally traveling at a velocity of 10 m/s when it is subjected to the acceleration shown. Determine the car's maximum velocity and the time t' when it stops. When $t = 0$, $s = 0$.



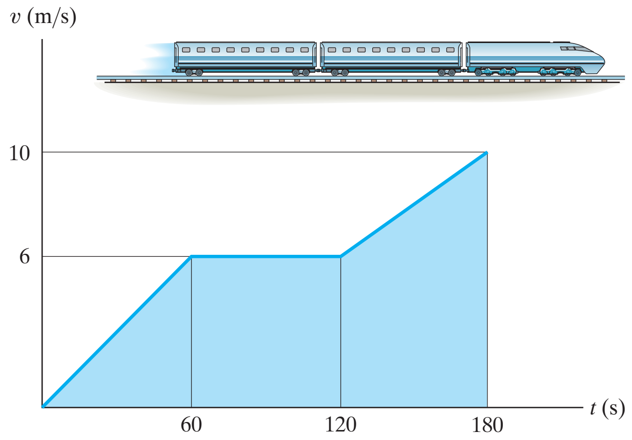
Prob. 12–48

12–49. The s–t graph for a train has been determined experimentally. From the data, construct the v–t and a–t graphs for the motion.



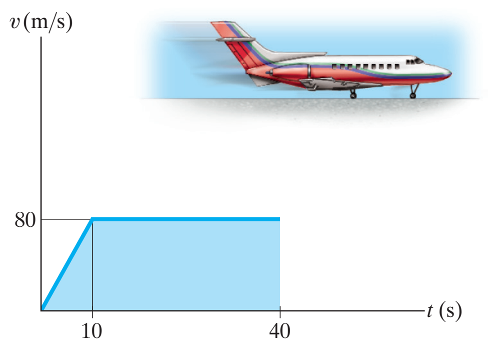
Prob. 12–49

12–50. The v–t graph for a train has been experimentally determined. From the data, construct the s–t and a–t graphs for the motion for $0 \le t \le 180$ s. When $t = 0$, $s = 0$.



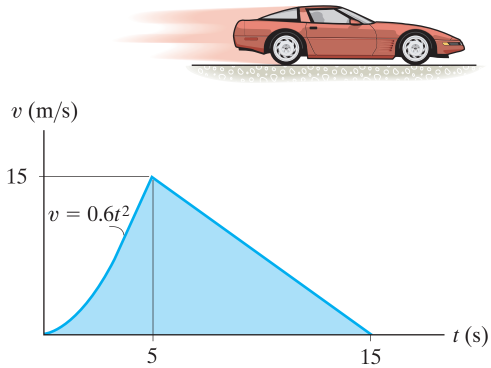
Prob. 12–50

12–51. From experimental data, the motion of a jet plane while traveling along a runway is defined by the v–t graph shown. Construct the s–t and a–t graphs for the motion.



Prob. 12–51

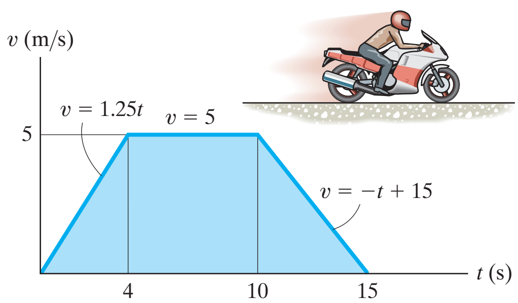
***12–52.** The v–t graph for the motion of a car as it moves along a straight road is shown. Draw the s–t and a–t graphs. Also determine the average speed and the distance traveled for the 15-s time interval. When $t = 0$, $s = 0$.



Prob. 12–52

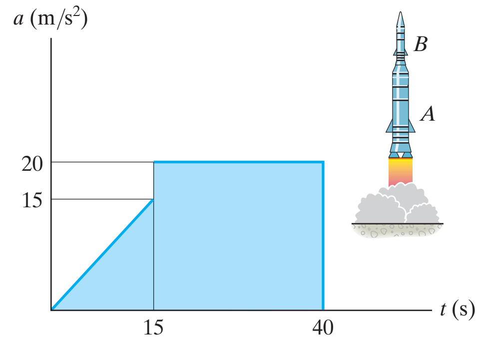
12–53. A motorcycle starts from rest at $s = 0$ and travels along a straight road with the speed shown by the v–t graph. Determine the total distance the motorcycle travels until it stops when $t = 15$ s. Also plot the a–t and s–t graphs.

12–54. A motorcycle starts from rest at $s = 0$ and travels along a straight road with the speed shown by the v–t graph. Determine the motorcycle's acceleration and position when $t = 8$ s and $t = 12$ s.



Probs. 12–53/54

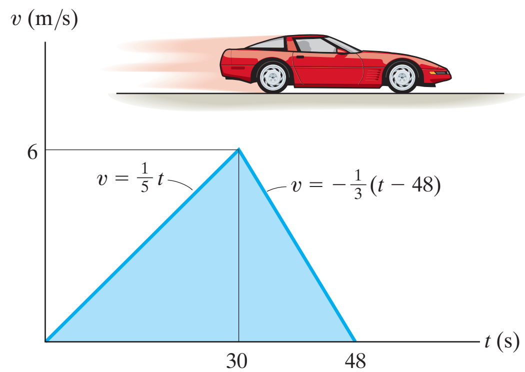
12–55. A two-stage rocket is fired vertically from rest with the acceleration shown. After 15 s the first stage A burns out and the second stage B ignites. Plot the v–t and s–t graphs which describe the motion of the second stage for $0 \leq t \leq 40$ s.



Prob. 12–55

***12–56.** A car travels along a straight road with the speed shown by the v–t graph. Plot the a–t graph.

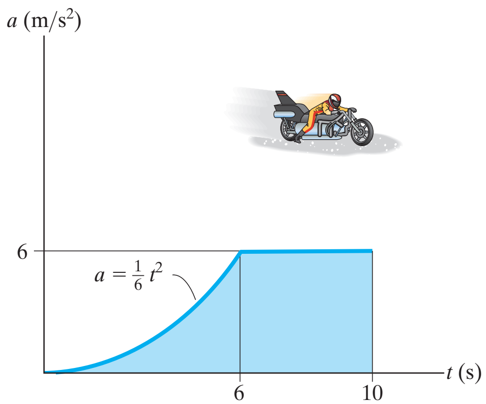
12–57. A car travels along a straight road with the speed shown by the v–t graph. Determine the total distance the car travels until it stops when $t = 48$ s. Also plot the s–t graph.



Probs. 12–56/57

12–58. Two cars start from rest side by side and travel along a straight road. Car A accelerates at $4\ \mathrm{m/s^2}$ for 10 s and then maintains a constant speed. Car B accelerates at $5\ \mathrm{m/s^2}$ until reaching a constant speed of 25 m/s and then maintains this speed. Construct the a–t, v–t, and s–t graphs for each car until $t = 15$ s. What is the distance between the two cars when $t = 15$ s?

12–59. A motorcyclist starting from rest travels along a straight road and for 10 s has an acceleration as shown. Draw the v–t graph that describes the motion and find the distance traveled in 10 s.



Prob. 12–59

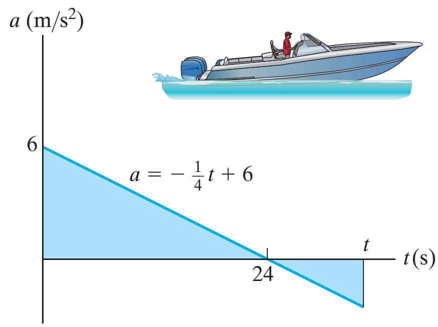
***12–60.** The speed of a train during the first minute has been recorded as follows:

t (s)	0	20	40	60
v (m/s)	0	16	21	24

Plot the v–t graph, approximating the curve as straight-line segments between the given points. Determine the total distance traveled.

12–61. A particle travels along a curve defined by the equation $s = (t^3 - 3t^2 + 2t)$ m, where t is in seconds. Draw the s–t, v–t, and a–t graphs for the particle for $0 \le t \le 3$ s.

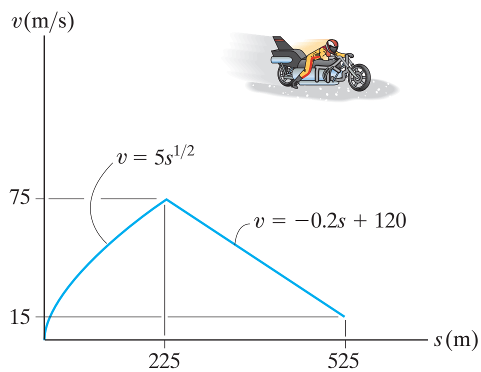
12–62. The boat is originally traveling at a speed of 8 m/s when it is subjected to the acceleration shown in the graph. Determine the boat's maximum speed and the time t when it stops.



Prob. 12–62

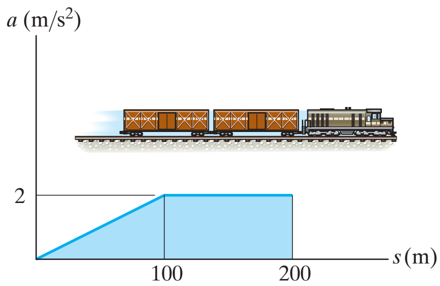
12–63. If the position of a particle is defined as $s = (5t - 3t^2)$ m, where t is in seconds, construct the s–t, v–t, and a–t graphs for $0 \le t \le 2.5$ s

***12–64.** The jet bike is moving along a straight road with the speed described by the v–s graph. Construct the a–s graph.



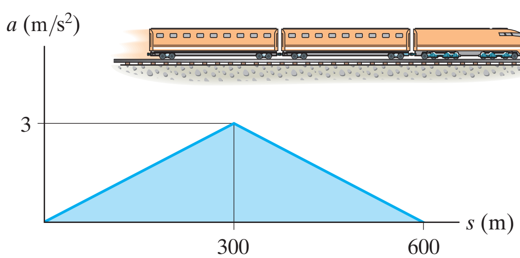
Prob. 12–64

12–65. The a–s graph for a freight train is given for the first 200 m of its motion. Plot the v–s graph. The train starts from rest.



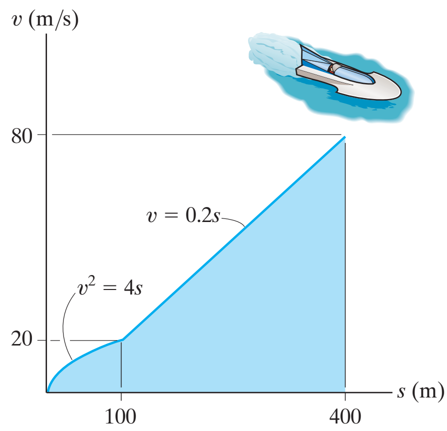
Prob. 12–65

12–66. The motion of a train is described by the a–s graph shown. Draw the v–s graph if $v = 0$ at $s = 0$.



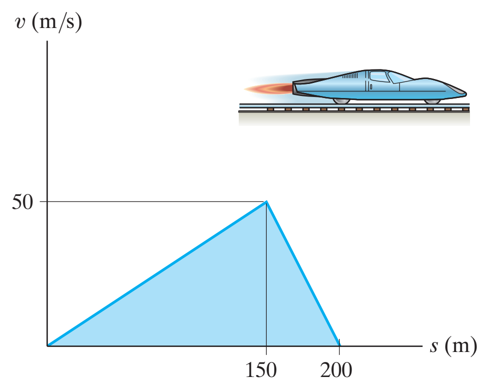
Prob. 12–66

12–67. The boat travels along a straight line with the speed described by the graph. Construct the s–t and a–s graphs. Also, determine the time required for the boat to travel a distance $s = 400$ m if $s = 0$ when $t = 0$.



Prob. 12–67

***12–68.** The v–s graph for a test vehicle is shown. Determine its acceleration when $s = 100$ m and when $s = 175$ m.



Prob. 12–68

12

12.4 GENERAL CURVILINEAR MOTION

Curvilinear motion occurs when a particle moves along a curved path. Since this path is often described in three dimensions, vector analysis will be used to formulate the particle's position, velocity, and acceleration.* In this section the general aspects of curvilinear motion are discussed, and in subsequent sections we will consider three types of coordinate systems often used to analyze this motion.

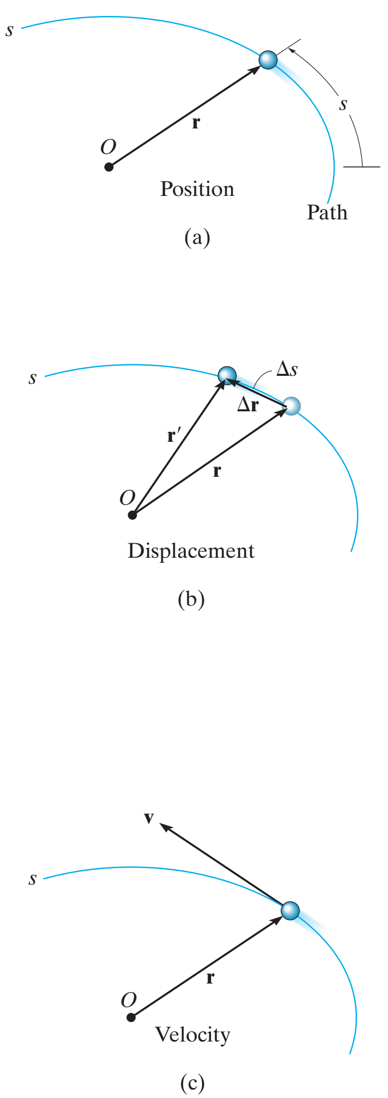


Fig. 12–16

Position. Consider a particle located at a point on a space curve defined by the path function $s(t)$, Fig. 12–16*a*. The position of the particle, measured from a fixed point O, will be designated by the *position vector* $\mathbf{r} = \mathbf{r}(t)$.

Displacement. If the particle moves a distance Δs along the curve to a new position, defined by $\mathbf{r}' = \mathbf{r} + \Delta\mathbf{r}$, Fig. 12–16*b*, then the *displacement* $\Delta\mathbf{r}$ represents the change in the particle's position and is determined by vector subtraction; i.e., $\Delta\mathbf{r} = \mathbf{r}' - \mathbf{r}$.

Velocity. If $\Delta\mathbf{r}$ occurs during the time Δt, then the *average velocity* of the particle is

$$\mathbf{v}_{avg} = \frac{\Delta\mathbf{r}}{\Delta t}$$

The *instantaneous velocity* is determined from this equation by letting $\Delta t \rightarrow 0$, and consequently the direction of $\Delta\mathbf{r}$ *approaches* the *tangent* to the curve. Hence, $\mathbf{v} = \lim_{\Delta t \rightarrow 0}(\Delta\mathbf{r}/\Delta t)$ or

$$\boxed{\mathbf{v} = \frac{d\mathbf{r}}{dt}} \qquad (12\text{–}7)$$

Since $d\mathbf{r}$ will be tangent to the curve, the *direction* of $\mathbf{v}$ is also *tangent to the curve*, Fig. 12–16*c*. The *magnitude* of $\mathbf{v}$, which is called the *speed*, is obtained by realizing that the length of the straight-line segment $\Delta\mathbf{r}$ in Fig. 12–16*b* approaches the arc length Δs as $\Delta t \rightarrow 0$, and so we have $v = \lim_{\Delta t \rightarrow 0}(\Delta r/\Delta t) = \lim_{\Delta t \rightarrow 0}(\Delta s/\Delta t)$, or

$$\boxed{v = \frac{ds}{dt}} \qquad (12\text{–}8)$$

Thus, the *speed* can be obtained by differentiating the path function s with respect to time.

* A summary of some of the important concepts of vector analysis is given in Appendix B.

Acceleration. If the particle has a velocity $\mathbf{v}$ at time t and a velocity $\mathbf{v}' = \mathbf{v} + \Delta\mathbf{v}$ at $t + \Delta t$, Fig. 12–16*d*, then the *average acceleration* of the particle during the time interval Δt is

$$\mathbf{a}_{\text{avg}} = \frac{\Delta\mathbf{v}}{\Delta t}$$

where $\Delta\mathbf{v} = \mathbf{v}' - \mathbf{v}$. To study this time rate of change, the two velocity vectors in Fig. 12–16*d* are plotted in Fig. 12–16*e* such that their tails are located at the fixed point O' and their arrowheads touch points on a curve. This curve is called a ***hodograph***, and when constructed, it describes the locus of points for the arrowhead of the velocity vector in the same manner as the *path* describes the locus of points for the arrowhead of the position vector, Fig. 12–16*a*.

To obtain the *instantaneous acceleration*, let $\Delta t \to 0$, and so $\mathbf{a} = \lim_{\Delta t \to 0} (\Delta\mathbf{v}/\Delta t)$, or

$$\boxed{\mathbf{a} = \frac{d\mathbf{v}}{dt}} \qquad (12\text{–}9)$$

Substituting Eq. 12–7 into this result, we can also write

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}$$

By definition of the derivative, $\mathbf{a}$ acts *tangent to the hodograph*, Fig. 12–16*f*, and, *in general it is not tangent to the path of motion*, Fig. 12–16*g*.

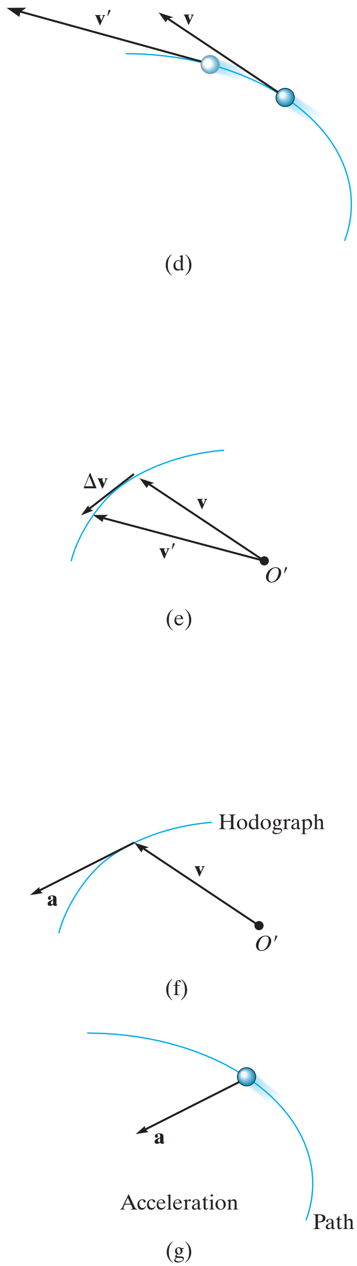


Fig. 12–16

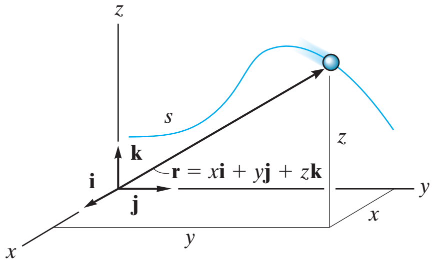
12

12.5 CURVILINEAR MOTION: RECTANGULAR COMPONENTS

Occasionally the motion of a particle can best be described along a path that is expressed in terms of its x, y, z coordinates.

Position. If the particle is at point (x, y, z) on the path shown in Fig. 12–17*a*, then its location is defined by the *position vector*

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{12–10}$$



Position

(a)

When the particle moves, the x, y, z components of $\mathbf{r}$ will be functions of time; i.e., $x = x(t)$, $y = y(t)$, $z = z(t)$, so that $\mathbf{r} = \mathbf{r}(t)$.

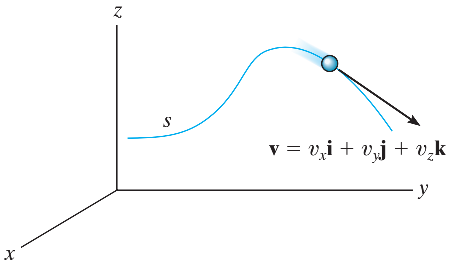
At any instant the *magnitude* of $\mathbf{r}$ is determined from Eq. B–3 in Appendix B as

$$r = \sqrt{x^2 + y^2 + z^2}$$

And the *direction* of $\mathbf{r}$ is specified by the unit vector $\mathbf{u}_r = \mathbf{r}/r$.

Velocity. The time derivative of $\mathbf{r}$ yields the velocity of the particle. Hence,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(x\mathbf{i}) + \frac{d}{dt}(y\mathbf{j}) + \frac{d}{dt}(z\mathbf{k})$$



Velocity

(b)

Fig. 12–17

When taking this derivative, it is necessary to account for changes in *both* the magnitude and direction of each of the vector's components. For example, the derivative of the $\mathbf{i}$ component of $\mathbf{r}$ is

$$\frac{d}{dt}(x\mathbf{i}) = \frac{dx}{dt}\mathbf{i} + x\frac{d\mathbf{i}}{dt}$$

The last term is zero, because the x, y, z reference frame is *fixed*, and therefore the *direction* (and the *magnitude*) of $\mathbf{i}$ does not change with time. Differentiation of the $\mathbf{j}$ and $\mathbf{k}$ components are carried out in a similar manner, and so the final result is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} \tag{12–11}$$

where

$$v_x = \dot{x} \quad v_y = \dot{y} \quad v_z = \dot{z} \tag{12–12}$$

The "dot" notation $\dot{x}, \dot{y}, \dot{z}$ represents the first time derivatives of $x = x(t)$, $y = y(t)$, $z = z(t)$, respectively.

The velocity has a *magnitude* that is found from

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

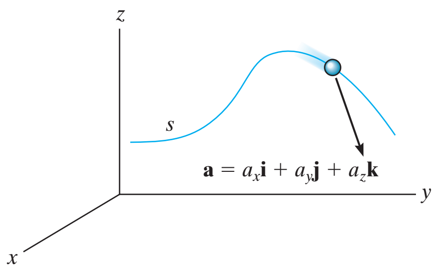
and a *direction* that is specified by the unit vector $\mathbf{u}_v = \mathbf{v}/v$. As discussed in Sec. 12.4, this direction is *always tangent to the path*, as shown in Fig. 12–17*b*.

Acceleration. The acceleration of the particle is obtained by taking the time derivative of Eq. 12–11 (or the second time derivative of Eq. 12–10). We have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \tag{12–13}$$

where

$$\begin{aligned} a_x &= \dot{v}_x = \ddot{x} \\ a_y &= \dot{v}_y = \ddot{y} \\ a_z &= \dot{v}_z = \ddot{z} \end{aligned} \tag{12–14}$$



Acceleration

(c)

Fig. 12–17

Here a_x, a_y, a_z represent the time derivatives of $v_x = v_x(t)$, $v_y = v_y(t)$, $v_z = v_z(t)$, or the second time derivatives of $x = x(t)$, $y = y(t)$, $z = z(t)$.

The acceleration has a *magnitude*

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

and a *direction* specified by the unit vector $\mathbf{u}_a = \mathbf{a}/a$. Since $\mathbf{a}$ represents the time rate of *change* in both the magnitude and direction of the velocity, in general $\mathbf{a}$ will *not* be tangent to the path, Fig. 12–17*c*.

IMPORTANT POINTS

- Curvilinear motion will generally cause changes in *both* the magnitude and direction of the particle's position, velocity, and acceleration.
- The velocity is always directed *tangent* to the path.
- In general, the acceleration is *not* tangent to the path, but rather, it is tangent to the hodograph.
- If the motion is described using rectangular coordinates, then the components along the x, y, z axes do not change direction, only their magnitude and sense (algebraic sign) will change.

PROCEDURE FOR ANALYSIS

Coordinate System.

- A rectangular coordinate system should be used to solve problems in cases where the motion can conveniently be expressed in terms of its x, y, z components.

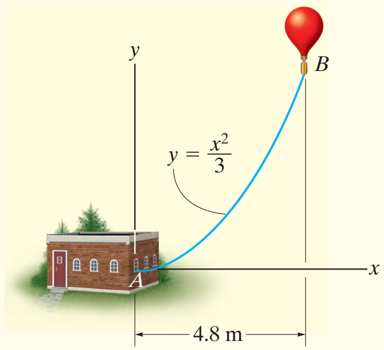
Kinematic Quantities.

- Since *rectilinear or straight-line motion* occurs along *each coordinate axis*, then $v = ds/dt$ and $a = dv/dt$; or in cases where the motion is not expressed as a function of time, the equation $a\,ds = v\,dv$ can be used.
- In two dimensions, the equation of the path $y = f(x)$ can be used to *relate* the x and y components of velocity and acceleration by applying the chain rule of calculus. A review of this concept is given in Appendix C.
- Once the x, y, z components of $\mathbf{v}$ and $\mathbf{a}$ have been determined, the magnitudes of these vectors are found from the Pythagorean theorem, Eq. B–3, and their coordinate direction angles from the components of their unit vectors, Eqs. B–4 and B–5.

Refer to the companion website for Lecture Summary and Quiz videos.

EXAMPLE 12.9

At any instant the horizontal position of the weather balloon in Fig. 12–18*a* is defined by $x = (2.4t)$ m, where t is in seconds. If the equation of the path is $y = x^2/3$, determine the magnitude and direction of the balloon's velocity and acceleration when $t = 2$ s.



(a)

SOLUTION

Velocity. The velocity component in the x direction is

$$v_x = \dot{x} = \frac{d}{dt}(2.4t) = 2.4 \text{ m/s} \rightarrow$$

To find the relationship between the velocity components we will use the chain rule of calculus. When $t = 2$ s, $x = 2.4(2) = 4.8$ m, Fig. 12–18*a*, and so

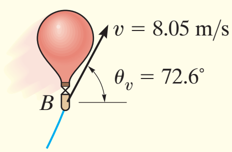
$$v_y = \dot{y} = \frac{d}{dt}(x^2/3) = 2x\dot{x}/3 = 2(4.8)(2.4)/3 = 7.68 \text{ m/s} \uparrow$$

When $t = 2$ s, the magnitude of velocity is therefore

$$v = \sqrt{(2.4 \text{ m/s})^2 + (7.68 \text{ m/s})^2} = 8.05 \text{ m/s} \qquad \textit{Ans.}$$

The velocity is tangent to the path, Fig. 12–18*b*, where

$$\theta_v = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\frac{7.68}{2.4} = 72.6^\circ \qquad \textit{Ans.}$$



(b)

Acceleration. The relationship between the acceleration components is determined using the chain rule. (See Appendix C.) We have

$$a_x = \dot{v}_x = \frac{d}{dt}(2.4) = 0$$

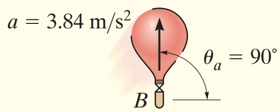
$$a_y = \dot{v}_y = \frac{d}{dt}(2x\dot{x}/3) = 2(\dot{x})\dot{x}/3 + 2x(\ddot{x})/3$$
$$= 2(2.4)^2/3 + 2(4.8)(0)/3 = 3.84 \text{ m/s}^2 \uparrow$$

Thus,

$$a = \sqrt{(0)^2 + (3.84)^2} = 3.84 \text{ m/s}^2 \qquad \textit{Ans.}$$

The direction of **a**, as shown in Fig. 12–18*c*, is

$$\theta_a = \tan^{-1}\frac{3.84}{0} = 90^\circ \qquad \textit{Ans.}$$



(c)

Fig. 12–18

NOTE: It is also possible to obtain v_y and a_y by first expressing $y = f(t) = (2.4t)^2/3 = 1.92t^2$ and then taking successive time derivatives.

12

EXAMPLE 12.10

For a short time, the path of the plane in Fig. 12–19*a* is described by $y = (0.001x^2)$ m. If the plane is rising with a constant upward velocity of 10 m/s, determine the magnitudes of the velocity and acceleration of the plane when it reaches an altitude of $y = 100$ m.

SOLUTION

When $y = 100$ m, then $100 = 0.001x^2$ or $x = 316.2$ m. Also, due to constant velocity $v_y = 10$ m/s, and so the time is

$$y = v_y t; \qquad 100\text{ m} = (10\text{ m/s})\,t \qquad t = 10\text{ s}$$

Velocity. Using the chain rule (see Appendix C, Eq. C–1) to find the relationship between the velocity components, we have

$$y = 0.001x^2$$

$$v_y = \dot{y} = \frac{dy}{dx}\dot{x} = \frac{d}{dt}(0.001x^2) = (0.002x)\dot{x} = 0.002xv_x \qquad (1)$$

Thus

$$10\text{ m/s} = 0.002(316.2\text{ m})(v_x)$$
$$v_x = 15.81\text{ m/s}$$

The magnitude of the velocity is therefore

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(15.81\text{ m/s})^2 + (10\text{ m/s})^2} = 18.7\text{ m/s} \qquad \textit{Ans.}$$

Acceleration. Using the chain rule, or Eq. C–2, the time derivative of Eq. 1 gives the relation between the acceleration components.

$$a_y = \dot{v}_y = (0.002\dot{x})\dot{x} + 0.002x(\ddot{x}) = 0.002(v_x^2 + xa_x)$$

When $x = 316.2$ m, $v_x = 15.81$ m/s, $\dot{v}_y = a_y = 0$, so that

$$0 = 0.002[(15.81\text{ m/s})^2 + 316.2\text{ m}(a_x)]$$
$$a_x = -0.791\text{ m/s}^2$$

The magnitude of the plane's acceleration is therefore

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(-0.791\text{ m/s}^2)^2 + (0\text{ m/s}^2)^2}$$
$$= 0.791\text{ m/s}^2 \qquad \textit{Ans.}$$

These results are shown in Fig. 12–19*b*.

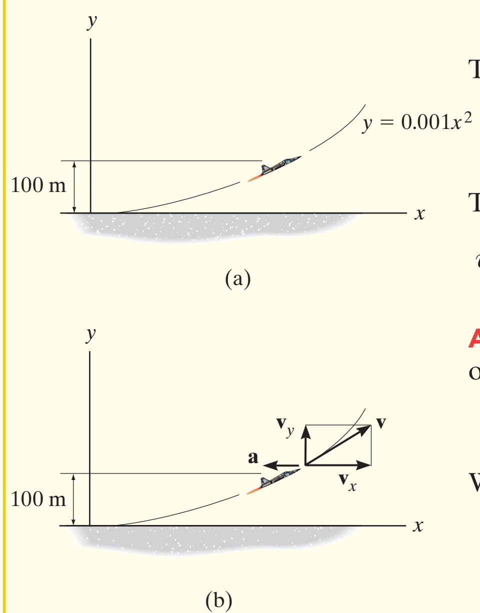


Fig. 12–19

Refer to the companion website for a self quiz of these Example problems.

12.6 MOTION OF A PROJECTILE

It is convenient to analyze the free-flight motion of a projectile in terms of its rectangular components. To illustrate, consider a projectile launched at point (x_0, y_0), with an initial velocity of $\mathbf{v}_0$, having components $(\mathbf{v}_0)_x$ and $(\mathbf{v}_0)_y$, Fig. 12–20. When air resistance is neglected, the only force acting on the projectile is its weight, and this causes the projectile to have a *constant downward acceleration* of $a_c = g = 9.81\ \text{m/s}^2$.*

Each picture in this sequence is taken after the same time period. In the horizontal direction the distance between the ball is the same because its velocity component is constant. The vertical distance between the ball is different because of the vertical deceleration-acceleration due to gravity.

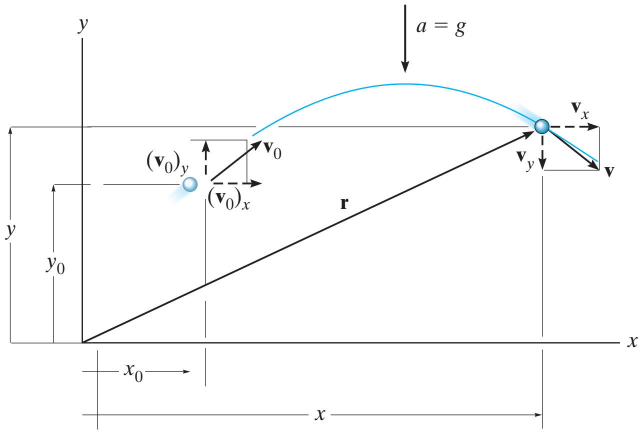


Fig. 12–20

Horizontal Motion.

Since $a_x = 0$, application of the constant acceleration equations, 12–4 to 12–6, yields

$(\overset{+}{\rightarrow})$ $\quad v = v_0 + a_c t; \qquad v_x = (v_0)_x$

$(\overset{+}{\rightarrow})$ $\quad x = x_0 + v_0 t + \frac{1}{2} a_c t^2; \qquad x = x_0 + (v_0)_x t$

$(\overset{+}{\rightarrow})$ $\quad v^2 = v_0^2 + 2a_c(x - x_0); \qquad v_x = (v_0)_x$

The first and last equations simply indicate that *the horizontal component of velocity always remains constant during the motion*.

Vertical Motion.

Since $a_y = -g$, then applying Eqs. 12–4 to 12–6, we get

$(+\uparrow)$ $\quad v = v_0 + a_c t; \qquad v_y = (v_0)_y - gt$

$(+\uparrow)$ $\quad y = y_0 + v_0 t + \frac{1}{2} a_c t^2; \qquad y = y_0 + (v_0)_y t - \frac{1}{2} g t^2$

$(+\uparrow)$ $\quad v^2 = v_0^2 + 2a_c(y - y_0); \qquad v_y^2 = (v_0)_y^2 - 2g(y - y_0)$

Since the last equation can be formulated on the basis of eliminating the time t from the first two equations, then *only two of the above three equations are independent of one another*.

*This assumes that the earth's gravitational field does not vary with altitude.

Once thrown, the basketball follows a parabolic trajectory.

To summarize, problems involving the motion of a projectile can have at most three unknowns since only three independent equations can be written; that is, *one* equation in the *horizontal direction* and *two* in the *vertical direction*. Once $\mathbf{v}_x$ and $\mathbf{v}_y$ are obtained, the resultant velocity $\mathbf{v}$, which is *always tangent* to the path, can be determined by the *vector sum* of $\mathbf{v}_x$ and $\mathbf{v}_y$, as shown in Fig. 12–20.

PROCEDURE FOR ANALYSIS

Coordinate System.

- Establish the *x, y* coordinate axes and sketch the trajectory of the particle. Between any *two points* on the path specify the given problem data and identify the *three unknowns*. In all cases the acceleration of gravity acts downward and equals $9.81\ \text{m/s}^2$. The particle's initial and final velocities should be represented in terms of their *x* and *y* components.
- Positive and negative position, velocity, and acceleration components always act in accordance with their associated coordinate directions.

Kinematic Equations.

- Depending upon the known data and what is to be determined, a choice should be made as to which three of the following four equations should be applied between the two points on the path to obtain the most direct solution to the problem.

Horizontal Motion.

- The *velocity* in the horizontal or *x* direction is *constant*, i.e., $v_x = (v_0)_x$, and

$$x = x_0 + (v_0)_x t$$

Vertical Motion.

- In the vertical or *y* direction *only two* of the following three equations can be used for the solution.

$$v_y = (v_0)_y + a_c t$$

$$y = y_0 + (v_0)_y t + \tfrac{1}{2} a_c t^2$$

$$v_y^2 = (v_0)_y^2 + 2a_c(y - y_0)$$

Refer to the companion website for Lecture Summary and Quiz videos.

EXAMPLE 12.11

A sack slides off the ramp, shown in Fig. 12–21, with a horizontal velocity of 12 m/s. If the height of the ramp is 6 m from the floor, determine the time needed for the sack to strike the floor and the range R where the sack strikes the ground.

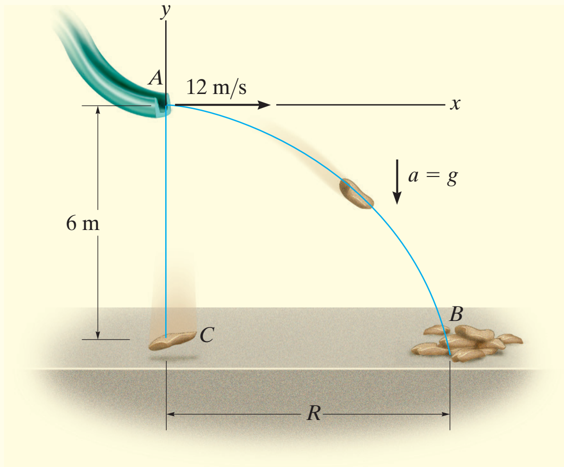


Fig. 12–21

SOLUTION

Coordinate System. The origin of coordinates is established at the beginning of the path, point A, Fig. 12–21. The initial velocity of the sack has components $(v_A)_x = 12\text{ m/s}$ and $(v_A)_y = 0$. Also, between points A and B the acceleration is $a_y = -9.81\text{ m/s}^2$. Since $(v_B)_x = (v_A)_x = 12\text{ m/s}$, the three unknowns are $(v_B)_y$, R, and the time of flight t_{AB}. Here we do not need to determine $(v_B)_y$.

Vertical Motion. The vertical distance from A to B is known, and therefore we can obtain a direct solution for t_{AB} by using the equation

$$(+\uparrow) \qquad y_B = y_A + (v_A)_y t_{AB} + \tfrac{1}{2}a_c t_{AB}^2$$

$$-6\text{ m} = 0 + 0 + \tfrac{1}{2}(-9.81\text{ m/s}^2)t_{AB}^2$$

$$t_{AB} = 1.11\text{ s} \qquad \textit{Ans.}$$

Horizontal Motion. Since t_{AB} has been calculated, R is determined as follows:

$$(\overset{+}{\rightarrow}) \qquad x_B = x_A + (v_A)_x t_{AB}$$

$$R = 0 + 12\text{ m/s}\,(1.11\text{ s})$$

$$R = 13.3\text{ m} \qquad \textit{Ans.}$$

NOTE: The calculation for t_{AB} also indicates that if the sack was released *from rest* at A, it would take the same amount of time to strike the floor at C, Fig. 12–21.

12

EXAMPLE 12.12

The chipping machine is designed to eject wood chips at $v_O = 7.5$ m/s as shown in Fig. 12–22. If the tube is oriented at 30° from the horizontal, determine how high, h, the chips strike the pile if they land on the pile 6 m from the tube.

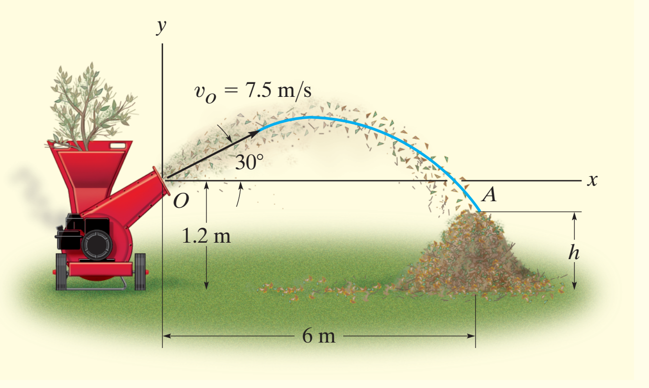


Fig. 12–22

SOLUTION

Coordinate System. The three unknowns are the height h, time of flight t_{OA}, and vertical component of velocity $(v_A)_y$. [Note that $(v_A)_x = (v_O)_x$.] With the origin of coordinates at O, Fig. 12–22, the initial velocity of a chip has components of

$$(v_O)_x = (7.5 \cos 30°)\text{ m/s} = 6.495\text{ m/s} \rightarrow$$

$$(v_O)_y = (7.5 \sin 30°)\text{ m/s} = 3.75\text{ m/s}\uparrow$$

Also, $(v_A)_x = (v_O)_x = 6.495$ m/s and $a_y = -9.81\text{ m/s}^2$. Since we do not need to determine $(v_A)_y$, we have

Horizontal Motion.

$$(\overset{+}{\rightarrow}) \qquad x_A = x_O + (v_O)_x t_{OA}$$

$$6\text{ m} = 0 + (6.495\text{ m/s})t_{OA}$$

$$t_{OA} = 0.9238\text{ s}$$

Vertical Motion. Relating t_{OA} to the initial and final elevations of a chip, we have

$$(+\uparrow) \quad y_A = y_O + (v_O)_y t_{OA} + \tfrac{1}{2}a_c t_{OA}^2$$

$$(h - 1.2\text{ m}) = 0 + (3.75\text{ m/s})(0.9238\text{ s}) + \tfrac{1}{2}(-9.81\text{ m/s}^2)(0.9238\text{ s})^2$$

$$h = 0.479\text{ m}$$ *Ans.*

EXAMPLE 12.13

The track for this racing event was designed so that riders jump off the slope at 30°, from a height of 1 m. During a race it was observed that the rider shown in Fig. 12–23*a* remained in mid air for 1.5 s. Determine the speed at which he was traveling off the ramp, the horizontal distance he travels before striking the ground, and the maximum height he attains. Neglect the size of the bike and rider.

(a)

SOLUTION

Coordinate System. As shown in Fig. 12–23*b*, the origin of the coordinates is established at *A*. Between the end points of the path *AB* the three unknowns are the initial speed v_A, range *R*, and the vertical component of velocity $(v_B)_y$.

Vertical Motion. Since the time of flight and the vertical distance between the ends of the path are known, we can determine v_A.

$(+\uparrow)$

$$y_B = y_A + (v_A)_y t_{AB} + \tfrac{1}{2}a_c t_{AB}^2$$

$$-1\text{ m} = 0 + v_A \sin 30°(1.5\text{ s}) + \tfrac{1}{2}(-9.81\text{ m/s}^2)(1.5\text{ s})^2$$

$$v_A = 13.38\text{ m/s} = 13.4\text{ m/s} \qquad \textit{Ans.}$$

Horizontal Motion. The range *R* can now be determined.

$(\overset{+}{\rightarrow})$

$$x_B = x_A + (v_A)_x t_{AB}$$

$$R = 0 + 13.38\cos 30°\text{ m/s}(1.5\text{ s})$$

$$= 17.4\text{ m} \qquad \textit{Ans.}$$

In order to find the maximum height *h* we will consider the path *AC*, Fig. 12–23*b*. Here the three unknowns are the time of flight t_{AC}, the horizontal distance from *A* to *C*, and the height *h*. At the maximum height $(v_C)_y = 0$, and since v_A is known, we can determine *h directly* without considering t_{AC} using the following equation.

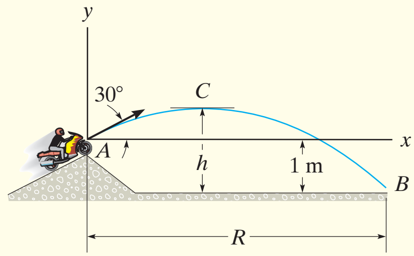
$$(v_C)_y^2 = (v_A)_y^2 + 2a_c[y_C - y_A]$$

$$0^2 = (13.38\sin 30°\text{ m/s})^2 + 2(-9.81\text{ m/s}^2)[(h - 1\text{ m}) - 0]$$

$$h = 3.28\text{ m} \qquad \textit{Ans.}$$

NOTE: Show that the bike will strike the ground at *B* with a velocity having components of

$$(v_B)_x = 11.6\text{ m/s}\rightarrow, \quad (v_B)_y = 8.02\text{ m/s}\downarrow$$



(b)

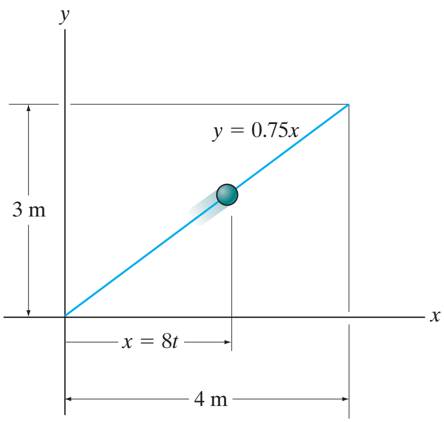
Fig. 12–23

Refer to the companion website for a self quiz of these Example problems.

FUNDAMENTAL PROBLEMS

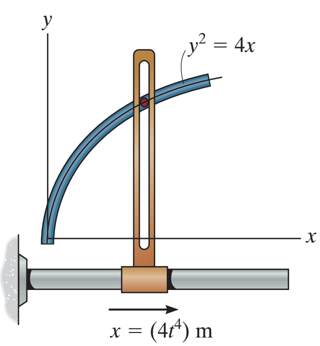
F12–15. If the x and y components of a particle's velocity are $v_x = (32t)$ m/s and $v_y = 8$ m/s, determine the equation of the path $y = f(x)$, if $x = 0$ and $y = 0$ when $t = 0$.

F12–16. A particle is traveling along the straight path. If its position along the x axis is $x = (8t)$ m, where t is in seconds, determine its speed when $t = 2$ s.



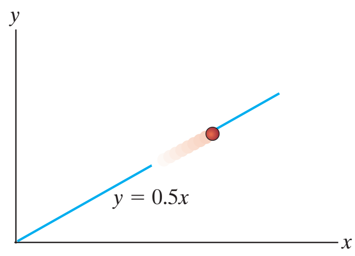
Prob. F12–16

F12–17. A particle is constrained to travel along the path. If $x = (4t^4)$ m, where t is in seconds, determine the magnitudes of the particle's velocity and acceleration when $t = 0.5$ s.



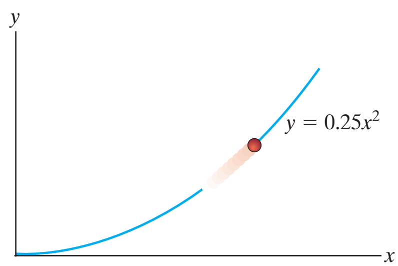
Prob. F12–17

F12–18. A particle travels along a straight-line path $y = 0.5x$. If the x component of the particle's velocity is $v_x = (2t^2)$ m/s, where t is in seconds, determine the magnitudes of the particle's velocity and acceleration when $t = 4$ s.



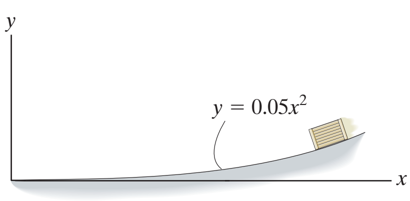
Prob. F12–18

F12–19. A particle is traveling along the parabolic path $y = 0.25x^2$. If $x = 8$ m, $v_x = 8$ m/s, and $a_x = 4$ m/s^2 when $t = 2$ s, determine the magnitudes of the particle's velocity and acceleration at this instant.



Prob. F12–19

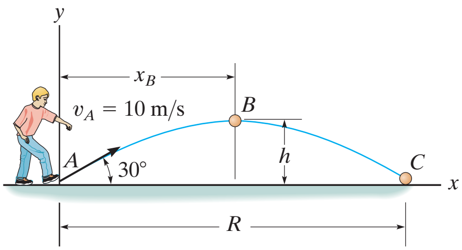
F12–20. The box slides down the path described by the equation $y = (0.05x^2)$ m, where x is in meters. If the box has x components of velocity and acceleration of $v_x = -3$ m/s and $a_x = -1.5$ m/s^2 at $x = 5$ m, determine the y components of the velocity and the acceleration of the box at this instant.



Prob. F12–20

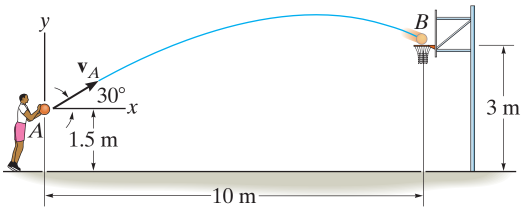
F12–21. The ball is kicked from point A with the initial velocity $v_A = 10\text{ m/s}$. Determine the maximum height h it reaches.

F12–22. The ball is kicked from point A with the initial velocity $v_A = 10\text{ m/s}$. Determine the range R, and the speed when the ball strikes the ground.



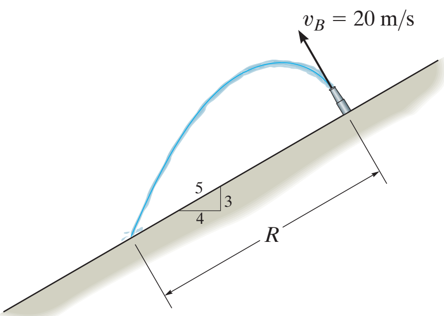
Probs. F12–21/22

F12–23. Determine the speed at which the basketball at A must be thrown at the angle of 30° so that it makes it to the basket at B.



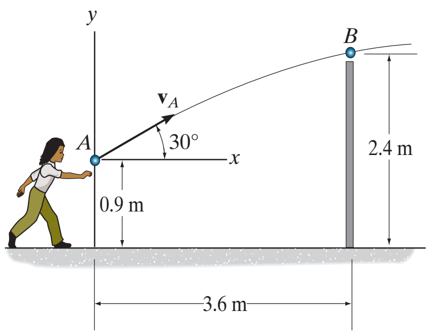
Prob. F12–23

F12–24. Water is sprayed at an angle of 90° from the slope at 20 m/s. Determine the range R.



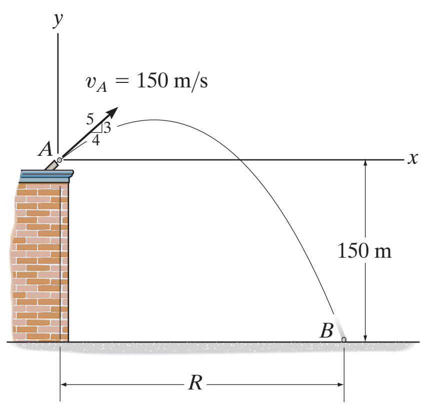
Prob. F12–24

F12–25. A ball is thrown from A. If it is required to clear the wall at B, determine the minimum magnitude of its initial velocity $\mathbf{v}_A$.



Prob. F12–25

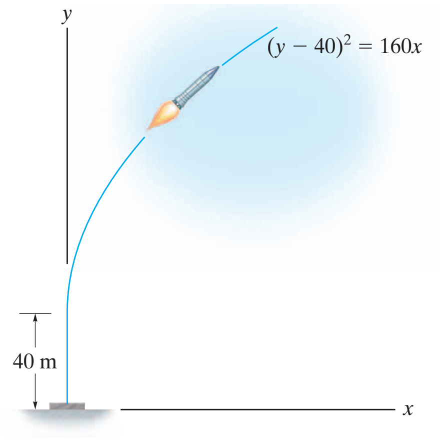
F12–26. A projectile is fired with an initial velocity of $v_A = 150\text{ m/s}$ off the roof of the building. Determine the range R where it strikes the ground at B.



Prob. F12–26

PROBLEMS

12–69. When a rocket reaches an altitude of 40 m it begins to travel along the parabolic path $(y - 40)^2 = 160x$, where the coordinates are measured in meters. If the component of velocity in the vertical direction is constant at $v_y = 180$ m/s, determine the magnitudes of the rocket's velocity and acceleration when it reaches an altitude of 80 m.



Prob. 12–69

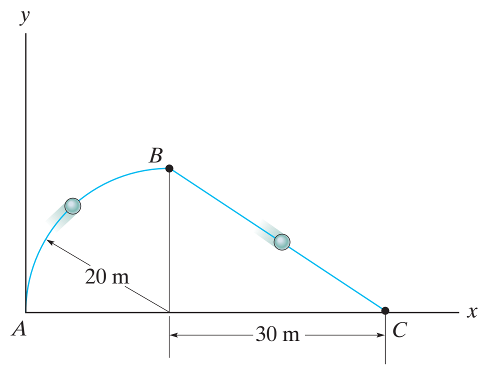
12–70. If the velocity of a particle is defined as $\mathbf{v}(t) = \{0.8t^2\mathbf{i} + 12t^{1/2}\mathbf{j} + 5\mathbf{k}\}$ m/s, determine the magnitude and coordinate direction angles α, β, γ of the particle's acceleration when $t = 2$ s.

12–71. The velocity of a particle is $\mathbf{v} = \{3\mathbf{i} + (6 - 2t)\mathbf{j}\}$ m/s, where t is in seconds. If $\mathbf{r} = \mathbf{0}$ when $t = 0$, determine the displacement of the particle during the time interval $t = 1$ s to $t = 3$ s.

***12–72.** A particle travels along the parabolic path $y = bx^2$. If its component of velocity along the y axis is $v_y = ct^2$, determine the x and y components of the particle's acceleration. Here b and c are constants.

12–73. A particle travels along the circular path $x^2 + y^2 = r^2$. If the y component of the particle's velocity is $v_y = 2r \cos 2t$, determine the x and y components of its acceleration at any instant.

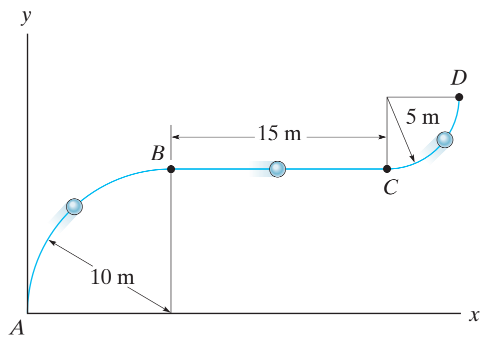
12–74. A particle travels along the curve from A to B in 5 s. It takes 8 s for it to go from B to C and then 10 s to go from C to A. Determine its average speed when it goes around the closed path.



Prob. 12–74

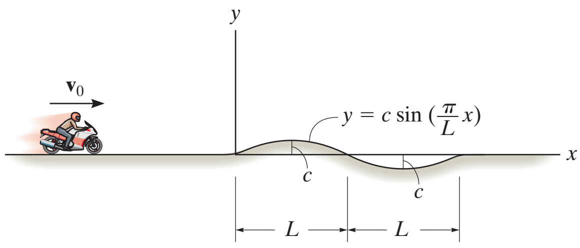
12–75. The velocity of a particle is given by $v = \{16t^2\mathbf{i} + 4t^3\mathbf{j} + (5t + 2)\mathbf{k}\}$ m/s, where t is in seconds. If the particle is at the origin when $t = 0$, determine the magnitude of the particle's acceleration when $t = 2$ s. Also, what is the x, y, z coordinate position of the particle at this instant?

***12–76.** A particle travels along the curve from A to B in 2 s. It takes 4 s for it to go from B to C and then 3 s to go from C to D. Determine its average speed when it goes from A to D.



Prob. 12–76

12–77. The motorcycle travels with constant speed v_0 along the path that, for a short distance, takes the form of a sine curve. Determine the x and y components of its velocity at any instant on the curve.

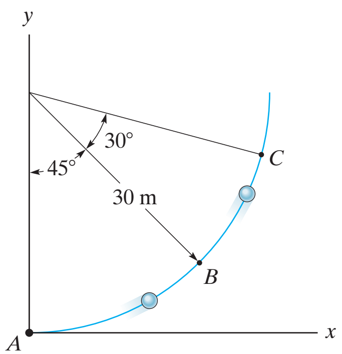


Prob. 12–77

12–78. Show that if a projectile is fired at an angle θ from the horizontal with an initial velocity $\mathbf{v}_0$, the *maximum* range the projectile can travel is given by $R_{max} = v_0^2/g$, where g is the acceleration of gravity. What is the angle θ for this condition?

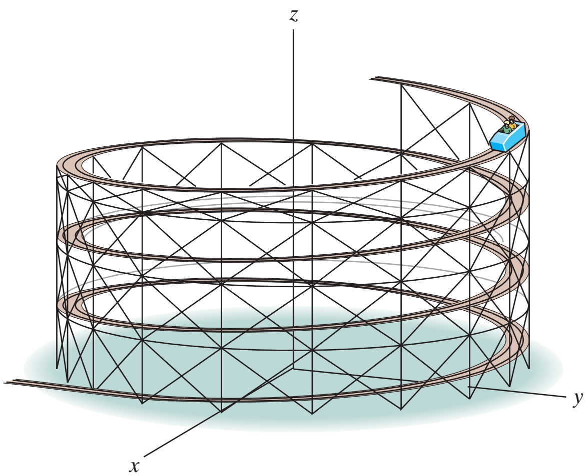
12–79. A rocket is fired from rest at $x = 0$ and travels along a parabolic trajectory described by $y^2 = [120(10^3)x]$ m. If the x component of acceleration is $a_x = (\frac{1}{4}t^2)$ m/s^2, where t is in seconds, determine the magnitudes of the rocket's velocity and acceleration when $t = 10$ s.

***12–80.** A particle travels along the curve from A to B in 1 s. If it takes 3 s for it to go from A to C, determine its *average velocity* when it goes from B to C.



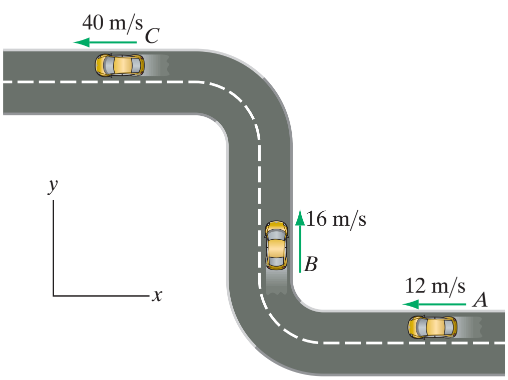
Prob. 12–80

12–81. The roller coaster car travels down the helical path at constant speed such that the parametric equations that define its position are $x = c \sin kt$, $y = c \cos kt$, $z = h - bt$, where c, h, and b are constants. Determine the magnitudes of its velocity and acceleration.



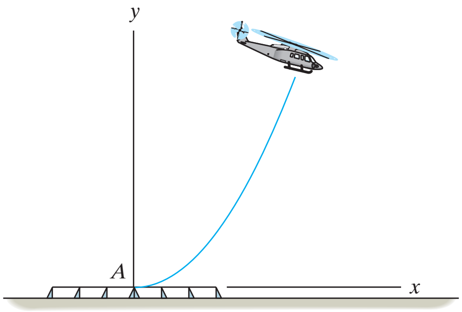
Prob. 12–81

12–82. A car traveling along the road has the velocities indicated in the figure when it arrives at points A, B, and C. If it takes 10 s to go from A to B, and then 15 s to go from B to C, determine the average acceleration between points A and B and between points A and C.



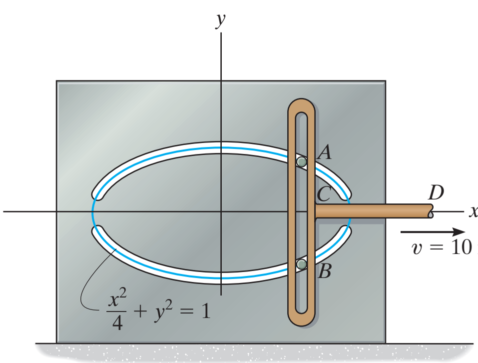
Prob. 12–82

12–83. The flight path of the helicopter as it takes off from A is defined by the parametric equations $x = (2t^2)$ m and $y = (0.04t^3)$ m, where t is the time in seconds. Determine the distance the helicopter is from point A and the magnitudes of its velocity and acceleration when $t = 10$ s.



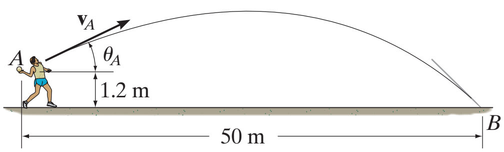
Prob. 12–83

***12–84.** Pegs A and B are restricted to move in the elliptical slots due to the motion of the slotted link. If the link moves with a constant speed of 10 m/s, determine the magnitudes of the velocity and acceleration of peg A when $x = 1$ m.



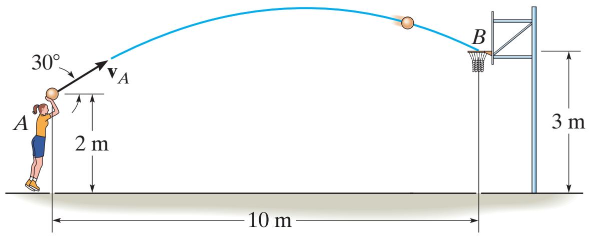
Prob. 12–84

12–85. It is observed that the time for the ball to strike the ground at B is 2.5 s. Determine the speed v_A and angle θ_A at which the ball was thrown.



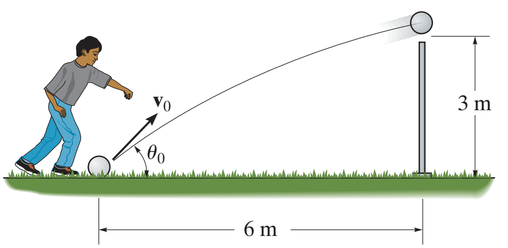
Prob. 12–85

12–86. Neglecting the size of the ball, determine the magnitude v_A of the basketball's initial velocity and its velocity when it passes through the basket.



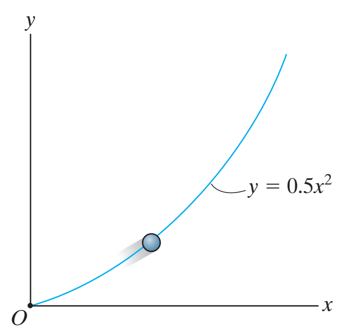
Prob. 12–86

12–87. Determine the minimum initial velocity v_0 and the corresponding angle θ_0 at which the ball must be kicked in order for it to just cross over the 3-m-high fence.



Prob. 12–87

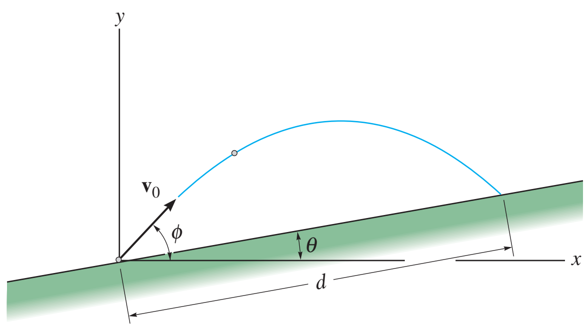
***12–88.** The particle travels along the path defined by the parabola $y = 0.5x^2$. If the component of velocity along the x axis is $v_x = (5t)$ m/s, where t is in seconds, determine the particle's distance from the origin O and the magnitude of its acceleration when $t = 1$ s. When $t = 0$, $x = 0$, $y = 0$.



Prob. 12–88

12–89. A projectile is given a velocity $\mathbf{v}_0$ at an angle ϕ above the horizontal. Determine the distance d to where it strikes the sloped ground. The acceleration due to gravity is g.

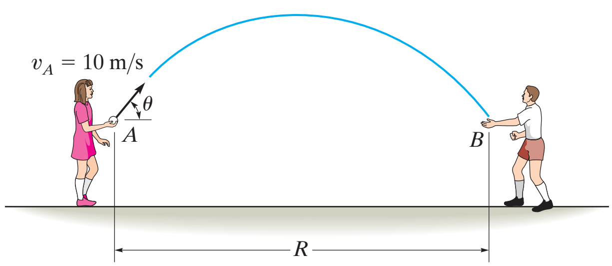
12–90. A projectile is given a velocity $\mathbf{v}_0$. Determine the angle ϕ at which it should be launched so that d is a maximum. The acceleration due to gravity is g.



Probs. 12–89/90

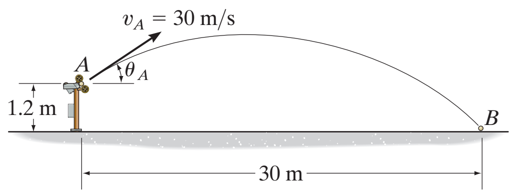
12–91. The girl at A can throw a ball at $v_A = 10$ m/s. Calculate the maximum possible range $R = R_{max}$ and the associated angle θ at which it should be thrown. Assume the ball is caught at B at the same elevation from which it is thrown.

***12–92.** Show that the girl at A can throw the ball to the boy at B by launching it at equal angles measured up or down from a 45° inclination. If $v_A = 10$ m/s, determine the range R if this value is 15°, i.e., $\theta_1 = 45° - 15° = 30°$ and $\theta_2 = 45° + 15° = 60°$. Assume the ball is caught at the same elevation from which it is thrown.



Probs. 12–91/92

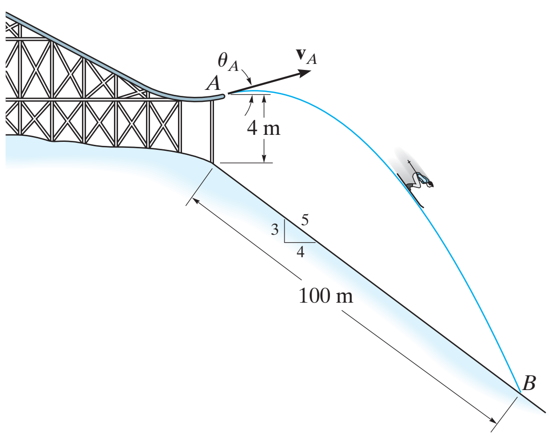
12–93. The pitching machine is adjusted so that the baseball is launched with a speed of $v_A = 30$ m/s. If the ball strikes the ground at B, determine the two possible angles θ_A at which it was launched.



Prob. 12–93

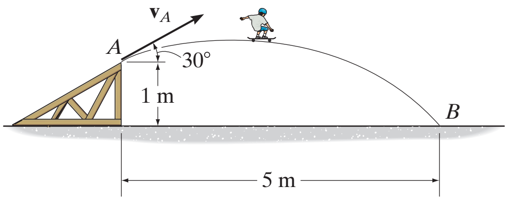
12–94. It is observed that the skier leaves the ramp A at an angle $\theta_A = 25°$ with the horizontal. If he strikes the ground at B, determine his initial speed v_A and the time of flight t_{AB}.

12–95. It is observed that the skier leaves the ramp A at an angle $\theta_A = 25°$ with the horizontal. If he strikes the ground at B, determine his initial speed v_A and the speed at which he strikes the ground.



Probs. 12–94/95

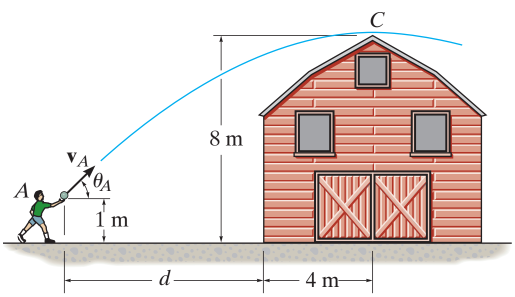
***12–96.** The skateboard rider leaves the ramp at A with an initial velocity v_A at a 30° angle. If he strikes the ground at B, determine v_A and the time of flight.



Prob. 12–96

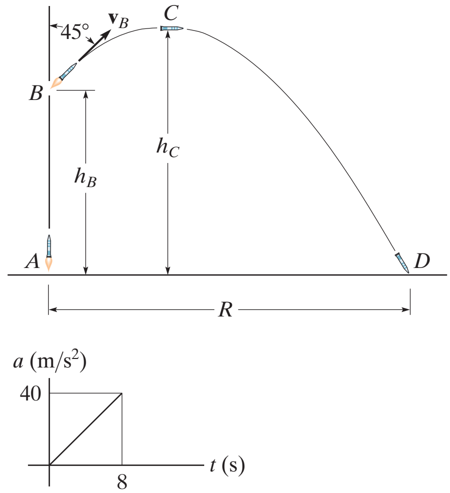
12–97. The boy at A attempts to throw a ball over the roof of a barn with an initial speed of $v_A = 15$ m/s. Determine the angle θ_A at which the ball must be thrown so that it reaches its maximum height at C. Also, find the distance d where the boy should stand to make the throw.

12–98. The boy at A attempts to throw a ball over the roof of a barn such that it is launched at an angle $\theta_A = 40°$. Determine the minimum speed v_A at which he must throw the ball so that it reaches its maximum height at C. Also, find the distance d where the boy must stand so that he can make the throw.



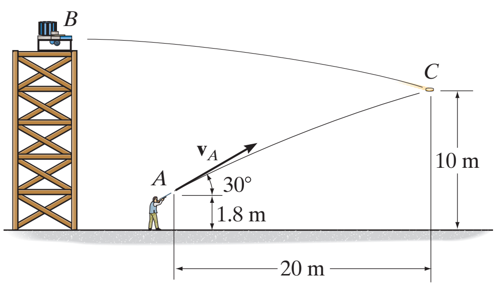
Probs. 12–97/98

12–99. The missile at A takes off from rest and rises vertically to B, where its fuel runs out in 8 s. If the acceleration varies with time as shown, determine the missile's height h_B and speed v_B. If by internal controls the missile is then suddenly pointed 45° as shown, and allowed to travel in free flight, determine the maximum height attained, h_C, and the range R to where it crashes at D.



Prob. 12–99

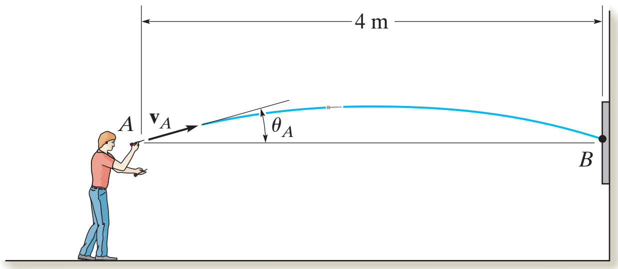
***12–100.** A projectile is fired from the platform at B. The shooter fires his gun from point A at an angle of 30°. Determine the muzzle speed of the bullet if it hits the projectile at C.



Prob. 12–100

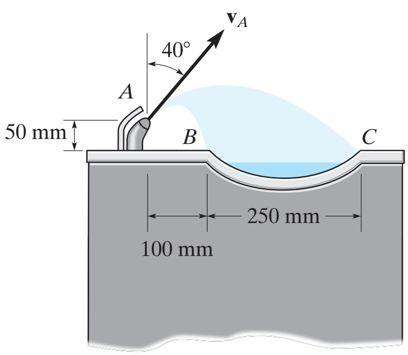
12–101. If the dart is thrown with a speed of 10 m/s, determine the shortest possible time before it strikes the target. Also, what is the corresponding angle θ_A at which it should be thrown, and what is the velocity of the dart when it strikes the target?

12–102. If the dart is thrown with a speed of 10 m/s, determine the longest possible time when it strikes the target. Also, what is the corresponding angle θ_A at which it should be thrown, and what is the velocity of the dart when it strikes the target?



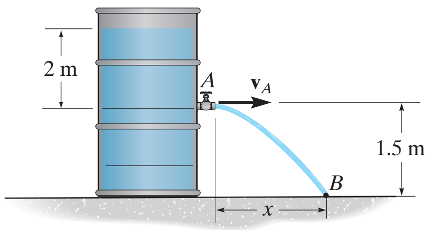
Probs. 12–101/102

12–103. The drinking fountain is designed such that the nozzle is located a distance away from the edge of the basin as shown. Determine the maximum and minimum speed at which water can be ejected from the nozzle so that it does not splash over the sides of the basin at B and C.



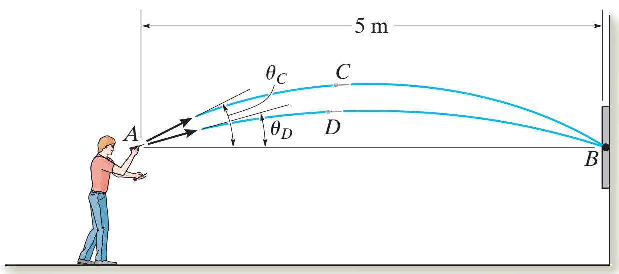
Prob. 12–103

***12–104.** The velocity of a water jet discharging from an orifice can be obtained from $v = \sqrt{2gh}$, where $h = 2$ m is the depth of the orifice from the free water surface. Determine the time for a particle of water leaving the orifice to reach a point B and the horizontal distance x where it hits the surface.



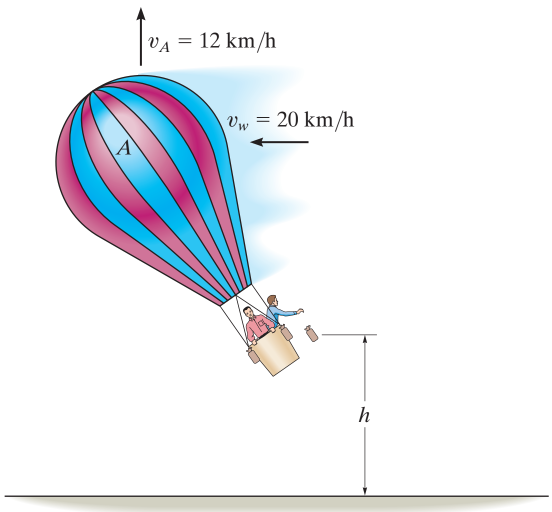
Prob. 12–104

12–105. The man at A wishes to throw two darts at the target at B so that they arrive at the *same time*. If each dart is thrown with a speed of 10 m/s, determine the angles θ_C and θ_D at which they should be thrown and the time between each throw. Note that the first dart must be thrown at θ_C $(>\theta_D)$, then the second dart is thrown at θ_D.



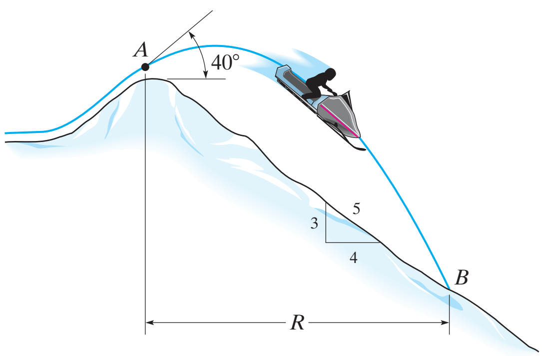
Prob. 12–105

12–106. The balloon A is ascending at the rate $v_A = 12$ km/h and is being carried horizontally by the wind at $v_w = 20$ km/h. If a ballast bag is dropped from the balloon at the instant $h = 50$ m, determine the time needed for it to strike the ground. Assume that the bag was released from the balloon with the same velocity as the balloon. Also, with what speed does the bag strike the ground?



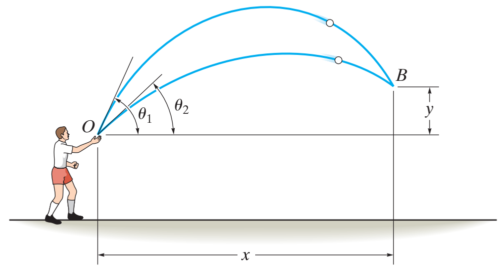
Prob. 12–106

12–107. The snowmobile is traveling at 10 m/s when it leaves the embankment at A. Determine the time of flight from A to B and the range R of the trajectory.



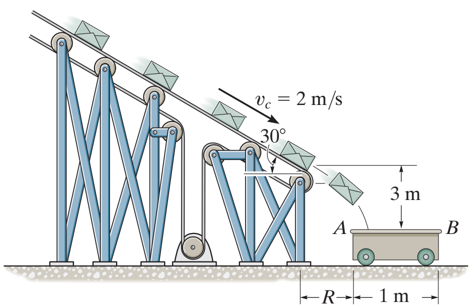
Prob. 12–107

***12–108.** A boy throws a ball at O in the air with a speed v_0 at an angle θ_1. If he then throws another ball with the same speed v_0 at an angle $\theta_2 < \theta_1$, determine the time between the throws so that the balls collide in midair at B.



Prob. 12–108

12–109. Small packages traveling on the conveyor belt fall off into a l-m-long loading car. If the conveyor is running at a constant speed of $v_C = 2$ m/s, determine the smallest and largest distance R at which the end A of the car may be placed from the conveyor so that the packages enter the car.



Prob. 12–109

12.7 CURVILINEAR MOTION: NORMAL AND TANGENTIAL COMPONENTS

When the path along which a particle travels is *known*, then it is often convenient to describe the motion using n and t axes that are normal and tangent to the path, respectively, and at the instant considered have their origin located at the particle.

Planar Motion. To establish these axes, consider the particle in Fig. 12–24*a*, which is at position s, measured from point O. The t axis is *tangent* to the curve at the particle and is positive in the direction of *increasing* s. We will designate this positive direction with the unit vector $\mathbf{u}_t$. A unique choice for the *normal axis* can be made by noting that geometrically the curve is constructed from a series of differential arc segments ds, Fig. 12–24*b*. Each segment ds is formed from the arc of an associated circle having a *radius of curvature* ρ (rho) and *center of curvature* O'. The normal axis n is perpendicular to the t axis and its positive direction is *towards* the center of curvature O', Fig. 12–24*a*. This direction, which is *always* on the concave side of the curve, will be designated by the unit vector $\mathbf{u}_n$. The plane which contains the n and t axes is referred to as the embracing or ***osculating plane***, and in this case it is fixed in the plane of motion.

Velocity. As indicated in Sec. 12.4, the particle's velocity $\mathbf{v}$ has a *direction* that is *always tangent to the path*, Fig. 12–24*c*, and a *magnitude* that is determined by taking the time derivative of the path function $s = s(t)$, i.e., $v = ds/dt$ (Eq. 12–8). Hence,

$$\mathbf{v} = v\mathbf{u}_t \qquad (12\text{–}15)$$

where

$$v = \dot{s} \qquad (12\text{–}16)$$

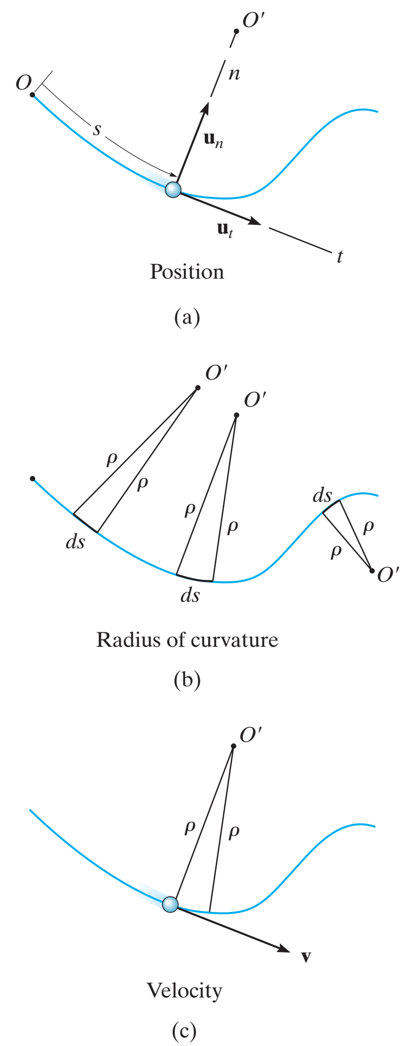
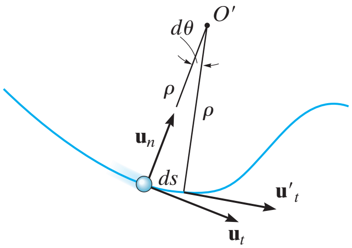


Fig. 12–24



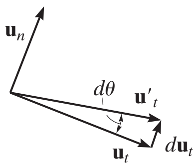
(d)

Acceleration. The acceleration of the particle is the time rate of change of the velocity. Therefore,

$$\mathbf{a} = \dot{\mathbf{v}} = \dot{v}\mathbf{u}_t + v\dot{\mathbf{u}}_t \tag{12–17}$$

To determine the time derivative $\dot{\mathbf{u}}_t$, note that as the particle moves along the arc ds in time dt, $\mathbf{u}_t$ preserves its magnitude of unity; however, its *direction* changes, and becomes $\mathbf{u}'_t$, Fig. 12–24*d*. As shown in Fig. 12–24*e*, we require $\mathbf{u}'_t = \mathbf{u}_t + d\mathbf{u}_t$, where $d\mathbf{u}_t$ acts between the arrowheads of $\mathbf{u}_t$ and $\mathbf{u}'_t$. Since $u_t = u'_t = 1$, then $d\mathbf{u}_t$ has a *magnitude* of $du_t = (1)\,d\theta$, and its *direction* is defined by $\mathbf{u}_n$. Consequently, $d\mathbf{u}_t = d\theta\mathbf{u}_n$, and therefore the time derivative becomes $\dot{\mathbf{u}}_t = \dot{\theta}\mathbf{u}_n$. Since $ds = \rho d\theta$, Fig. 12–24*d*, then $\dot{\theta} = \dot{s}/\rho$, and therefore

$$\dot{\mathbf{u}}_t = \dot{\theta}\mathbf{u}_n = \frac{\dot{s}}{\rho}\mathbf{u}_n = \frac{v}{\rho}\mathbf{u}_n$$



(e)

Substituting this into Eq. 12–17, $\mathbf{a}$ can be written as the sum of its two components,

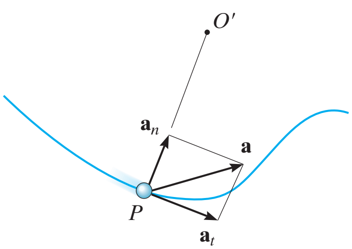
$$\boxed{\mathbf{a} = a_t\mathbf{u}_t + a_n\mathbf{u}_n} \tag{12–18}$$

where

$$\boxed{a_t = \dot{v}} \quad \text{or} \quad \boxed{a_t ds = v\,dv} \tag{12–19}$$

and

$$\boxed{a_n = \frac{v^2}{\rho}} \tag{12–20}$$



Acceleration

(f)

Fig. 12–24 (cont.)

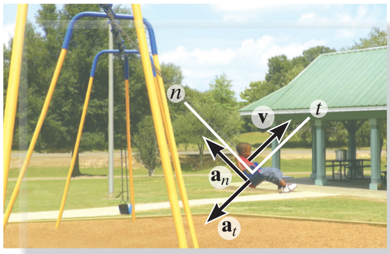
These two mutually perpendicular components are shown in Fig. 12–24*f*. Therefore, the *magnitude* of acceleration is

$$a = \sqrt{a_t^2 + a_n^2} \tag{12–21}$$

To better understand these results, consider the following two special cases of motion.

1. If the particle moves along a straight line, then $\rho \rightarrow \infty$ and from Eq. 12–20, $a_n = 0$. Therefore, $a = a_t = \dot{v}$, and so the *tangential component of acceleration represents the time rate of change in the magnitude of the velocity.*
2. If the particle moves along a curve with a constant speed, then $a_t = \dot{v} = 0$ and $a = a_n = v^2/\rho$. Therefore, the *normal component of acceleration represents the time rate of change in the direction of the velocity*. Since $\mathbf{a}_n$ *always* acts towards the center of curvature, this component is sometimes referred to as the *centripetal* (or center seeking) *acceleration*.

As a result of these interpretations, a particle moving along the curved path in Fig. 12–25 will have an acceleration directed as shown.



As the boy swings upward with a velocity **v**, his motion can be analyzed using n–t coordinates. As he rises, the magnitude of his velocity (speed) is decreasing, and so a_t will be negative. The rate at which the direction of his velocity changes is a_n, which is always positive, that is, towards the center of rotation.

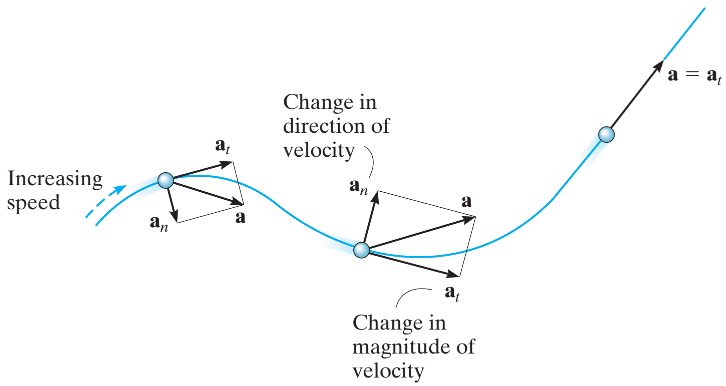


Fig. 12–25

Three-Dimensional Motion. If the particle moves along a space curve, Fig. 12–26, then at a given instant the t axis is uniquely specified; however, an infinite number of straight lines can be constructed normal to the tangent axis. As in the case of planar motion, however, we will choose the positive n axis directed toward the path's center of curvature O'. This axis is referred to as the *principal normal* to the curve. With the n and t axes so defined, Eqs. 12–15 through 12–21 can then be used to determine **v** and **a**. Since $\mathbf{u}_t$ and $\mathbf{u}_n$ are always perpendicular to one another and lie in the osculating plane, for spatial motion a third unit vector, $\mathbf{u}_b$, defines the *binormal axis b* which is perpendicular to $\mathbf{u}_t$ and $\mathbf{u}_n$, Fig. 12–26.

Since the three unit vectors are related to one another by the vector cross product, e.g., $\mathbf{u}_b = \mathbf{u}_t \times \mathbf{u}_n$, Fig. 12–26, it may be possible to use this relation to establish the direction of one of the axes if the directions of the other two are known. For example, no motion occurs in the $\mathbf{u}_b$ direction, and if this direction and $\mathbf{u}_t$ are known, then $\mathbf{u}_n$ can be determined from $\mathbf{u}_n = \mathbf{u}_b \times \mathbf{u}_t$, Fig. 12–26.

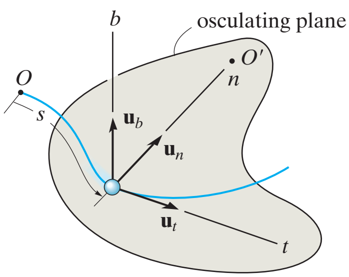


Fig. 12–26

12

Once the rotation is constant, the riders will then have only a normal component of acceleration.

Motorists traveling along this cloverleaf interchange experience a normal acceleration due to the change in direction of their velocity. A tangential component of acceleration occurs when the cars' speed is increased or decreased.

Refer to the companion website for Lecture Summary and Quiz videos.

PROCEDURE FOR ANALYSIS

Coordinate System.

- Provided the *path* of the particle is *known*, we can establish a set of n and t coordinates having a *fixed origin*, which is coincident with the particle at the instant considered.
- The positive tangent axis acts in the direction of motion and the positive normal axis is directed toward the path's center of curvature.

Velocity.

- The particle's *velocity* is always tangent to the path.
- The magnitude of velocity is found from the time derivative of the path function.

$$v = \dot{s}$$

Tangential Acceleration.

- The tangential component of acceleration represents the time rate of change in the *magnitude* of velocity. This component acts in the positive s direction if the particle's speed is increasing or in the opposite direction if the speed is decreasing.
- The relations between a_t, v, t, and s are the same as for rectilinear motion, namely,

$$a_t = \dot{v} \quad a_t\,ds = v\,dv$$

- If a_t is constant, $a_t = (a_t)_c$, then

$$s = s_0 + v_0 t + \tfrac{1}{2}(a_t)_c t^2$$
$$v = v_0 + (a_t)_c t$$
$$v^2 = v_0^2 + 2(a_t)_c(s - s_0)$$

Normal Acceleration.

- The normal component of acceleration represents the time rate of change in the *direction* of the velocity. This component is *always* directed toward the center of curvature of the path, i.e., along the positive n axis.
- The magnitude of this component is determined from

$$a_n = \frac{v^2}{\rho}$$

- If the path is expressed as $y = f(x)$, the radius of curvature ρ at any point on the path is determined from the equation

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|}$$

The derivation of this result is given in any standard calculus text.

EXAMPLE 12.14

The skier has a speed of 6 m/s at point A, Fig. 12–27a, which is increasing at 2 m/s^2. Determine the direction of the velocity and the direction and magnitude of acceleration at this instant. Neglect the size of the skier in the calculation.

SOLUTION

Coordinate System. Although the path has been expressed in terms of its x and y coordinates, we can still establish the origin of the n, t axes at the fixed point A on the path and determine the components of $\mathbf{v}$ and $\mathbf{a}$ along these axes, Fig. 12–27a.

Velocity. By definition, the velocity is always tangent to the path. Since $y = \frac{1}{20}x^2$, $dy/dx = \frac{1}{10}x$, then at $x = 10$ m, $dy/dx = 1$. Hence, at A, $\mathbf{v}$ makes an angle of $\theta = \tan^{-1}1 = 45^\circ$ with the x axis, Fig. 12–27b. Therefore,

$$v_A = 6\ \text{m/s} \quad 45^\circ \swarrow \qquad \textit{Ans.}$$

The acceleration is determined from $\mathbf{a} = \dot{v}\mathbf{u}_t + (v^2/\rho)\mathbf{u}_n$. However, we must first determine the radius of curvature of the path at A (10 m, 5 m). Since $d^2y/dx^2 = \frac{1}{10}$, then

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|} = \left.\frac{\left[1 + \left(\frac{1}{10}x\right)^2\right]^{3/2}}{\left|\frac{1}{10}\right|}\right|_{x=10\text{ m}} = 28.28\text{ m}$$

The acceleration is therefore

$$\mathbf{a}_A = \dot{v}\mathbf{u}_t + \frac{v^2}{\rho}\mathbf{u}_n$$

$$= 2\mathbf{u}_t + \frac{(6\text{ m/s})^2}{28.28\text{ m}}\mathbf{u}_n$$

$$= \{2\mathbf{u}_t + 1.273\mathbf{u}_n\}\text{ m/s}^2$$

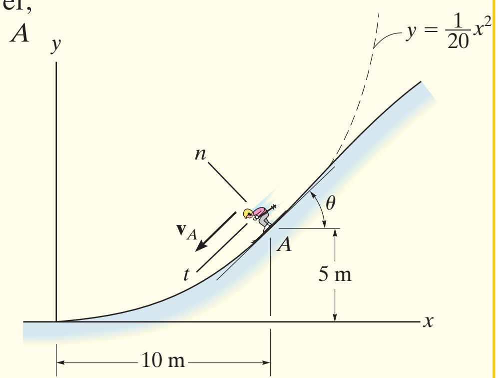
As shown in Fig. 12–27b,

$$a = \sqrt{(2\text{ m/s}^2)^2 + (1.273\text{ m/s}^2)^2} = 2.37\text{ m/s}^2$$

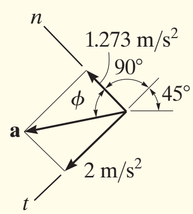
$$\phi = \tan^{-1}\frac{2}{1.273} = 57.5^\circ$$

Thus, $45^\circ + 90^\circ + 57.5^\circ - 180^\circ = 12.5^\circ$, so that

$$a = 2.37\text{ m/s}^2 \quad 12.5^\circ \swarrow \qquad \textit{Ans.}$$



(a)



(b)

Fig. 12–27

12

EXAMPLE 12.15

A race car C travels around the horizontal circular track that has a radius of 90 m, Fig. 12–28. If the car increases its speed at a constant rate of 2.1 m/s^2, starting from rest, determine the time needed for it to reach an acceleration of 2.4 m/s^2. What is its speed at this instant?

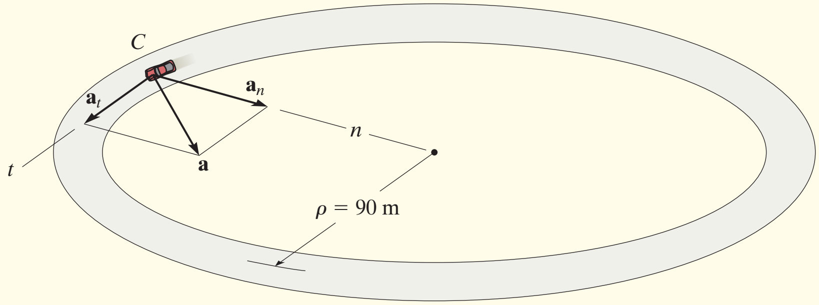


Fig. 12–28

SOLUTION

Coordinate System. The origin of the n and t axes is coincident with the car at the instant considered. The t axis is in the direction of motion, and the positive n axis is directed toward the center of the circle. This coordinate system is selected since the path is known.

Acceleration. The magnitude of acceleration can be related to its components using $a = \sqrt{a_t^2 + a_n^2}$. Here $a_t = 2.1$ m/s^2. Since $a_n = v^2/\rho$, the velocity as a function of time must be determined first.

$$v = v_0 + (a_t)_c t$$
$$v = 0 + 2.1t$$

Thus

$$a_n = \frac{v^2}{\rho} = \frac{(2.1t)^2}{90} = 0.049t^2 \text{ m/s}^2$$

The time needed for the acceleration to reach 2.4 m/s^2 is therefore

$$a = \sqrt{a_t^2 + a_n^2}$$
$$2.4 \text{ m/s}^2 = \sqrt{(2.1 \text{ m/s}^2)^2 + (0.049t^2)^2}$$

Solving for the positive value of t yields

$$0.049t^2 = \sqrt{(2.4 \text{ m/s}^2)^2 - (2.1 \text{ m/s}^2)^2}$$
$$t = 4.87 \text{ s} \qquad \textit{Ans.}$$

Velocity. The speed at time $t = 4.87$ s is

$$v = 2.1t = 2.1(4.87) = 10.2 \text{ m/s} \qquad \textit{Ans.}$$

NOTE: Remember the velocity will always be tangent to the path, whereas the acceleration will be directed within the curvature of the path.

EXAMPLE 12.16

If the box in Fig. 12–29*a* starts from rest at A and increases its speed, such that $a_t = (0.2t)\ \text{m/s}^2$, where t is in seconds, determine the time needed for the box to reach B and the magnitude of its acceleration when it arrives at this point.

SOLUTION

Coordinate System. The position of the box at any instant is defined from the fixed point A using the position or path coordinate s, Fig. 12–29*a*. The acceleration is to be determined at B, so the origin of the n, t axes is at this point.

Time. Since $v_A = 0$ when $t = 0$, then

$$a_t = \dot{v} = 0.2t \tag{1}$$

$$\int_0^v dv = \int_0^t 0.2t\,dt$$

$$v = 0.1t^2 \tag{2}$$

Here $s_B = 3 + 2\pi(2)/4 = 6.142$ m, Fig. 12–29*a*, and since $s_A = 0$ when $t = 0$ we have

$$v = \frac{ds}{dt} = 0.1t^2$$

$$\int_0^{6.142\text{ m}} ds = \int_0^{t_B} 0.1t^2 dt$$

$$6.142\text{ m} = 0.0333t_B^3$$

$$t_B = 5.690\text{ s} \qquad \textit{Ans.}$$

Acceleration. Substituting t_B into Eqs. 1 and 2 yields

$$(a_B)_t = \dot{v}_B = 0.2(5.690) = 1.138\ \text{m/s}^2$$
$$v_B = 0.1(5.69)^2 = 3.238\ \text{m/s}$$

At B, $\rho_B = 2$ m, so that

$$(a_B)_n = \frac{v_B^2}{\rho_B} = \frac{(3.238\ \text{m/s})^2}{2\text{ m}} = 5.242\ \text{m/s}^2$$

The magnitude of $\mathbf{a}_B$, Fig. 12–29*b*, is therefore

$$a_B = \sqrt{(1.138\ \text{m/s}^2)^2 + (5.242\ \text{m/s}^2)^2} = 5.36\ \text{m/s}^2 \qquad \textit{Ans.}$$

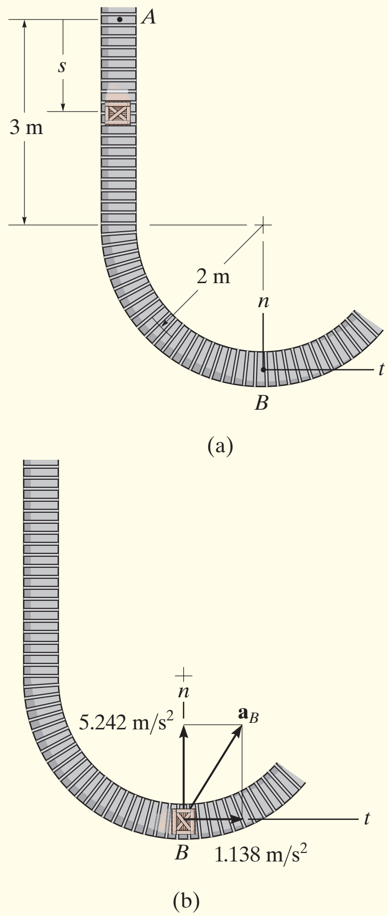


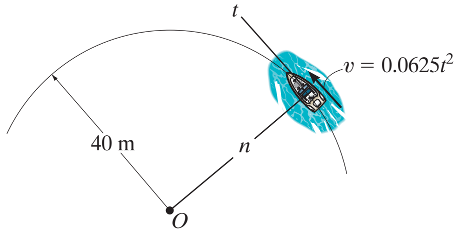
Fig. 12–29

Refer to the companion website for a self quiz of these Example problems.

12

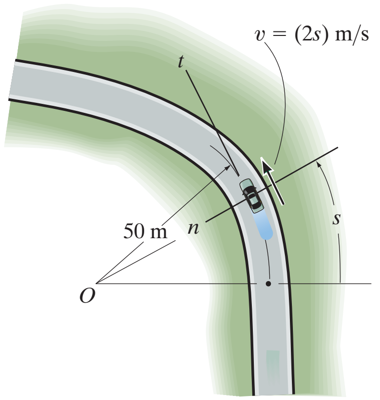
FUNDAMENTAL PROBLEMS

F12–27. The boat is traveling along the circular path with a speed of $v = (0.0625t^2)$ m/s, where t is in seconds. Determine the magnitude of its acceleration when $t = 10$ s.



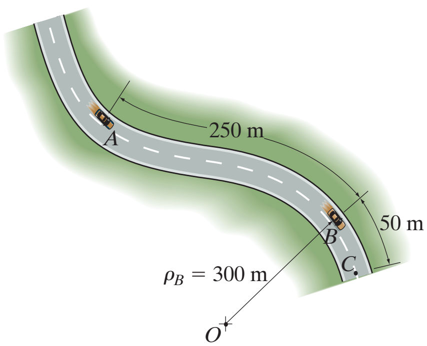
Prob. F12–27

F12–28. The car is traveling along the road with a speed of $v = (2\,s)$ m/s, where s is in meters. Determine the magnitude of its acceleration when $s = 10$ m.



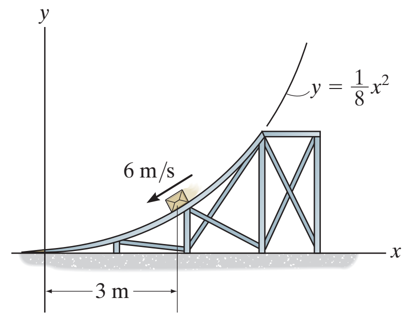
Prob. F12–28

F12–29. If the car decelerates uniformly along the curved road from 25 m/s at A to 15 m/s at C, determine the acceleration of the car at B.



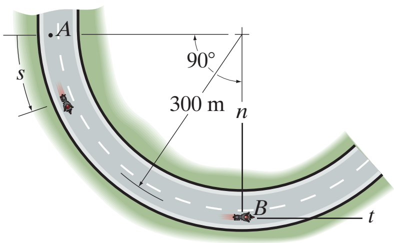
Prob. F12–29

F12–30. When $x = 3$ m, the crate has a speed of 6 m/s which is increasing at 2 m/s^2. Determine the direction of the crate's velocity and the magnitude of its acceleration at this instant.



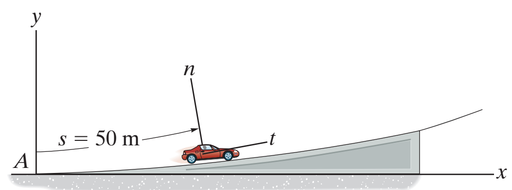
Prob. F12–30

F12–31. If the motorcycle has a deceleration of $a_t = -(0.001s)$ m/s^2 and its speed at position A is 25 m/s, determine the magnitude of its acceleration when it passes point B.



Prob. F12–31

F12–32. The car travels up the hill with a speed of $v = (0.2s)$ m/s, where s is in meters, measured from A. Determine the magnitude of its acceleration when it is at $s = 50$ m, where $\rho = 500$ m.

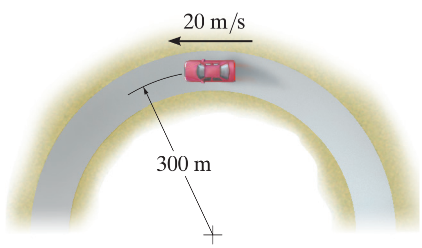


Prob. F12–32

12

PROBLEMS

12–110. The car travels along the curve having a radius of 300 m. If its speed is uniformly increased from 15 m/s to 27 m/s in 3 s, determine the magnitude of its acceleration at the instant its speed is 20 m/s.



Prob. 12–110

12–111. Determine the maximum constant speed a race car can have if the acceleration of the car cannot exceed 7.5 m/s^2 while rounding a track having a radius of curvature of 200 m.

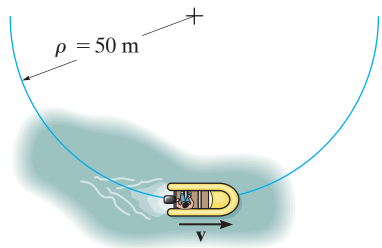
***12–112.** At a given instant, a car travels along a circular curved road with a speed of 20 m/s while decreasing its speed at the rate of 3 m/s^2. If the magnitude of the car's acceleration is 5 m/s^2, determine the radius of curvature of the road.

12–113. The motion of a particle is defined by the equations $x = (2t + t^2)$ m and $y = (t^2)$ m, where t is in seconds. Determine the normal and tangential components of the particle's velocity and acceleration when $t = 2$ s.

12–114. The position of a particle is defined by $\mathbf{r} = \{4(t - \sin t)\mathbf{i} + (2t^2 - 3)\mathbf{j}\}$ m, where t is in seconds and the argument for the sine is in radians. Determine the speed of the particle and its normal and tangential components of acceleration when $t = 1$ s.

12–115. Starting from rest, the motorboat travels around the circular path, $\rho = 50$ m, at a speed $v = (0.8t)$ m/s, where t is in seconds. Determine the magnitudes of the boat's velocity and acceleration when it has traveled 20 m.

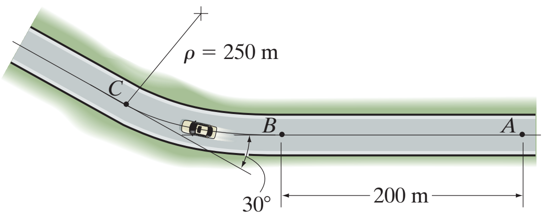
***12–116.** Starting from rest, the motorboat travels around the circular path, $\rho = 50$ m, at a speed $v = (0.2t^2)$ m/s, where t is in seconds. Determine the magnitudes of the boat's velocity and acceleration at the instant $t = 3$ s.



Probs. 12–115/116

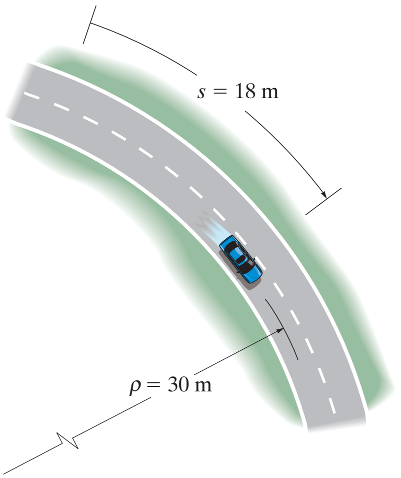
12–117. When the car reaches point A it has a speed of 25 m/s. If the brakes are applied, its speed is reduced by $a_t = (-\frac{1}{4}t^{1/2})$ m/s^2. Determine the magnitude of acceleration of the car just before it reaches point C.

12–118. When the car reaches point A, it has a speed of 25 m/s. If the brakes are applied, its speed is reduced by $a_t = (0.001s - 1)$ m/s^2. Determine the magnitude of acceleration of the car just before it reaches point C.



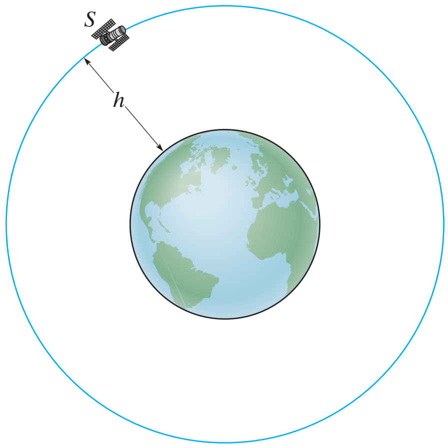
Probs. 12–117/118

12–119. The car travels along the circular path such that its speed is increased by $a_t = (0.5e^t)\ \text{m/s}^2$, where t is in seconds. Determine the magnitudes of its velocity and acceleration after the car has traveled $s = 18$ m starting from rest. Neglect the size of the car.



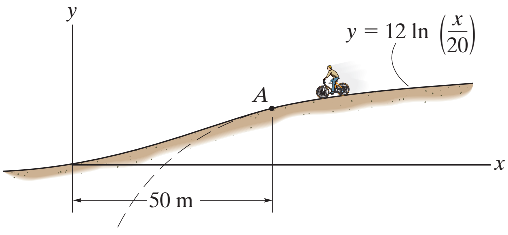
Prob. 12–119

***12–120.** The satellite S travels around the earth in a circular path with a constant speed of 20 Mm/h. If the acceleration is $2.5\ \text{m/s}^2$, determine the altitude h. Assume the earth's diameter to be 12 713 km.



Prob. 12–120

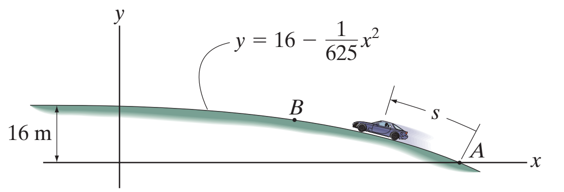
12–121. When the bicycle passes point A, it has a speed of 6 m/s, which is increasing at the rate of $\dot{v} = 0.5\ \text{m/s}^2$. Determine the magnitude of its acceleration when it is at point A.



Prob. 12–121

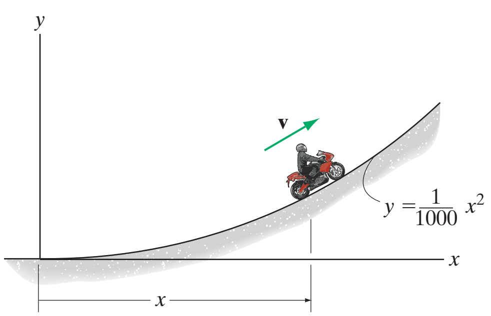
12–122. The car passes point A with a speed of 25 m/s after which its speed is defined by $v = (25 - 0.15s)$ m/s, where s is in meters. Determine the magnitude of the car's acceleration when it reaches point B, where $s = 51.5$ m and $x = 50$ m.

12–123. If the car passes point A with a speed of 20 m/s and begins to increase its speed at a constant rate of $a_t = 0.5\ \text{m/s}^2$, determine the magnitude of the car's acceleration when $s = 101.68$ m and $x = 0$.



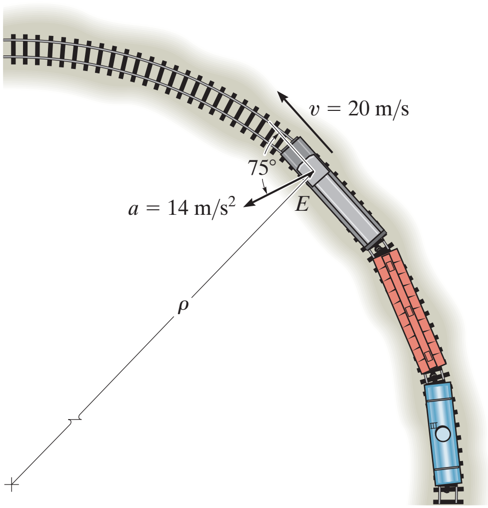
Probs. 12–122/123

***12–124.** The motorcycle travels up the hill at a constant speed of 15 m/s. Determine the magnitude of its acceleration as a function of x.



Prob. 12–124

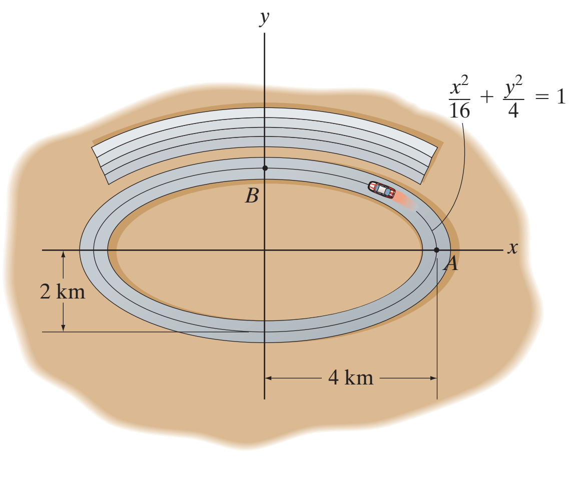
12–125. At a given instant the train engine at E has a speed of 20 m/s and an acceleration of 14 m/s^2 acting in the direction shown. Determine the rate of increase in the train's speed and the radius of curvature ρ of the path.



Prob. 12–125

12–126. A racing car travels with a constant speed of 240 km/h around the elliptical race track. Determine the acceleration experienced by the driver at A.

12–127. The racing car travels with a constant speed of 240 km/h around the elliptical race track. Determine the acceleration experienced by the driver at B.

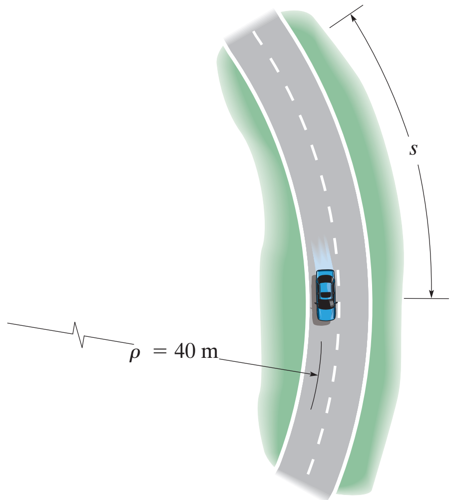


Probs. 12–126/127

***12–128.** A particle travels around a circular path having a radius of 50 m. If it is initially traveling with a speed of 10 m/s and its speed then increases at a rate of $\dot{v} = (0.05\,v)\ \text{m/s}^2$, determine the magnitude of the particle's acceleration four seconds later.

12–129. The car has an initial speed $v_0 = 20$ m/s at $s = 0$. If it increases its speed along the circular track at $a_t = (0.8s)\ \text{m/s}^2$, where s is in meters, determine the time needed for the car to travel $s = 25$ m.

12–130. The car starts from rest at $s = 0$ and increases its speed at $a_t = 4\ \text{m/s}^2$. Determine the time when the magnitude of acceleration becomes 20 m/s^2. At what position s does this occur?



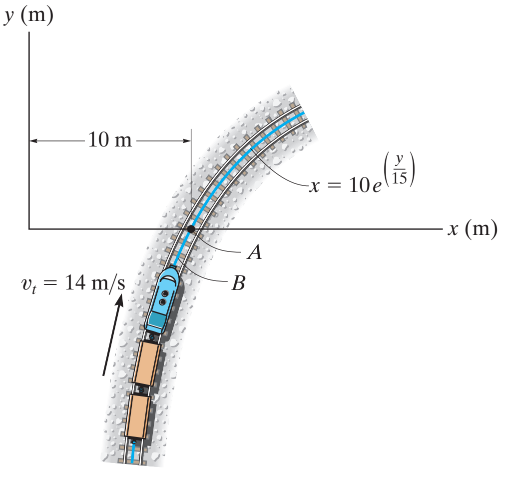
Probs. 12–129/130

12–131. The position of a particle traveling along a curved path is $s = (3t^3 - 4t^2 + 4)$ m, where t is in seconds. When $t = 2$ s, the particle is at a position on the path where the radius of curvature is 25 m. Determine the magnitude of the particle's acceleration at this instant.

***12–132.** A boat is traveling along a circular path having a radius of 20 m. Determine the magnitude of the boat's acceleration when the speed is $v = 5$ m/s and the rate of increase in the speed is $\dot{v} = 2\ \text{m/s}^2$.

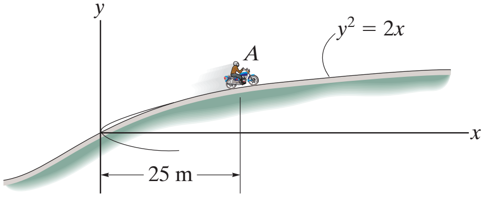
12–133. Starting from rest, a bicyclist travels around a horizontal circular path, $\rho = 10$ m, at a speed of $v = (0.09t^2 + 0.1t)$ m/s, where t is in seconds. Determine the magnitudes of her velocity and acceleration when she has traveled $s = 3$ m.

12–134. A train is traveling with a constant speed of 14 m/s along the curved path. Determine the magnitude of the acceleration of the front of the train, B, at the instant it reaches point A $(y = 0)$.



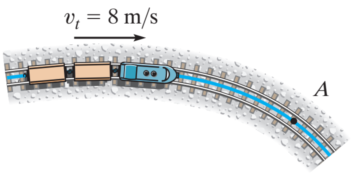
Prob. 12–134

12–135. The motorcycle is traveling at a constant speed of 60 km/h. Determine the magnitude of its acceleration when it is at point A.



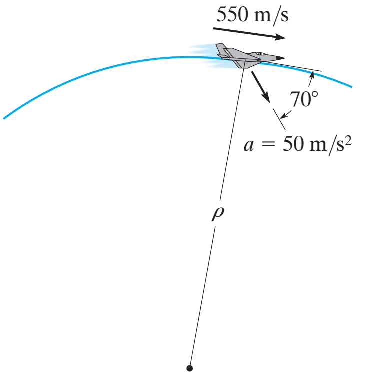
Prob. 12–135

***12–136.** When $t = 0$, the train has a speed of 8 m/s, which is increasing at 0.5 m/s^2. Determine the magnitude of the acceleration of the engine when it reaches point A, at $t = 20$ s. Here the radius of curvature of the tracks is $\rho_A = 400$ m.



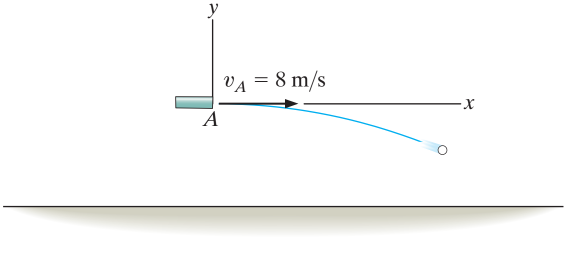
Prob. 12–136

12–137. At a given instant the jet plane has a speed of 550 m/s and an acceleration of 50 m/s^2 acting in the direction shown. Determine the rate of increase in the plane's speed, and also the radius of curvature ρ of the path.



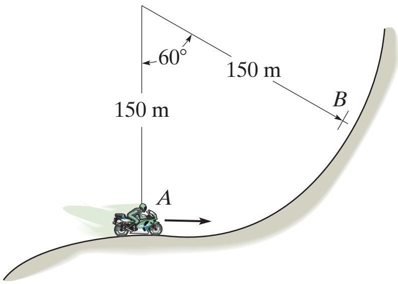
Prob. 12–137

12–138. The ball is ejected horizontally from the tube with a speed of 8 m/s. Find the equation of the path, $y = f(x)$, and then find the ball's velocity and the normal and tangential components of acceleration when $t = 0.25$ s.



Prob. 12–138

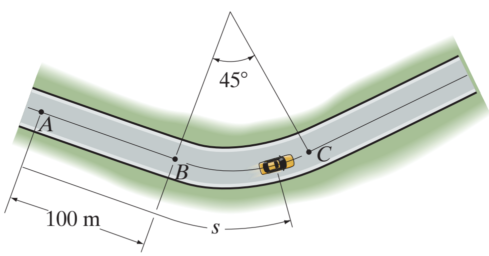
12–139. The motorcycle is traveling at 40 m/s when it is at A. If the speed is then decreased at $\dot{v} = -(0.05\,s)$ m/s^2, where s is in meters measured from A, determine its speed and acceleration when it reaches B.



Prob. 12–139

***12–140.** The car is traveling at a constant speed of 30 m/s. The driver then applies the brakes at A and thereby reduces the car's speed at the rate of $a_t = (-0.08v)$ m/s^2, where v is in m/s. Determine the magnitude of the acceleration of the car just before it reaches point C on the circular curve. It takes 15 s for the car to travel from A to C.

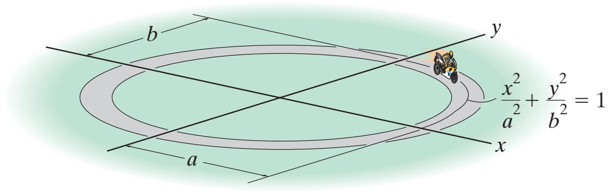
12–141. The car is traveling at a speed of 30 m/s. The driver then applies the brakes at A and thereby reduces the speed at the rate of $a_t = (-\frac{1}{8}t)$ m/s^2, where t is in seconds. Determine the magnitude of the acceleration of the car just before it reaches point C on the circular curve. It takes 15 s for the car to travel from A to C.



Probs. 12–140/141

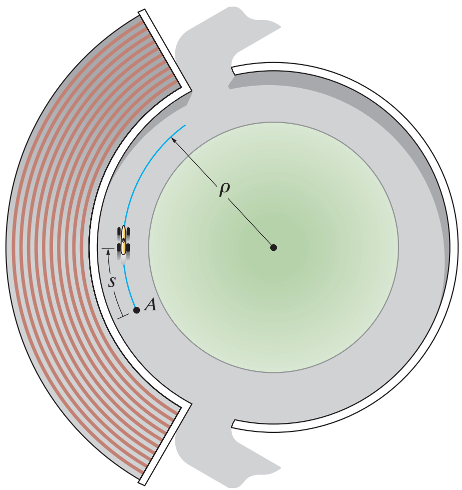
12–142. The motorcycle travels along the elliptical track at a constant speed v. Determine its greatest acceleration if $a > b$.

12–143. The motorcycle travels along the elliptical track at a constant speed v. Determine its smallest acceleration if $a > b$.



Probs. 12–142/143

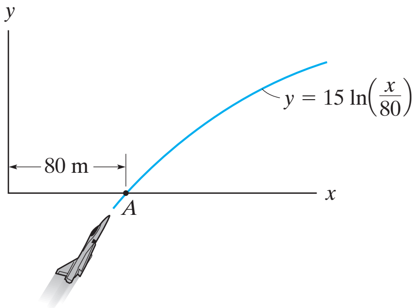
***12–144.** The race car has an initial speed $v_A = 15$ m/s at A. If it increases its speed along the circular track at the rate $a_t = (0.4s)$ m/s^2, where s is in meters, determine the time needed for the car to travel 20 m. Take $\rho = 150$ m.



Prob. 12–144

12–145. The jet plane is traveling with a speed of 120 m/s which is decreasing at 40 m/s^2 when it reaches point A. Determine the magnitude of its acceleration when it is at this point. Also, specify the direction of flight, measured from the x axis.

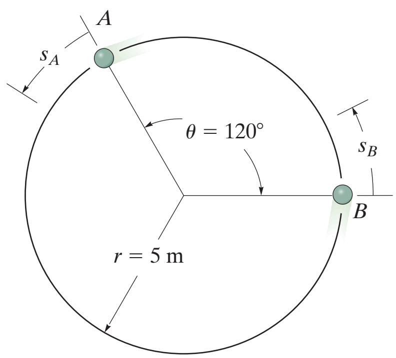
12–146. The jet plane is traveling with a constant speed of 110 m/s along the curved path. Determine the magnitude of the acceleration of the plane at the instant it reaches point $A(y = 0)$.



Probs. 12–145/146

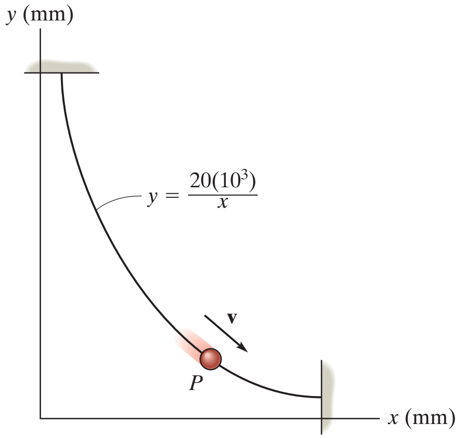
12–147. Particles A and B are traveling counterclockwise around a circular track at a constant speed of 8 m/s. If at the instant shown the speed of A begins to increase by $(a_t)_A = (0.4s_A)$ m/s^2, where s_A is in meters, determine the distance measured counterclockwise along the track from B to A when $t = 1$ s. What is the magnitude of the acceleration of each particle at this instant?

***12–148.** Particles A and B are traveling around a circular track at a speed of 8 m/s at the instant shown. If the speed of B is increasing by $(a_t)_B = 4$ m/s^2, and at the same instant A has an increase in speed of $(a_t)_A = 0.8t$ m/s^2, where t is in seconds, determine how long it takes for a collision to occur. What is the magnitude of the acceleration of each particle just before the collision occurs?



Probs. 12–147/148

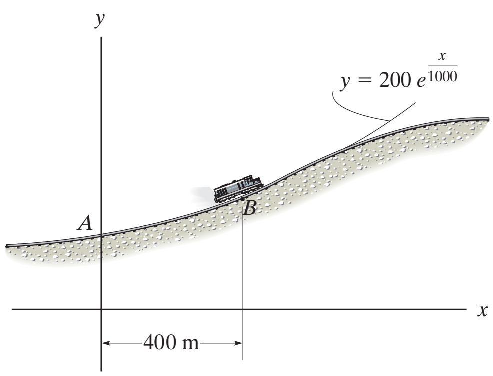
12–149. The particle travels with a constant speed of 300 mm/s along the curve. Determine the particle's acceleration when it is located at point (200 mm, 100 mm) and sketch this vector on the curve.



Prob. 12–149

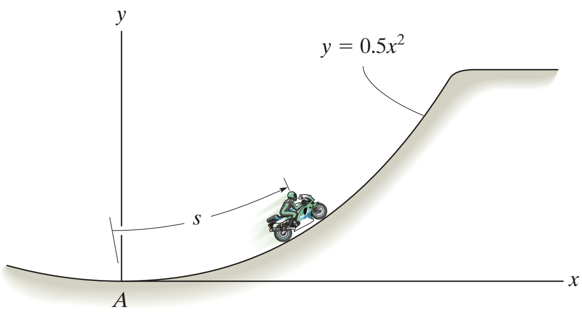
12–150. The train passes point B with a speed of 20 m/s which is decreasing at $a_t = -0.5$ m/s^2. Determine the magnitude of acceleration of the train at this point.

12–151. The train passes point A with a speed of 30 m/s and begins to decrease its speed at a constant rate of $a_t = -0.25$ m/s^2. Determine the magnitude of the acceleration of the train when it reaches point B, where $s_{AB} = 412$ m.



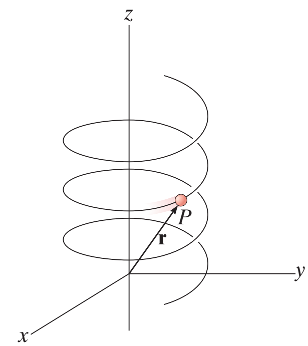
Probs. 12–150/151

*__12–152.__ The motorcycle is traveling at 1 m/s when it is at A. If the speed is then increased at $\dot{v} = 0.1\ \text{m/s}^2$, determine its speed and acceleration at the instant $t = 5$ s.



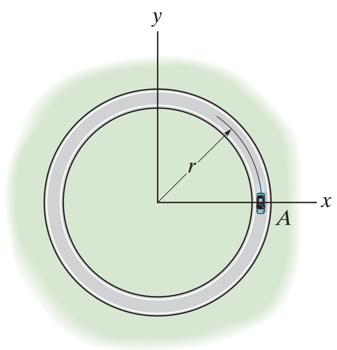
Prob. 12–152

■__12–153.__ A particle P travels along an elliptical spiral path such that its position vector $\mathbf{r}$ is defined by $\mathbf{r} = \{2\cos(0.1t)\mathbf{i} + 1.5\sin(0.1t)\mathbf{j} + (2t)\mathbf{k}\}$ m, where t is in seconds and the arguments for the sine and cosine are given in radians. When $t = 8$ s, determine the coordinate direction angles α, β, and γ, which the binormal axis to the osculating plane makes with the x, y, and z axes. *Hint*: Solve for the velocity $\mathbf{v}_P$ and acceleration $\mathbf{a}_P$ of the particle in terms of their $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components. The binormal is parallel to $\mathbf{v}_P \times \mathbf{a}_P$. Why?



Prob. 12–153

12–154. A car travels around a circular track having a radius of $r = 300$ m such that when it is at point A it has a velocity of 5 m/s, which is increasing at the rate of $\dot{v} = (0.06t)\ \text{m/s}^2$, where t is in seconds. Determine the magnitudes of its velocity and acceleration when it has traveled one-third the way around the track.

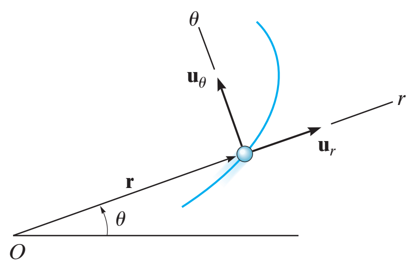


Prob. 12–154

*12.8 CURVILINEAR MOTION: CYLINDRICAL COMPONENTS

Sometimes the motion of the particle is constrained on a path that is best described using cylindrical coordinates. If motion is restricted to the plane, then polar coordinates are used.

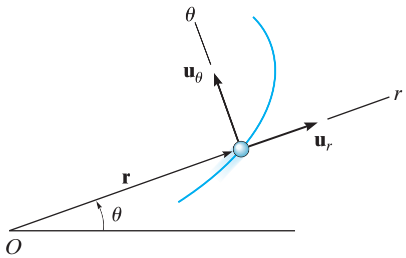
Polar Coordinates. We can specify the location of the particle shown in Fig. 12–30*a* using a ***radial coordinate r***, which extends outward from the fixed origin O to the particle, and a ***transverse coordinate θ***, which is the counterclockwise angle between a fixed reference line and the r axis. This angle is generally measured in degrees or radians, where $1\ \text{rad} = 180°/\pi$. The positive directions of the r and θ coordinates are defined by the unit vectors $\mathbf{u}_r$ and $\mathbf{u}_\theta$, respectively. Here $\mathbf{u}_r$ is in the direction of increasing r when θ is held fixed, and $\mathbf{u}_\theta$ is in a direction of increasing θ when r is held fixed.



Position

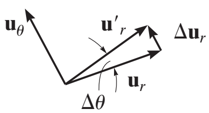
(a)

Fig. 12–30

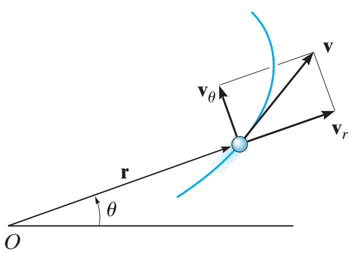


Position

(a)



(b)



Velocity

(c)

Fig. 12–30 (cont.)

Position. At any instant the position of the particle, Fig. 12–30*a*, is defined by the position vector

$$\mathbf{r} = r\mathbf{u}_r \tag{12–22}$$

Velocity. The velocity **v** is obtained by taking the time derivative of **r**. Using a dot to represent the time derivative, we have

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{u}_r + r\dot{\mathbf{u}}_r$$

To evaluate $\dot{\mathbf{u}}_r$, notice that $\mathbf{u}_r$ only changes its direction with respect to time, since the magnitude of this vector is always one unit. During the time Δt, a change Δr will not cause a change in the direction of $\mathbf{u}_r$; however, a change $\Delta\theta$ will cause $\mathbf{u}_r$ to become $\mathbf{u}'_r$, where $\mathbf{u}'_r = \mathbf{u}_r + \Delta\mathbf{u}_r$, Fig. 12–30*b*. The time change in $\mathbf{u}_r$ is then $\Delta\mathbf{u}_r$. For small angles $\Delta\theta$ this vector has a magnitude $\Delta u_r \approx 1(\Delta\theta)$ and acts in the $\mathbf{u}_\theta$ direction. Therefore, $\Delta\mathbf{u}_r = \Delta\theta\mathbf{u}_\theta$, and so

$$\dot{\mathbf{u}}_r = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{u}_r}{\Delta t} = \left(\lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}\right)\mathbf{u}_\theta$$

$$\dot{\mathbf{u}}_r = \dot{\theta}\mathbf{u}_\theta \tag{12–23}$$

Substituting this result into the above equation, the velocity can then be written in component form as

$$\boxed{\mathbf{v} = v_r\mathbf{u}_r + v_\theta\mathbf{u}_\theta} \tag{12–24}$$

where

$$\boxed{\begin{aligned} v_r &= \dot{r} \\ v_\theta &= r\dot{\theta} \end{aligned}} \tag{12–25}$$

These components are shown graphically in Fig. 12–30*c*. The *radial component* $\mathbf{v}_r$ is a measure of the rate of increase or decrease in the length of the radial coordinate, i.e., $\dot{r}$; whereas the *transverse component* $\mathbf{v}_\theta$ can be interpreted as the velocity along the circumference of a circle having a radius r. In particular, the term $\dot{\theta} = d\theta/dt$ is called the ***angular velocity***, since it indicates the time rate of change of the angle θ. Common units used for this measurement are rad/s.

Since $\mathbf{v}_r$ and $\mathbf{v}_\theta$ are mutually perpendicular, the *magnitude* of velocity or speed is simply

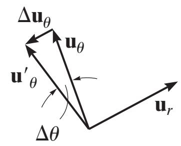
$$v = \sqrt{(\dot{r})^2 + (r\dot{\theta})^2} \tag{12–26}$$

and the *direction* of **v** is, of course, tangent to the path, Fig. 12–30*c*.

Acceleration. Taking the time derivatives of Eq. 12–24, using Eqs. 12–25, we obtain the particle's acceleration,

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{r}\mathbf{u}_r + \dot{r}\dot{\mathbf{u}}_r + \dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\dot{\mathbf{u}}_\theta$$

To evaluate $\dot{\mathbf{u}}_\theta$, it is necessary only to find the change in the direction of $\mathbf{u}_\theta$ since its magnitude is always unity. During the time Δt, a change Δr will not change the direction of $\mathbf{u}_\theta$; however, a change $\Delta\theta$ will cause $\mathbf{u}_\theta$ to become $\mathbf{u}'_\theta$, where $\mathbf{u}'_\theta = \mathbf{u}_\theta + \Delta\mathbf{u}_\theta$, Fig. 12–30*d*. The time change in $\mathbf{u}_\theta$ is thus $\Delta\mathbf{u}_\theta$. For small angles this vector has a magnitude $\Delta u_\theta \approx 1(\Delta\theta)$ and acts in the $-\mathbf{u}_r$ direction; i.e., $\Delta\mathbf{u}_\theta = -\Delta\theta\mathbf{u}_r$. Thus,



(d)

$$\dot{\mathbf{u}}_\theta = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{u}_\theta}{\Delta t} = -\left(\lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}\right)\mathbf{u}_r$$

$$\dot{\mathbf{u}}_\theta = -\dot{\theta}\mathbf{u}_r \qquad (12\text{–}27)$$

Substituting this result and Eq. 12–23 into the above equation for **a**, we can write the acceleration in component form as

$$\boxed{\mathbf{a} = a_r\mathbf{u}_r + a_\theta\mathbf{u}_\theta} \qquad (12\text{–}28)$$

where

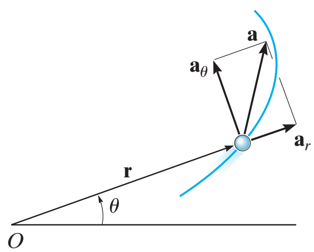
$$\boxed{\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 \\ a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} \end{aligned}} \qquad (12\text{–}29)$$

The term $\ddot{\theta} = d^2\theta/dt^2 = d/dt(d\theta/dt)$ is called the ***angular acceleration*** since it measures the change made in the angular velocity during an instant of time. Units for this measurement are rad/s^2.

Since $\mathbf{a}_r$ and $\mathbf{a}_\theta$ are perpendicular to one another, the *magnitude* of acceleration is

$$a = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2} \qquad (12\text{–}30)$$

The *direction* is determined from the vector addition of its two components, and as shown in Fig. 12–30*e*, in general **a** will *not* be tangent to the path.



Acceleration

(e)

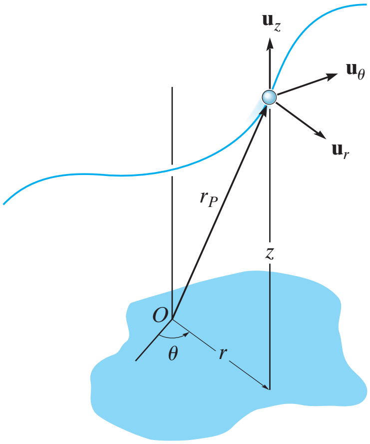


Fig. 12–31

Cylindrical Coordinates. If the particle moves along a space curve, Fig. 12–31, then its location can be specified by the three *cylindrical coordinates*, r, θ, z, where the z coordinate is identical to that used for rectangular coordinates. Since the unit vector defining its direction, $\mathbf{u}_z$, is constant, the time derivatives of this vector are zero, and therefore the position, velocity, and acceleration of the particle can be written in terms of its cylindrical coordinates as

$$\mathbf{r}_P = r\mathbf{u}_r + z\mathbf{u}_z$$

$$\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{u}_z \qquad (12\text{–}31)$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta + \ddot{z}\mathbf{u}_z \qquad (12\text{–}32)$$

Time Derivatives. The above equations require that we obtain the time derivatives $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, and $\ddot{\theta}$ in order to determine the r and θ components of $\mathbf{v}$ and $\mathbf{a}$. Two types of problems generally occur:

1. If the polar coordinates are specified as time parametric equations, $r = r(t)$ and $\theta = \theta(t)$, then the time derivatives can be found directly.
2. If the time parametric equations are not given, then the path $r = f(\theta)$ must be known. Using the chain rule of calculus we can then find the relation between $\dot{r}$ and $\dot{\theta}$, and between $\ddot{r}$ and $\ddot{\theta}$.

The spiral motion of this girl can be followed by using cylindrical components. Here the radial coordinate r is constant, the transverse coordinate θ will increase with time as the girl rotates about the vertical, and her altitude z will decrease with time.

PROCEDURE FOR ANALYSIS

Coordinate System.

- Polar coordinates are a suitable choice for solving problems where the angular motion of the radial coordinate r is given to describe the particle's motion. Also, some paths of motion can conveniently be described in terms of these coordinates.
- To use polar coordinates, the origin is established at a fixed point, and the radial line r is directed to the particle.
- The transverse coordinate θ is measured from a fixed reference line to the radial line.

Velocity and Acceleration.

- Once r and the four time derivatives $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, and $\ddot{\theta}$ have been evaluated at the instant considered, their values are substituted into Eqs. 12–25 and 12–29 to obtain the radial and transverse components of $\mathbf{v}$ and $\mathbf{a}$.
- If it is necessary to take the time derivatives of $r = f(\theta)$, then the chain rule must be used. See Appendix C.
- Motion in three dimensions requires a simple extension of the above procedure to include $\dot{z}$ and $\ddot{z}$.

EXAMPLE 12.17

The amusement park ride shown in Fig. 12–32*a* consists of a chair that is rotating in a horizontal circular path of radius r such that the arm OB has an angular velocity $\dot{\theta}$ and angular acceleration $\ddot{\theta}$. Determine the radial and transverse components of velocity and acceleration of the car. Neglect its size in the calculation.

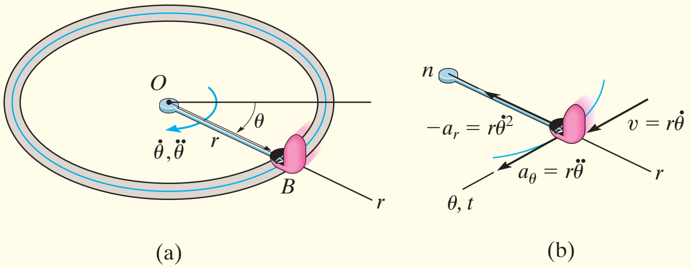


Fig. 12–32

SOLUTION

Coordinate System. Since the angular motion of the arm is known, polar coordinates will be used for the solution, Fig. 12–32*a*.

Velocity and Acceleration. It is first necessary to specify the first and second time derivatives of r and θ. Since r is *constant*, we have

$$r = r \qquad \dot{r} = 0 \qquad \ddot{r} = 0$$

Thus,

$$v_r = \dot{r} = 0 \qquad \textit{Ans.}$$

$$v_\theta = r\dot{\theta} \qquad \textit{Ans.}$$

$$a_r = \ddot{r} - r\dot{\theta}^2 = -r\dot{\theta}^2 \qquad \textit{Ans.}$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = r\ddot{\theta} \qquad \textit{Ans.}$$

These results are shown in Fig. 12–32*b*.

NOTE: The *n*, *t* axes are also shown in Fig. 12–32*b*, which in this special case of circular motion happen to be *collinear* with the r and θ axes, respectively. Since $v = v_\theta = v_t = r\dot{\theta}$, then by comparison,

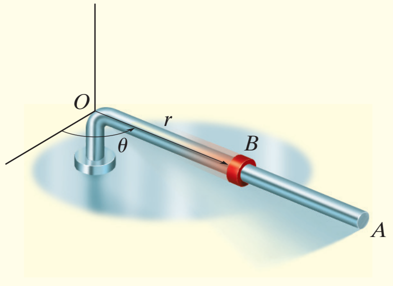
$$-a_r = a_n = \frac{v^2}{\rho} = \frac{(r\dot{\theta})^2}{r} = r\dot{\theta}^2$$

$$a_\theta = a_t = \frac{dv}{dt} = \frac{d}{dt}(r\dot{\theta}) = \frac{dr}{dt}\dot{\theta} + r\frac{d\dot{\theta}}{dt} = 0 + r\ddot{\theta}$$

12

EXAMPLE 12.18

The rod OA in Fig. 12–33a rotates in the horizontal plane such that $\theta = (t^3)$ rad. At the same time, the collar B is sliding outward along OA so that $r = (100t^2)$ mm. If t is in seconds, determine the velocity and acceleration of the collar when $t = 1$ s.



(a)

SOLUTION

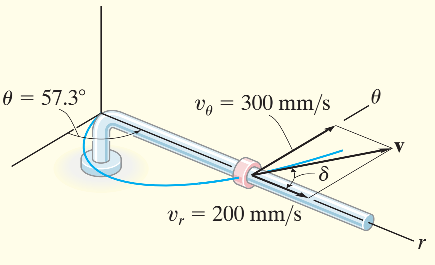
Coordinate System. Since time parametric equations of the path are given, it is not necessary to relate r to θ.

Velocity and Acceleration. Determining the time derivatives of r and θ and evaluating them when $t = 1$ s, we have

$$r = 100t^2\Big|_{t=1\text{ s}} = 100\text{ mm} \quad \theta = t^3\Big|_{t=1\text{ s}} = 1\text{ rad} = 57.3^\circ$$

$$\dot{r} = 200t\Big|_{t=1\text{ s}} = 200\text{ mm/s} \quad \dot{\theta} = 3t^2\Big|_{t=1\text{ s}} = 3\text{ rad/s}$$

$$\ddot{r} = 200\Big|_{t=1\text{ s}} = 200\text{ mm/s}^2 \quad \ddot{\theta} = 6t\Big|_{t=1\text{ s}} = 6\text{ rad/s}^2$$



(b)

As shown in Fig. 12–33b,

$$\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta$$
$$= 200\mathbf{u}_r + 100(3)\mathbf{u}_\theta = \{200\mathbf{u}_r + 300\mathbf{u}_\theta\}\text{ mm/s}$$

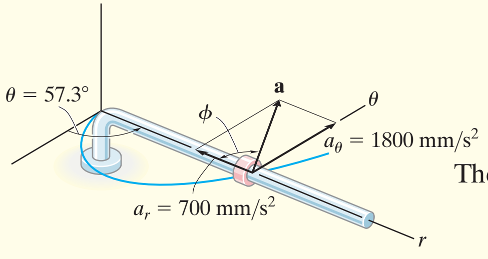
The magnitude of $\mathbf{v}$ is

$$v = \sqrt{(200)^2 + (300)^2} = 361\text{ mm/s} \qquad \textit{Ans.}$$

$$\delta = \tan^{-1}\left(\frac{300}{200}\right) = 56.3^\circ \quad \delta + 57.3^\circ = 114^\circ \qquad \textit{Ans.}$$

As shown in Fig. 12–33c,

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta$$
$$= [200 - 100(3)^2]\mathbf{u}_r + [100(6) + 2(200)3]\mathbf{u}_\theta$$
$$= \{-700\mathbf{u}_r + 1800\mathbf{u}_\theta\}\text{ mm/s}^2$$



(c)

Fig. 12–33

The magnitude of $\mathbf{a}$ is

$$a = \sqrt{(-700)^2 + (1800)^2} = 1930\text{ mm/s}^2 \qquad \textit{Ans.}$$

$$\phi = \tan^{-1}\left(\frac{1800}{700}\right) = 68.7^\circ \quad (180^\circ - \phi) + 57.3^\circ = 169^\circ \qquad \textit{Ans.}$$

NOTE: The velocity is tangent to the path; however, the acceleration is directed within the curvature of the path, as expected.

EXAMPLE 12.19

The searchlight in Fig. 12–34*a* casts a spot of light along the wall that is located 100 m from the searchlight. Determine the magnitudes of the velocity and acceleration at which the light appears to travel across the wall at the instant $\theta = 45°$. The searchlight rotates at a constant rate of $\dot{\theta} = 4$ rad/s.

SOLUTION

Coordinate System. Polar coordinates will be used to solve this problem since the angular rate of the searchlight is given. To find the necessary time derivatives it is first necessary to relate r to θ. From Fig. 12–34*a*,

$$r = 100/\cos\theta = 100\sec\theta$$

Velocity and Acceleration. Using the chain rule, noting that $d(\sec\theta) = \sec\theta\tan\theta\, d\theta$, and $d(\tan\theta) = \sec^2\theta\, d\theta$, we have

$$\dot{r} = 100(\sec\theta\tan\theta)\dot{\theta}$$

$$\ddot{r} = 100(\sec\theta\tan\theta)\dot{\theta}(\tan\theta)\dot{\theta} + 100\sec\theta(\sec^2\theta)\dot{\theta}(\dot{\theta})$$
$$+\ 100\sec\theta\tan\theta(\ddot{\theta})$$
$$= 100\sec\theta\tan^2\theta\,(\dot{\theta})^2 + 100\sec^3\theta\,(\dot{\theta})^2 + 100(\sec\theta\tan\theta)\ddot{\theta}$$

Since $\dot{\theta} = 4$ rad/s $=$ constant, then $\ddot{\theta} = 0$, and the above equations, when $\theta = 45°$, become

$$r = 100\sec 45° = 141.4$$
$$\dot{r} = 400\sec 45°\tan 45° = 565.7$$
$$\ddot{r} = 1600\,(\sec 45°\tan^2 45° + \sec^3 45°) = 6788.2$$

As shown in Fig. 12–34*b*,

$$\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta$$
$$= 565.7\mathbf{u}_r + 141.4(4)\mathbf{u}_\theta$$
$$= \{565.7\mathbf{u}_r + 565.7\mathbf{u}_\theta\}\text{ m/s}$$
$$v = \sqrt{v_r^2 + v_\theta^2} = \sqrt{(565.7)^2 + (565.7)^2}$$
$$= 800\text{ m/s} \qquad \textit{Ans.}$$

As shown in Fig. 12–34*c*,

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta$$
$$= [6788.2 - 141.4(4)^2]\mathbf{u}_r + [141.4(0) + 2(565.7)4]\mathbf{u}_\theta$$
$$= \{4525.5\mathbf{u}_r + 4525.5\mathbf{u}_\theta\}\text{ m/s}^2$$
$$a = \sqrt{a_r^2 + a_\theta^2} = \sqrt{(4525.5)^2 + (4525.5)^2}$$
$$= 6400\text{ m/s}^2 \qquad \textit{Ans.}$$

NOTE: It is also possible to find a without having to calculate $\ddot{r}$ (or a_r). As shown in Fig. 12–34*d*, since $a_\theta = 4525.5$ m/s^2, then by vector resolution, $a = 4525.5/\cos 45° = 6400$ m/s^2.

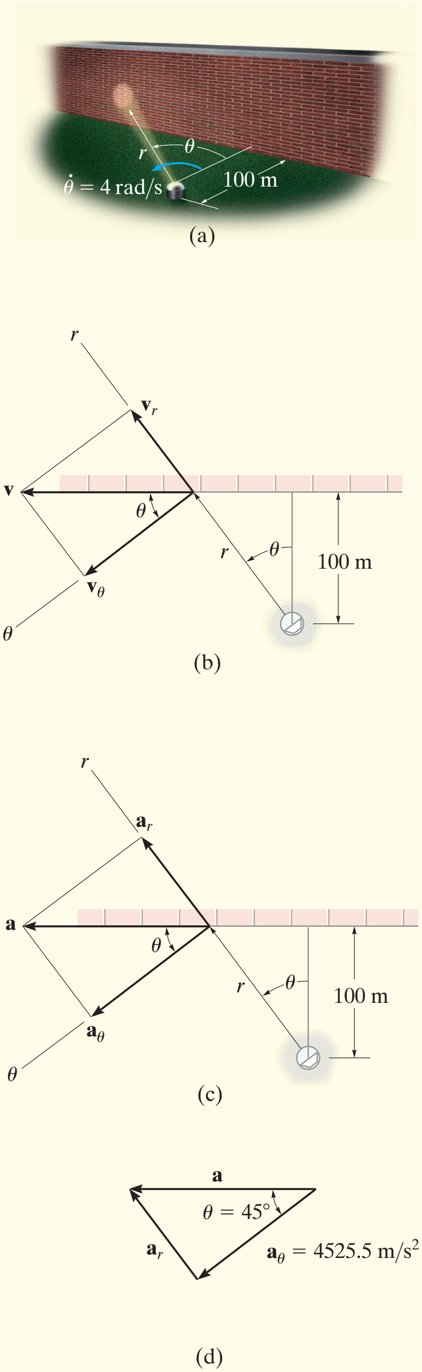
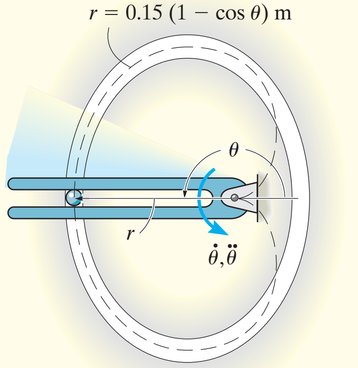


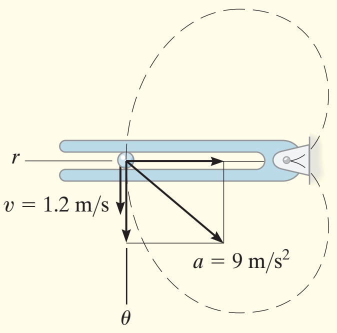
Fig. 12–34

EXAMPLE 12.20

Due to the rotation of the forked rod, the ball in Fig. 12–35*a* travels around the slotted path, a portion of which is in the shape of a cardioid, $r = 0.15(1 - \cos\theta)$ m, where θ is in radians. If the ball's velocity is $v = 1.2$ m/s and its acceleration is $a = 9$ m/s^2 at the instant $\theta = 180°$, determine the angular velocity $\dot{\theta}$ and angular acceleration $\ddot{\theta}$ of the fork.



(a)



(b)

Fig. 12–35

SOLUTION

Coordinate System. This path is most unusual, and mathematically it is best expressed using polar coordinates, as done here, rather than using rectangular coordinates. Also, since $\dot{\theta}$ and $\ddot{\theta}$ must be determined, then r, θ coordinates are an obvious choice.

Velocity and Acceleration. The time derivatives of r and θ can be determined using the chain rule.

$$r = 0.15(1 - \cos\theta)$$
$$\dot{r} = 0.15(\sin\theta)\dot{\theta}$$
$$\ddot{r} = 0.15(\cos\theta)(\dot{\theta})^2 + 0.15(\sin\theta)\ddot{\theta}$$

Evaluating these results at $\theta = 180°$, we have

$$r = 0.3\text{ m} \qquad \dot{r} = 0 \qquad \ddot{r} = -0.15\dot{\theta}^2$$

Since $v = 1.2$ m/s, using Eq. 12–26 to determine $\dot{\theta}$ yields

$$v = \sqrt{(\dot{r})^2 + (r\dot{\theta})^2}$$
$$1.2 = \sqrt{(0)^2 + (0.3\dot{\theta})^2}$$
$$\dot{\theta} = 4\text{ rad/s}$$ *Ans.*

In a similar manner, $\ddot{\theta}$ can be found using Eq. 12–30.

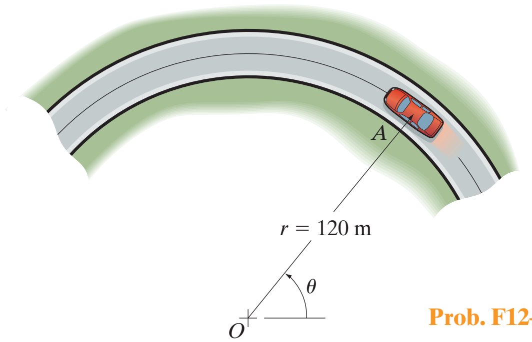
$$a = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2}$$
$$9 = \sqrt{[-0.15(4)^2 - 0.3(4)^2]^2 + [0.3\ddot{\theta} + 2(0)(4)]^2}$$
$$(9)^2 = (-7.2)^2 + 0.09\ddot{\theta}^2$$
$$\ddot{\theta} = 18\text{ rad/s}^2$$ *Ans.*

Vectors **a** and **v** are shown in Fig. 12–35*b*.

12

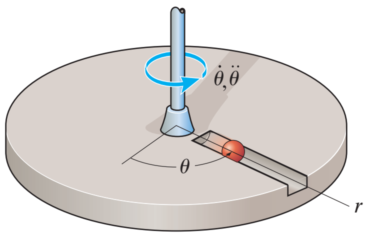
FUNDAMENTAL PROBLEMS

F12–33. The car has a speed of 16.5 m/s. Determine the angular velocity $\dot{\theta}$ of the radial line OA at this instant.



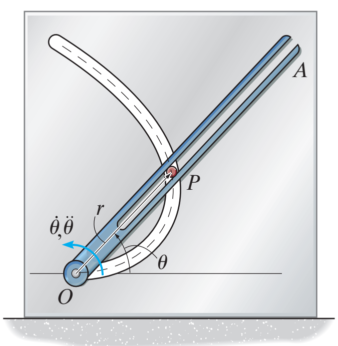
Prob. F12–33

F12–34. The platform is rotating about the vertical axis such that at any instant its angular position is $\theta = (4t^{3/2})$ rad, where t is in seconds. A ball rolls outward along the radial groove so that its position is $r = (0.1t^3)$ m, where t is in seconds. Determine the magnitudes of the velocity and acceleration of the ball when $t = 1.5$ s.



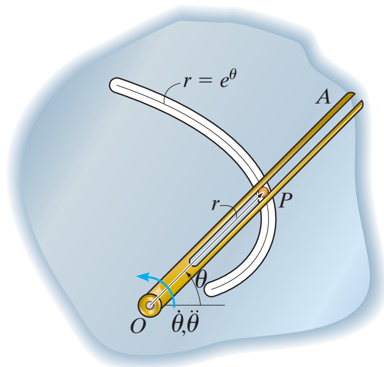
Prob. F12–34

F12–35. Peg P is driven by the fork link OA along the curved path described by $r = (0.6\theta)$ m. At the instant $\theta = \pi/4$ rad, the angular velocity and angular acceleration of the link are $\dot{\theta} = 3$ rad/s and $\ddot{\theta} = 1$ rad/s^2. Determine the magnitude of the peg's acceleration at this instant.



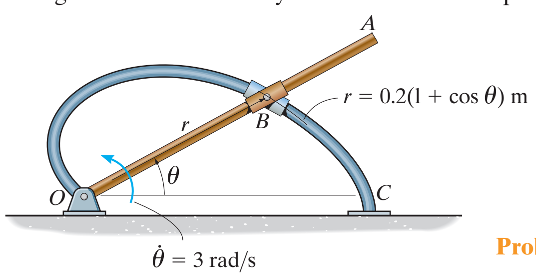
Prob. F12–35

F12–36. Peg P is driven by the forked link OA along the path described by $r = e^{\theta}$, where r is in meters. When $\theta = \frac{\pi}{4}$ rad, the link has an angular velocity and angular acceleration of $\dot{\theta} = 2$ rad/s and $\ddot{\theta} = 4$ rad/s^2. Determine the radial and transverse components of the peg's acceleration at this instant.



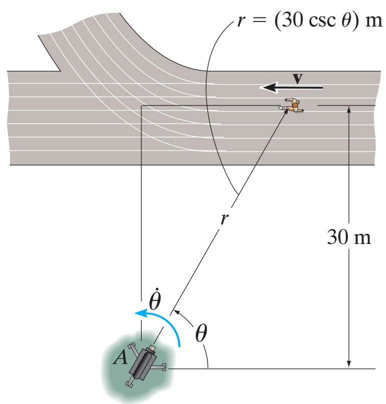
Prob. F12–36

F12–37. The collars are pin connected at B and are free to move along rod OA and the curved guide OC having the shape of a cardioid, $r = [0.2(1 + \cos\theta)]$ m. At $\theta = 30°$, the angular velocity of OA is $\dot{\theta} = 3$ rad/s. Determine the magnitude of the velocity of the collars at this point.



Prob. F12–37

F12–38. At the instant $\theta = 45°$, the athlete is running with a constant speed of 2 m/s. Determine the angular velocity at which the camera must turn in order to follow the motion.

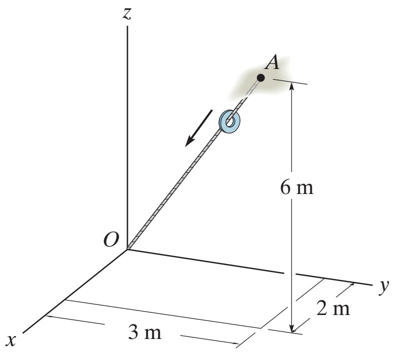


Prob. F12–38

12

PROBLEMS

12–155. The small washer is sliding down the cord OA. When it is at the midpoint, its speed is 28 m/s and its acceleration is 7 m/s^2. Express the velocity and acceleration of the washer at this point in terms of its cylindrical components.



Prob. 12–155

***12–156.** If a particle's position is described by the polar coordinates $r = 4(1 + \sin t)$ m and $\theta = (2e^{-t})$ rad, where t is in seconds and the argument for the sine is in radians, determine the radial and transverse components of the particle's velocity and acceleration when $t = 2$ s.

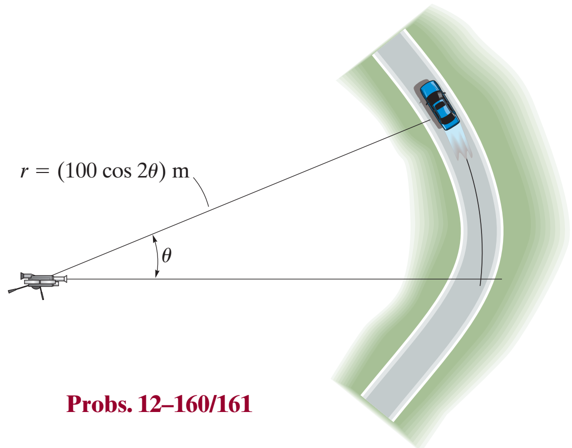
12–157. For a short time a rocket travels up and to the right at a constant speed of 800 m/s along the parabolic path $y = 600 - 35x^2$. Determine the radial and transverse components of velocity of the rocket at the instant $\theta = 60°$, where θ is measured counterclockwise from the x axis.

12–158. A particle travels around a limaçon, defined by the equation $r = b - a \cos \theta$, where a and b are constants. Determine the particle's radial and transverse components of velocity and acceleration as a funcion of θ and its time derivatives.

12–159. A particle is moving along a circular path having a radius of 150 mm such that its position as a function of time is given by $\theta = \sin 3t$, where θ and the argument for the sine are in radians, and t is in seconds. Determine the magnitude of the acceleration of the particle at $\theta = 30°$. The particle starts from rest at $\theta = 0°$.

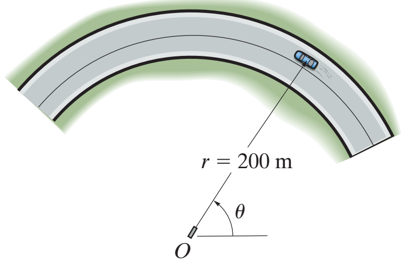
***12–160.** The driver of the car maintains a constant speed of 40 m/s. Determine the angular velocity of the camera tracking the car when $\theta = 15°$.

12–161. When $\theta = 15°$, the car has a speed of 50 m/s which is increasing at 6 m/s^2. Determine the angular velocity of the camera tracking the car at this instant.



Probs. 12–160/161

12–162. A radar gun at O rotates with the angular velocity of $\dot{\theta} = 0.1$ rad/s and angular acceleration of $\ddot{\theta} = 0.025$ rad/s^2, at the instant $\theta = 45°$, as it follows the motion of the car traveling along the circular road having a radius of $r = 200$ m. Determine the magnitudes of velocity and acceleration of the car at this instant.

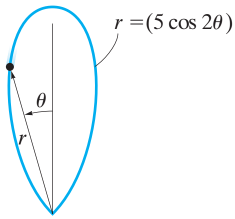


Prob. 12–162

12–163. If a particle moves along a path such that $r = (e^{at})$ m and $\theta = t$, where t is in seconds, plot the path $r = f(\theta)$, and determine the particle's radial and transverse components of velocity and acceleration.

***12–164.** If a particle's position is described by the polar coordinates $r = (2 \sin 2\theta)$ m and $\theta = (4t)$ rad, where t is in seconds, determine the radial and transverse components of its velocity and acceleration when $t = 1$ s.

12–165. A particle travels along the portion of the "four-leaf rose" defined by the equation $r = (5 \cos 2\theta)$ m. If the angular velocity of the radial coordinate line is $\dot{\theta} = (3t^2)$ rad/s, where t is in seconds, determine the radial and transverse components of the particle's velocity and acceleration at the instant $\theta = 30°$. When $t = 0$, $\theta = 0$.

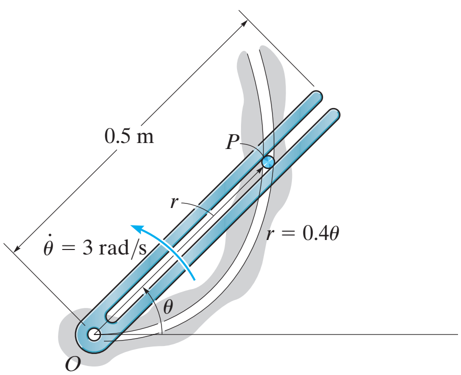


Prob. 12–165

12–166. The time rate of change of acceleration is referred to as the *jerk*, which is often used as a means of measuring passenger discomfort. Calculate this vector, $\dot{\mathbf{a}}$, in terms of its cylindrical components, using Eq. 12–32.

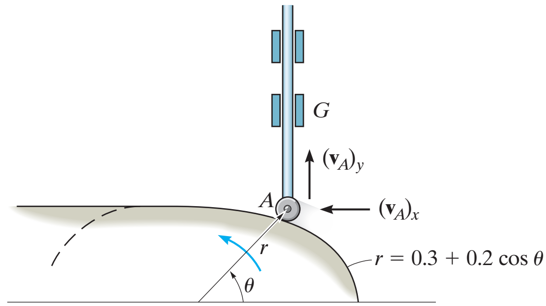
12–167. The position of a particle is described by $r = (300e^{-0.5t})$ mm and $\theta = (0.3t^2)$ rad, where t is in seconds. Determine the magnitudes of the particle's velocity and acceleration at the instant $t = 1.5$ s.

***12–168.** The slotted link is pinned at O, and as a result of the constant angular velocity $\dot{\theta} = 3$ rad/s it drives the peg P for a short distance along the spiral guide $r = (0.4\,\theta)$ m, where θ is in radians. Determine the radial and transverse components of the velocity and acceleration of P at the instant $\theta = \pi/3$ rad.



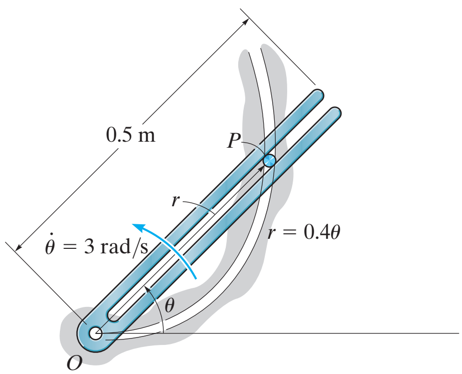
Prob. 12–168

12–169. The mechanism of a machine is constructed so that the roller at A follows the surface of the cam described by the equation $r = (0.3 + 0.2 \cos \theta)$ m. If $\dot{\theta} = 0.5$ rad/s and $\ddot{\theta} = 0$, determine the magnitudes of the roller's velocity and acceleration when $\theta = 30°$. Neglect the size of the roller. Also determine the velocity components $(\mathbf{v}_A)_x$ and $(\mathbf{v}_A)_y$ of the roller at this instant. The rod to which the roller is attached remains vertical and can slide up or down along the guides while the guides translate horizontally to the left.



Prob. 12–169

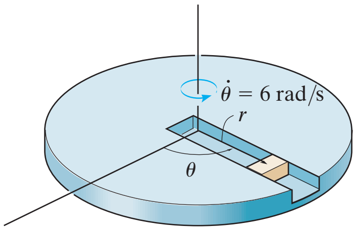
12–170. The slotted link is pinned at O, and as a result of the constant angular velocity $\dot{\theta} = 3$ rad/s it drives the peg P for a short distance along the spiral guide $r = (0.4\,\theta)$ m, where θ is in radians. Determine the velocity and acceleration of the particle at the instant it leaves the slot in the link, i.e., when $r = 0.5$ m.



Prob. 12–170

12–171. A particle moves along a circular path of radius 300 mm. If its angular velocity is $\dot{\theta} = (2t^2)$ rad/s, where t is in seconds, determine the magnitude of the particle's acceleration when $t = 2$ s.

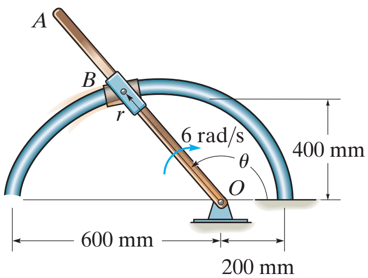
***12–172.** A block moves outward along the slot in the platform with a speed of $\dot{r} = (4t)$ m/s, where t is in seconds. The platform rotates at a constant rate of 6 rad/s. If the block starts from rest at the center, determine the magnitudes of its velocity and acceleration when $t = 1$ s.



Prob. 12–172

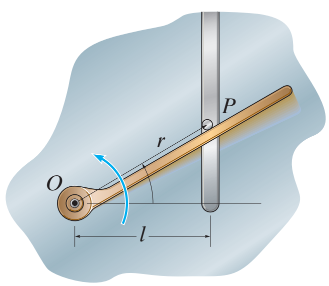
12–173. The rod OA rotates clockwise with a constant angular velocity of 6 rad/s. Two pin-connected slider blocks, located at B, move freely on OA and the curved rod whose shape is a limaçon described by the equation $r = 200(2 - \cos\theta)$ mm. Determine the speed of the slider blocks at the instant $\theta = 150°$.

12–174. Determine the magnitude of the acceleration of the slider blocks in Prob. 12–173 when $\theta = 150°$.



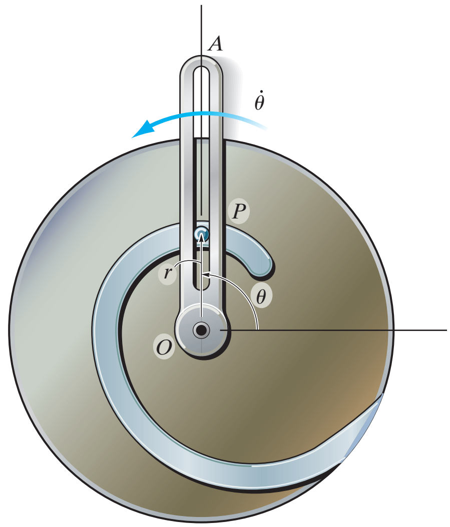
Probs. 12–173/174

12–175. The link is pinned at O, and as a result of its rotation it drives the peg P along the vertical guide. Calculate the magnitudes of the velocity and acceleration of P if $\theta = ct$ rad, where c is a constant.



Prob. 12–175

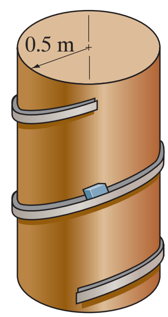
***12–176.** The motion of the pin P is controlled by the rotation of the grooved link OA. If the link is rotating at a constant angular rate of $\dot{\theta} = 6$ rad/s, determine the magnitudes of the velocity and acceleration of P at the instant $\theta = \pi/2$ rad. The spiral path is defined by the equation $r = (40\,\theta)$ mm, where θ is in radians.



Prob. 12–176

12–177. The box slides down the helical ramp with a constant speed of $v = 2$ m/s. Determine the magnitude of its acceleration. The ramp descends a vertical distance of 1 m for every full revolution. The mean radius of the ramp is $r = 0.5$ m.

12–178. The box slides down the helical ramp such that $r = 0.5$ m, $\theta = (0.5t^3)$ rad, and $z = (2–0.2t^2)$ m, where t is in seconds. Determine the magnitudes of the velocity and acceleration of the box at the instant $\theta = 2\pi$ rad.

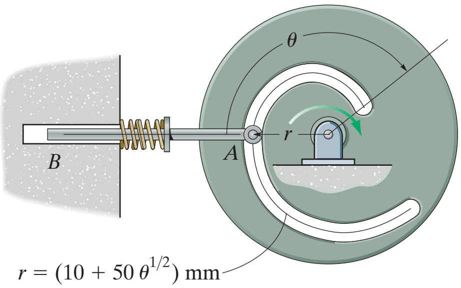


Probs. 12–177/178

12–179. A particle is moving along a circular path having a 400-mm radius. Its position as a function of time is given by $\theta = (2t^2)$ rad, where t is in seconds. Determine the magnitude of the particle's acceleration when $\theta = 30°$. The particle starts from rest when $\theta = 0°$.

***12–180.** If the circular plate rotates clockwise with a constant angular velocity of $\dot{\theta} = 1.5$ rad/s, determine the magnitudes of the velocity and acceleration of the follower rod AB when $\theta = (2/3\,\pi)$ rad.

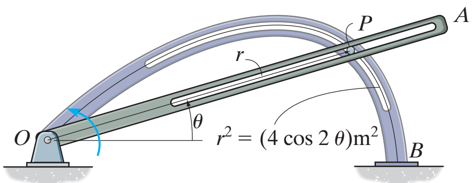
12–181. When $\theta = (2/3\pi)$ rad, the angular velocity and angular acceleration of the circular plate are $\dot{\theta} = 1.5$ rad/s and $\ddot{\theta} = 3$ rad/s^2, respectively. Determine the magnitudes of the velocity and acceleration of the rod AB at this instant.



Probs. 12–180/181

12–182. The motion of peg P is constrained by the lemniscate curved slot in OB and by the slotted arm OA. If OA rotates counterclockwise with a constant angular velocity of $\dot{\theta} = 3$ rad/s, determine the magnitudes of the velocity and acceleration of peg P at $\theta = 30°$.

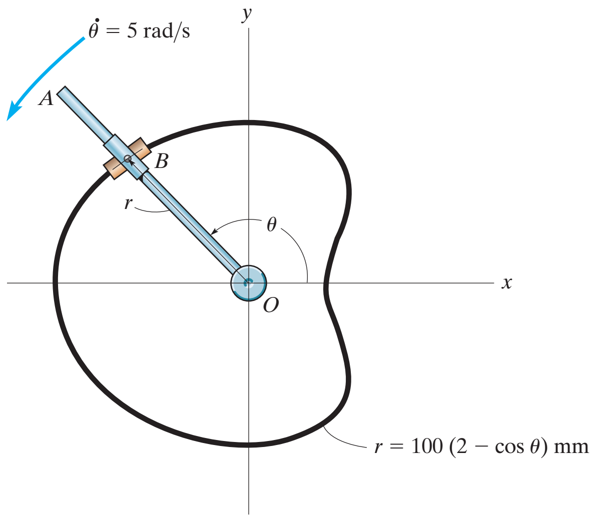
12–183. The motion of peg P is constrained by the lemniscate curved slot in OB and by the slotted arm OA. If OA rotates counterclockwise with an angular velocity of $\dot{\theta} = (3t^{3/2})$ rad/s, where t is in seconds, determine the magnitudes of the velocity and acceleration of peg P at $\theta = 30°$. When $t = 0$, $\theta = 0°$.



Probs. 12–182/183

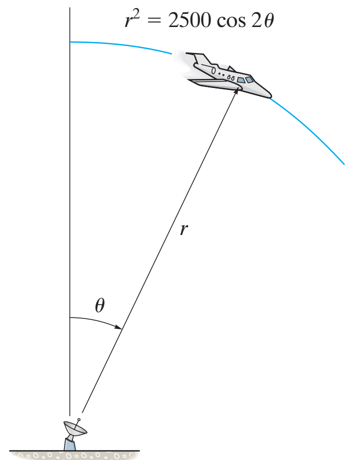
***12–184.** The rod OA rotates counterclockwise with a constant angular velocity of $\dot{\theta} = 5$ rad/s. Two pin-connected slider blocks, located at B, move freely on OA and the curved rod whose shape is a limaçon described by the equation $r = 100(2 - \cos\theta)$ mm. Determine the speed of the slider blocks at the instant $\theta = 120°$.

12–185. Determine the magnitude of the acceleration of the slider blocks in Prob. 12–184 when $\theta = 120°$.



Probs. 12–184/185

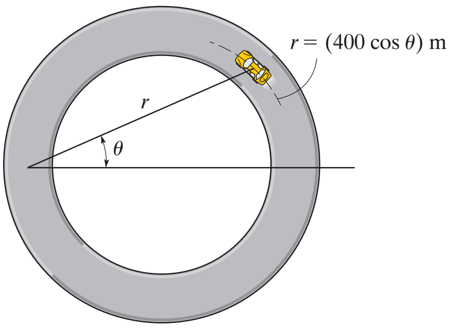
12–186. For a short time the jet plane moves along a path in the shape of a lemniscate, $r^2 = (2500 \cos 2\theta)$ km^2. At the instant $\theta = 30°$, the radar tracking device is rotating at $\dot{\theta} = 5(10^{-3})$ rad/s with $\ddot{\theta} = 2(10^{-3})$ rad/s^2. Determine the radial and transverse components of velocity and acceleration of the plane at this instant.



Prob. 12–186

12–187. The car travels around the circular track with a constant speed of 20 m/s. Determine the car's radial and transverse components of velocity and acceleration at the instant $\theta = \pi/4$ rad.

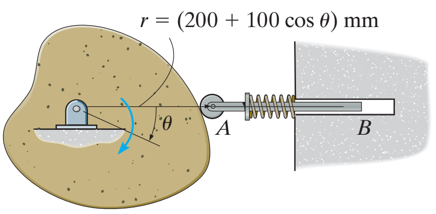
***12–188.** The car travels around the circular track such that its transverse component is $\theta = (0.006t^2)$ rad, where t is in seconds. Determine the car's radial and transverse components of velocity and acceleration at the instant $t = 4$ s.



Probs. 12–187/188

12–189. If the cam rotates clockwise with a constant angular velocity of $\dot{\theta} = 5$ rad/s, determine the magnitudes of the velocity and acceleration of the follower rod AB at the instant $\theta = 30°$. The surface of the cam has the shape of a limaçon defined by $r = (200 + 100 \cos \theta)$ mm.

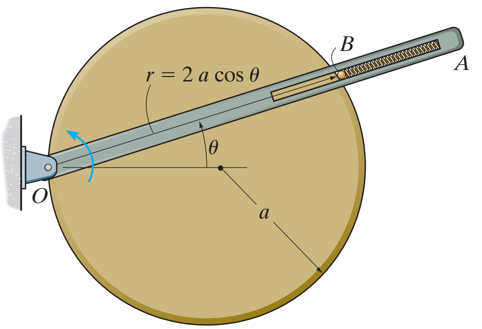
12–190. At the instant $\theta = 30°$, the cam rotates with a clockwise angular velocity of $\dot{\theta} = 5$ rad/s and angular acceleration of $\ddot{\theta} = 6$ rad/s^2. Determine the magnitudes of the velocity and acceleration of the follower rod AB at this instant. The surface of the cam has the shape of a limaçon defined by $r = (200 + 100 \cos \theta)$ mm.



Probs. 12–189/190

12–191. The slotted arm OA rotates counterclockwise about O with a constant angular velocity of $\dot{\theta}$. The motion of pin B is constrained such that it moves on the fixed circular surface and along the slot in OA. Determine the magnitudes of the velocity and acceleration of pin B as a function of θ.

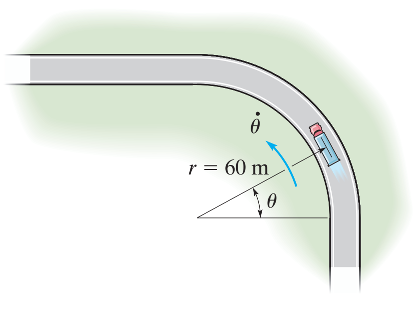
***12–192.** The slotted arm OA rotates counterclockwise about O such that when $\theta = \pi/4$, arm OA is rotating with an angular velocity of $\dot{\theta}$ and an angular acceleration of $\ddot{\theta}$. Determine the magnitudes of the velocity and acceleration of pin B at this instant. The motion of pin B is constrained such that it moves on the fixed circular surface and along the slot in OA.



Probs. 12–191/192

12–193. A truck is traveling along the horizontal circular curve of radius $r = 60$ m with a constant speed $v = 20$ m/s. Determine the angular rate of rotation $\dot{\theta}$ of the radial line r and the magnitude of the truck's acceleration.

12–194. A truck is traveling along the horizontal circular curve of radius $r = 60$ m with a speed of 20 m/s which is increasing at 3 m/s^2, Determine the truck's radial and transverse components of acceleration.



Probs. 12–193/194

12.9 ABSOLUTE DEPENDENT MOTION ANALYSIS OF TWO PARTICLES

In some types of problems the motion of one particle will *depend* on the corresponding motion of another particle. This dependency commonly occurs if the particles, here represented by blocks, are interconnected by inextensible cords which are wrapped around pulleys. For example, the movement of block A downward along the inclined plane in Fig. 12–36 will cause a corresponding movement of block B up the other incline.

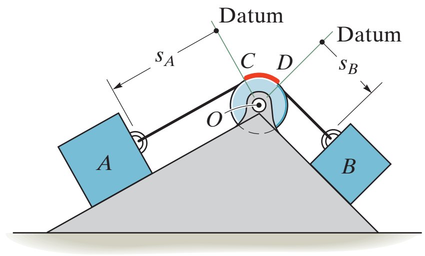


Fig. 12–36

Although this may seem obvious, we can show this mathematically by first specifying the location of the blocks using ***position coordinates*** s_A and s_B, where each of the coordinate axes is (1) measured from a *fixed* point (O) or *fixed* datum line, (2) measured along each inclined plane *in the direction of motion* of each block, and (3) has a positive sense from the fixed datums to A and to B. If the total cord length is l_T, then the two position coordinates can be related by the equation

$$s_A + l_{CD} + s_B = l_T$$

Here l_{CD} is the length of the cord passing over arc CD. Taking the time derivative of this expression, realizing that l_{CD} and l_T *remain constant*, while s_A and s_B measure the segments of the cord that change in length, we have

$$\frac{ds_A}{dt} + \frac{ds_B}{dt} = 0 \qquad \text{or} \qquad v_B = -v_A$$

The negative sign indicates that when block A has a velocity downward, i.e., in the direction of positive s_A, it causes a corresponding upward velocity of block B; i.e., B moves in the negative s_B direction.

In a similar manner, time differentiation of the velocities yields the relation between the accelerations, i.e.,

$$a_B = -a_A$$

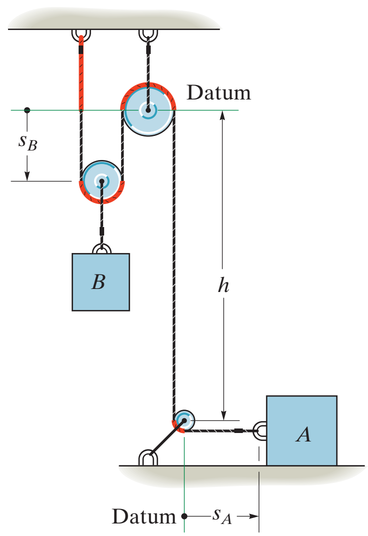
A more complicated example is shown in Fig. 12–37*a*. In this case, the position of block A is specified by s_A, and the position of the *end* of the cord from which block B is suspended is defined by s_B. As above, we have chosen position coordinates which (1) have their origin at fixed points or datums, (2) are measured in the direction of motion of each block, and (3) from the fixed datums are positive to the right for s_A and positive downward for s_B. During the motion, the length of the red colored segments of the cord in Fig. 12–37*a remains constant*. If l represents the total length of cord minus these segments, then the two position coordinates can be related by the equation

$$2s_B + h + s_A = l$$

Since l and h are constant during the motion, the two time derivatives yield

$$2v_B = -v_A \qquad 2a_B = -a_A$$

Hence, when B moves downward ($+s_B$), A moves to the left ($-s_A$) with twice the motion.



(a)

Fig. 12–37

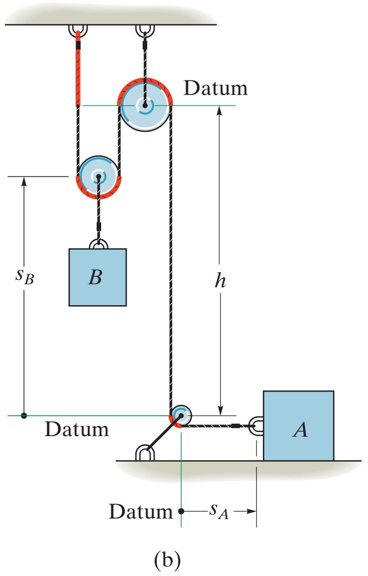


Fig. 12–37 (cont.)

This example can also be worked by defining the position of block B from the center of the bottom pulley (a fixed point), Fig. 12–37b. In this case

$$2(h - s_B) + h + s_A = l$$

Time differentiation yields

$$2v_B = v_A \qquad 2a_B = a_A$$

Here the signs are the same because when A moves to the right ($+s_A$), B moves up ($+s_B$).

The cable is wrapped around several pulleys on this crane in order to reduce the required force needed to hoist a load.

PROCEDURE FOR ANALYSIS

Position-Coordinate Equation.

- Establish each position coordinate with an origin located at a *fixed* point or datum.
- It is *not necessary* that the *origin* be the *same* for each of the coordinates; however, it is *important* that each coordinate be directed along the *path of motion* of the particle.
- Using geometry or trigonometry, relate the position coordinates to the total length of the cord, l_T, or to that portion of cord, l, which *excludes* the segments that do not change length as the particles move—such as arc segments wrapped over pulleys.
- If two or more cords are wrapped around pulleys, then the position of a point on one cord must be related to the position of a point on another cord using the above procedure. Separate equations are written for each cord and the positions of the two particles are then related by these equations (see Examples 12.22 and 12.23).

Time Derivatives.

- Two successive time derivatives of the position-coordinate equations will give the velocity and acceleration equations which relate the motions of the particles.
- The signs of the terms in these equations will be consistent with those that specify the positive and negative sense of the position coordinates.

Refer to the companion website for Lecture Summary and Quiz videos.

EXAMPLE 12.21

Determine the speed of block A in Fig. 12–38 if block B is moving upward at 1.8 m/s.

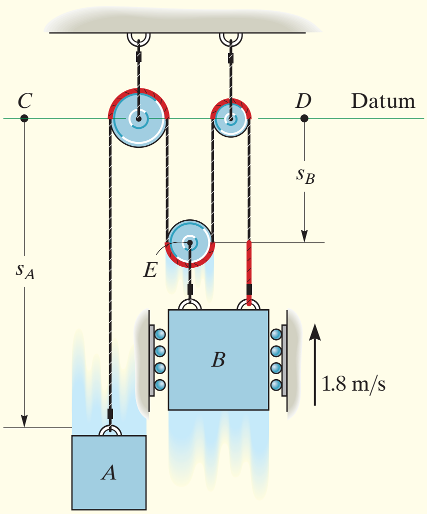


Fig. 12–38

SOLUTION

Position-Coordinate Equation. There is *one cord* in this system having segments which change length. Position coordinates s_A and s_B will be used since each is measured from a fixed point (C or D) and extends along each block's *path of motion*. Here, s_B is directed to point E since motion of B and E is the *same*.

The four red colored segments of the cord in Fig. 12–38 maintain a constant length and do not have to be considered as the blocks move. The remaining length of cord, l, is also constant and is related to the changing position coordinates s_A and s_B by the equation

$$s_A + 3s_B = l$$

Time Derivative. Taking the time derivative,

$$v_A + 3v_B = 0$$

so that when $v_B = -1.8$ m/s (upward),

$$v_A = 5.4 \text{ m/s} \downarrow \qquad \textit{Ans.}$$

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EXAMPLE 12.22

Determine the speed of block A in Fig. 12–39 if block B has an upward speed of 1.8 m/s.

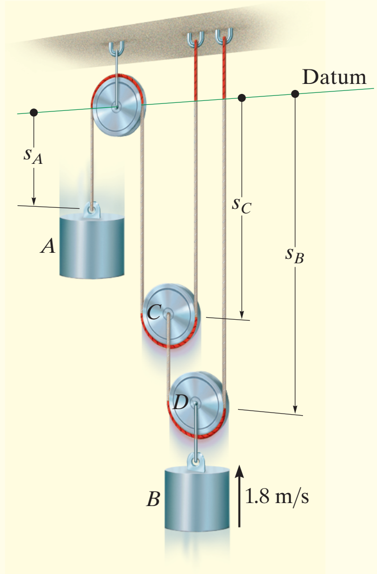


Fig. 12–39

SOLUTION

Position-Coordinate Equation. As shown, the positions of A and B are defined using coordinates s_A and s_B. Since the system has *two cords* with segments that change length, it will be necessary to use a third coordinate, s_C, in order to relate s_A to s_B. In other words, the length of one of the cords can be expressed in terms of s_A and s_C, and the length of the other cord can be expressed in terms of s_B and s_C.

The red colored segments of the cords in Fig. 12–39 do not have to be considered in the analysis. Why? For the remaining cord lengths, say l_1 and l_2, we have

$$s_A + 2s_C = l_1 \qquad s_B + (s_B - s_C) = l_2$$

Time Derivative. Taking the time derivative of these equations yields

$$v_A + 2v_C = 0 \quad 2v_B - v_C = 0$$

Eliminating v_C produces the relationship between the motions of each block.

$$v_A + 4v_B = 0$$

so that when $v_B = -1.8$ m/s (upward),

$$v_A = +7.2 \text{ m/s} = 7.2 \text{ m/s} \downarrow \qquad \textit{Ans.}$$

EXAMPLE 12.23

Determine the speed of block B in Fig. 12–40 if the end of the cord at A is pulled down with a speed of 2 m/s.

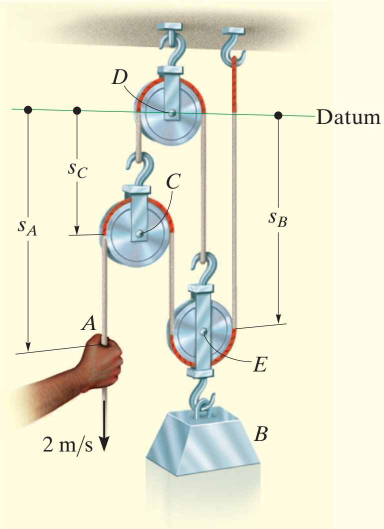


Fig. 12–40

SOLUTION

Position-Coordinate Equation. The position of A is defined by s_A, and the position of block B is specified by s_B since point E on the pulley will have the *same motion* as the block. Both coordinates are measured from a horizontal datum passing through the *fixed* pin at pulley D. Since the system consists of *two* cords, the coordinates s_A and s_B cannot be directly related. Instead, by establishing a third position coordinate, s_C, we can now express the length of one of the cords in terms of s_B and s_C, and the length of the other cord in terms of s_A, s_B, and s_C.

Excluding the four red colored segments of the cords in Fig. 12–40, the remaining constant cord lengths l_1 and l_2 (along with the hook and link dimensions) can be expressed as

$$s_C + s_B = l_1$$

$$(s_A - s_C) + (s_B - s_C) + s_B = l_2$$

Time Derivative. The time derivative of each equation gives

$$v_C + v_B = 0$$

$$v_A - 2v_C + 2v_B = 0$$

Eliminating v_C, we obtain

$$v_A + 4v_B = 0$$

so that when $v_A = 2$ m/s (downward),

$$v_B = -0.5 \text{ m/s} = 0.5 \text{ m/s} \uparrow \qquad \textit{Ans.}$$

12

EXAMPLE 12.24

The person at A is hoisting a safe S as shown in Fig. 12–41 by walking to the right with a constant velocity $v_A = 0.5$ m/s. Determine the velocity and acceleration of the safe when it reaches an elevation of 10 m. The rope is 30 m long and passes over a small pulley at D.

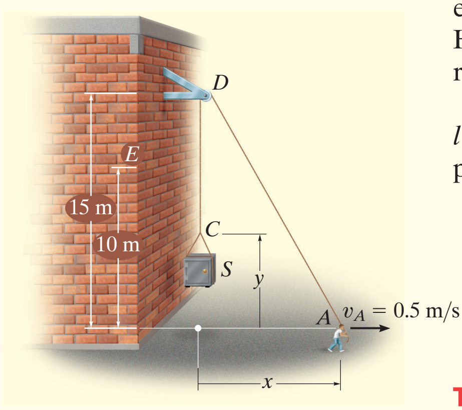


Fig. 12–41

SOLUTION

Position-Coordinate Equation. This problem is unlike the previous examples since rope segment DA changes *both direction and magnitude*. However, the ends of the rope, A and C, move along the x and y axes, respectively.

These coordinates may be related since the rope has a fixed length $l = 30$ m, which at all times is equal to the length of segment DA plus CD. Using the Pythagorean theorem to determine l_{DA}, we have $l_{DA} = \sqrt{(15)^2 + x^2}$. Also, $l_{CD} = 15 - y$, and so

$$l = l_{DA} + l_{CD}$$

$$30 = \sqrt{(15)^2 + x^2} + (15 - y)$$

$$y = \sqrt{225 + x^2} - 15 \qquad (1)$$

Time Derivatives. Taking the time derivative, using the chain rule (see Appendix C), where $v_S = dy/dt$ and $v_A = dx/dt$, we have

$$v_S = \frac{dy}{dt} = \left[\frac{1}{2}\frac{2x}{\sqrt{225 + x^2}}\right]\frac{dx}{dt}$$

$$= \frac{x}{\sqrt{225 + x^2}}v_A \qquad (2)$$

At $y = 10$ m, x is determined from Eq. 1, i.e., $x = 20$ m. Hence, from Eq. 2 with $v_A = 0.5$ m/s,

$$v_S = \frac{20}{\sqrt{225 + (20)^2}}(0.5) = 0.4 \text{ m/s} = 400 \text{ mm/s} \uparrow \qquad \textit{Ans.}$$

The acceleration is determined by taking the time derivative of Eq. 2. Since v_A is constant, then $a_A = dv_A/dt = 0$, and so

$$a_S = \frac{d^2y}{dt^2} = \left[\frac{-x(dx/dt)}{(225 + x^2)^{3/2}}\right]xv_A + \left[\frac{1}{\sqrt{225 + x^2}}\right]\left(\frac{dx}{dt}\right)v_A + \left[\frac{1}{\sqrt{225 + x^2}}\right]x\frac{dv_A}{dt} = \frac{225v_A^2}{(225 + x^2)^{3/2}}$$

At $x = 20$ m, with $v_A = 0.5$ m/s, the acceleration becomes

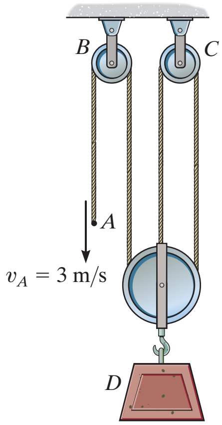
$$a_S = \frac{225(0.5 \text{ m/s})^2}{[225 + (20 \text{ m})^2]^{3/2}} = 0.00360 \text{ m/s}^2 = 3.60 \text{ mm/s}^2 \uparrow \qquad \textit{Ans.}$$

NOTE: The constant velocity at A causes the other end C of the rope to have an acceleration since $\mathbf{v}_A$ causes segment DA to change its direction as well as its length.

Refer to the companion website for a self quiz of these Example problems.

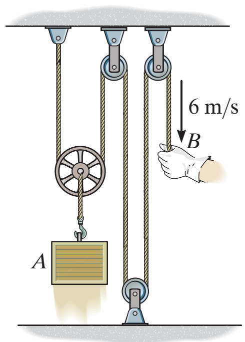
FUNDAMENTAL PROBLEMS

F12–39. Determine the velocity of block D if end A of the rope is pulled down with a speed of $v_A = 3 \text{ m/s}$.



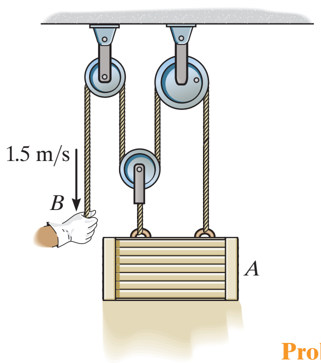
Prob. F12–39

F12–40. Determine the velocity of block A if end B of the rope is pulled down with a speed of 6 m/s.



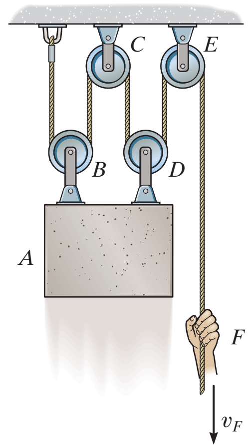
Prob. F12–40

F12–41. Determine the velocity of block A if end B of the rope is pulled down with a speed of 1.5 m/s.



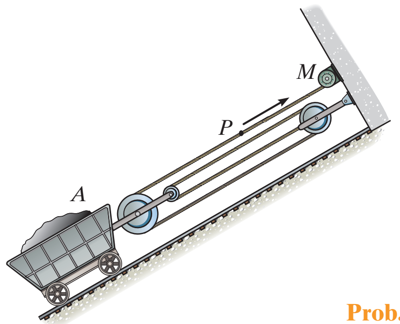
Prob. F12–41

F12–42. Determine the velocity of block A if end F of the rope is pulled down with a speed of $v_F = 3 \text{ m/s}$.



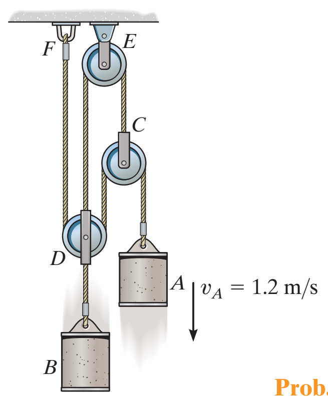
Prob. F12–42

F12–43. Determine the velocity of car A. The motor M winds in the cable such that $v_P = 4 \text{ m/s}$.



Prob. F12–43

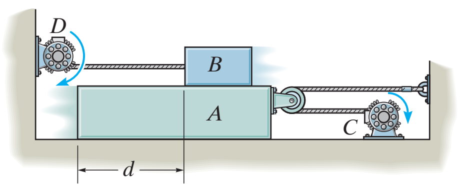
F12–44. Determine the velocity of block B if block A moves downward with a speed of $v_A = 1.2 \text{ m/s}$.



Prob. F12–44

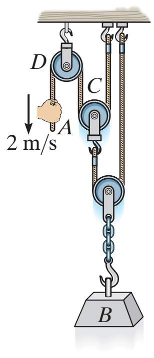
PROBLEMS

12–195. The motor at C pulls in the cable with an acceleration $a_C = (3t^2)\ \text{m/s}^2$, where t is in seconds. The motor at D draws in its cable at $a_D = 5\ \text{m/s}^2$. If both motors start at the same instant from rest when $d = 3$ m, determine (a) the time needed for $d = 0$, and (b) the velocity of block A relative to block B when this occurs.



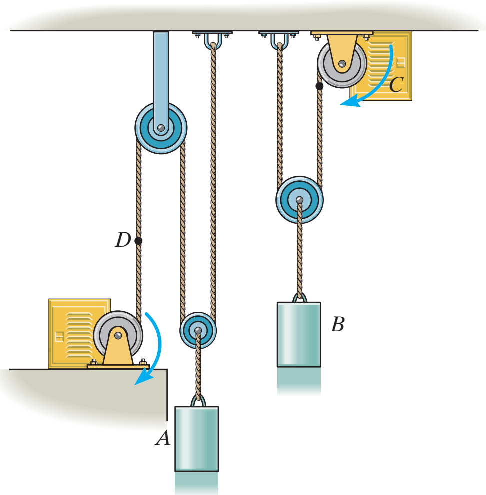
Prob. 12–195

*12–196. If the end of the cable at A is pulled down with a speed of 2 m/s, determine the speed at which block B rises.



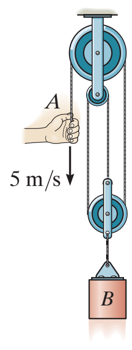
Prob. 12–196

12–197. The motor draws in the cable at C with a constant velocity of $v_C = 4\ \text{m/s}$. The motor draws in the cable at D with a constant acceleration of $a_D = 8\ \text{m/s}^2$. If $v_D = 0$ when $t = 0$, determine (a) the time needed for block A to rise 3 m, and (b) the relative velocity of block A with respect to block B when this occurs.



Prob. 12–197

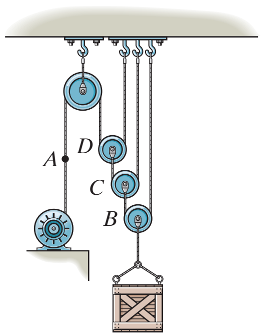
12–198. If the end of the cable at A is pulled down with a speed of 5 m/s, determine the speed at which block B rises.



Prob. 12–198

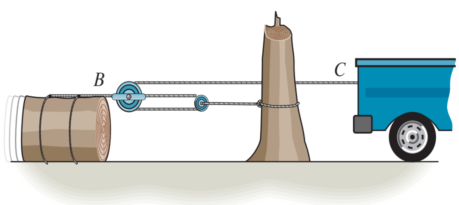
12–199. Determine the constant speed at which the cable at A must be drawn in by the motor in order to hoist the load 6 m in 1.5 s.

***12–200.** Starting from rest, the cable can be wound onto the drum of the motor at a rate of $v_A = (3t^2)$ m/s, where t is in seconds. Determine the time needed to lift the load 7 m.



Probs. 12–199/200

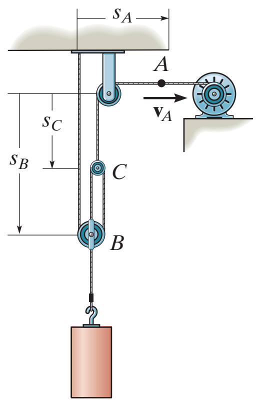
12–201. Determine the displacement of the log if the truck at C pulls the cable 1.2 m to the right.



Prob. 12–201

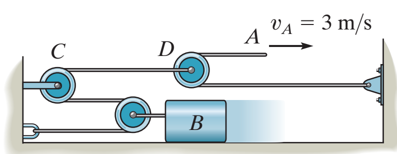
12–202. Determine the time needed for the load at B to attain a speed of 10 m/s, starting from rest, if the cable is drawn into the motor with an acceleration of 3 m/s^2.

12–203. The cable at A is being drawn toward the motor at $v_A = 8$ m/s. Determine the velocity of the block.



Probs. 12–202/203

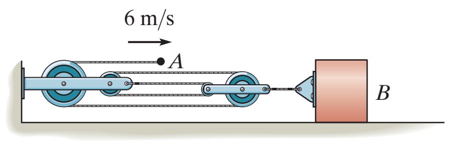
***12–204.** If the end A of the cable is moving at $v_A = 3$ m/s, determine the speed of block B.



Prob. 12–204

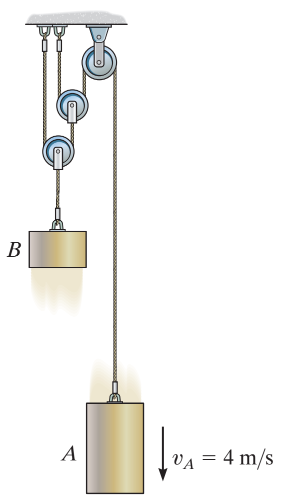
12

12–205. Determine the speed of the block at B.



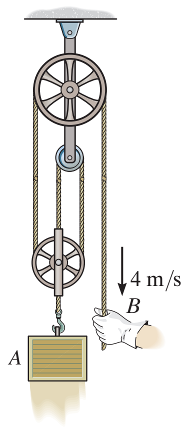
Prob. 12–205

12–206. Determine the speed of B if A is moving downward with a speed of $v_A = 4$ m/s at the instant shown.



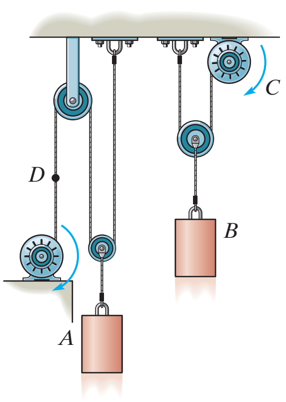
Prob. 12–206

12–207. Determine the speed of block A if the end of the rope is pulled down with a speed of 4 m/s.



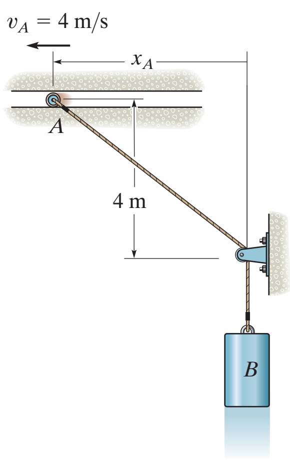
Prob. 12–207

***12–208.** The motor draws in the cable at C with a constant velocity of $v_C = 4$ m/s. The motor draws in the cable at D with a constant acceleration of $a_D = 8$ m/s^2. If $v_D = 0$ when $t = 0$, determine (a) the time needed for block A to rise 3 m, and (b) the relative velocity of block A with respect to block B when this occurs.



Prob. 12–208

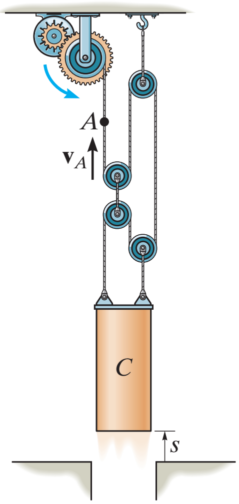
12–209. The roller at A is moving with a velocity of $v_A = 4$ m/s and has an acceleration of $a_A = 2$ m/s^2 when $x_A = 3$ m. Determine the velocity and acceleration of block B at this instant.



Prob. 12–209

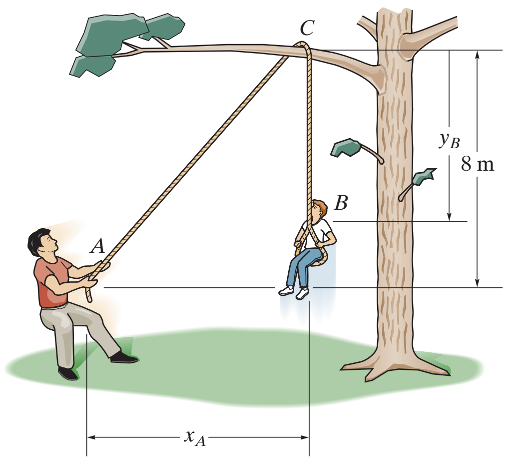
12–210. The cylinder C is being lifted using the cable and pulley system shown. If point A on the cable is being drawn toward the drum with a speed of 2 m/s, determine the speed of the cylinder.

12–211. The cylinder C can be lifted with a maximum acceleration of $a_C = 3$ m/s^2 without causing the cables to fail. Determine the speed at which point A is moving toward the drum when $s = 4$ m if the cylinder is lifted from rest in the shortest time possible.



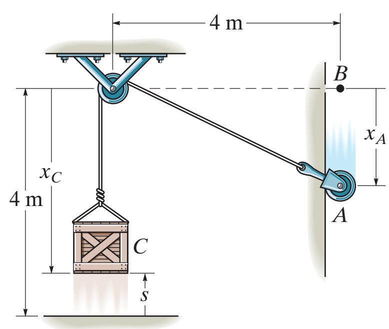
Probs. 12–210/211

***12–212.** The man pulls the boy up to the tree limb C by walking backward. If he starts from rest when $x_A = 0$ and moves backward with a constant acceleration $a_A = 0.2$ m/s^2, determine the speed of the boy at the instant $y_B = 4$ m. Neglect the size of the limb. When $x_A = 0$, $y_B = 8$ m, so that A and B are coincident, i.e., the rope is 16 m long.



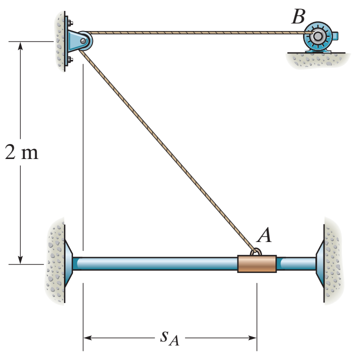
Prob. 12–212

12–213. The crate C is being lifted by moving the roller at A downward with a constant speed of $v_A = 2$ m/s along the guide. Determine the velocity and acceleration of the crate at the instant $s = 1$ m. When the roller is at B, the crate rests on the ground. Neglect the size of the pulley in the calculation. *Hint*: Relate the coordinates x_C and x_A using the problem geometry, then take the first and second time derivatives.



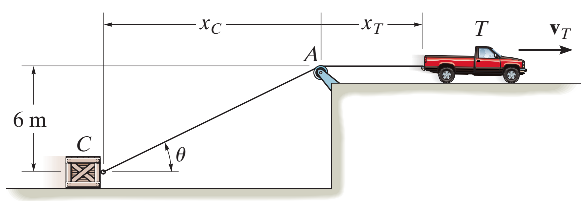
Prob. 12–213

12–214. The motor draws in the cord at B with an acceleration of $a_B = 2\ \text{m/s}^2$. When $s_A = 1.5$ m, $v_B = 6$ m/s. Determine the velocity and acceleration of the collar at this instant.



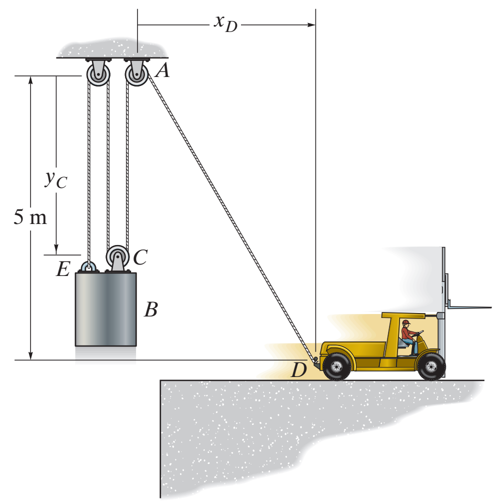
Prob. 12–214

12–215. If the truck travels at a constant speed of $v_T = 1.8$ m/s, determine the speed of the crate for any angle θ of the rope. The rope has a length of 30 m and passes over a pulley of negligible size at A. *Hint*: Relate the coordinates x_T and x_C to the length of the rope and take the time derivative. Then substitute the trigonometric relation between x_C and θ.



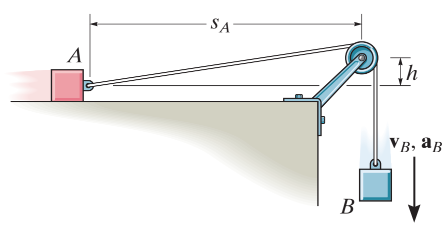
Prob. 12–215

***12–216.** The block B is suspended from a cable that is attached to the block at E, wraps around three pulleys, and is tied to the back of a truck. If the truck starts from rest when x_D is zero, and moves forward with a constant acceleration of $a_D = 0.5\ \text{m/s}^2$, determine the speed of the block at the instant $x_D = 2$ m. Neglect the size of the pulleys in the calculation. When $x_D = 0$, $y_C = 5$ m, so that points C and D are at the same elevation.



Prob. 12–216

12–217. If block B is moving down with a velocity v_B and has an acceleration a_B, determine the velocity and acceleration of block A in terms of the parameters shown.



Prob. 12–217

12.10 RELATIVE MOTION OF TWO PARTICLES USING TRANSLATING AXES

Throughout this chapter the absolute motion of a particle has been determined using a single fixed reference frame. There are many cases, however, where the path of motion for a particle is complicated, so that it may be easier to analyze the motion in parts by using two or more frames of reference. For example, the motion of a particle located at the tip of an airplane propeller, while the plane is in flight, is more easily described if one observes first the motion of the airplane from a fixed reference and then superimposes (vectorially) the circular motion of the particle measured from a reference attached to the airplane. In this section *translating frames of reference* will be considered for the analysis.

Position. Consider particles A and B, which move along the arbitrary paths shown in Fig. 12–42. The *absolute position* of each particle, $\mathbf{r}_A$ and $\mathbf{r}_B$, is measured from the common origin O of the *fixed* x, y, z reference frame. The origin of a translating frame of reference x', y', z' is attached to and moves with particle A. The position of B measured relative to A is denoted by the *relative-position vector* $\mathbf{r}_{B/A}$. The three vectors shown in Fig. 12–42 are related by vector addition.

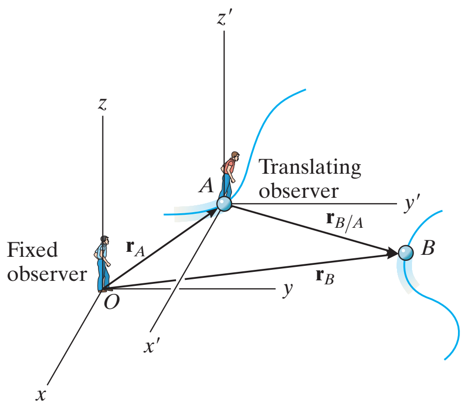


Fig. 12–42

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \tag{12–33}$$

Velocity. The equation that relates the velocities of the particles is determined by taking the time derivative of this equation; i.e.,

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \tag{12–34}$$

Here $\mathbf{v}_B = d\mathbf{r}_B/dt$ and $\mathbf{v}_A = d\mathbf{r}_A/dt$ refer to *absolute velocities*, since they are observed from the fixed frame; whereas the *relative velocity* $\mathbf{v}_{B/A} = d\mathbf{r}_{B/A}/dt$ is observed from the translating frame. It is important to note that since the x', y', z' axes translate, the *components* of $\mathbf{r}_{B/A}$ will *not* change direction and therefore the time derivative of these components will only have to account for the change in their magnitudes. Equation 12–34 therefore states that the velocity of B is equal to the velocity of A plus (vectorially) the velocity of "B with respect to A," as measured by the *translating observer* fixed to the x', y', z' reference frame.

Acceleration. The time derivative of the velocity equation yields a similar relationship between the *absolute* and *relative accelerations* of particles A and B.

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \tag{12–35}$$

Here $\mathbf{a}_{B/A}$ is the acceleration of B as seen by the observer located at A and translating with the x', y', z' reference frame.*

PROCEDURE FOR ANALYSIS

- Specify the particle (A) that is the origin for the translating x', y', z' axes. Usually this point has a *known* velocity or acceleration.
- Since vector addition forms a triangle, there can be at most *two unknowns*, represented by the magnitudes and/or directions of the vector quantities.
- These unknowns can be solved for either graphically, using trigonometry (law of sines, law of cosines), or, as we will show, by resolving each of the three vectors into rectangular or Cartesian components, thereby generating a set of scalar equations.

The pilots of these close-flying planes must be aware of their relative positions and velocities at all times in order to avoid a collision.

Refer to the companion website for Lecture Summary and Quiz videos.

* An easy way to remember the setup of these equations is to note the "cancellation" of the subscript A between the two terms, e.g., $\mathbf{a}_B = \mathbf{a}_{\not{A}} + \mathbf{a}_{B/\not{A}}$.

EXAMPLE 12.25

A train travels at a constant speed of 96 km/h and crosses over a road as shown in Fig. 12–43*a*. If the automobile *A* is traveling at 72 km/h along the road, determine the magnitude and direction of the velocity of the train relative to the automobile.

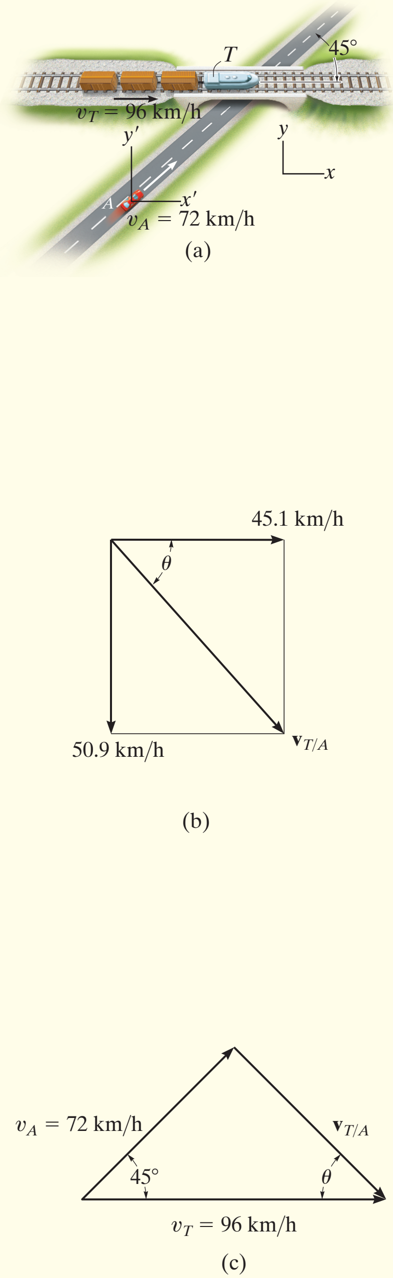


Fig. 12–43

SOLUTION I

Vector Analysis. The relative velocity $\mathbf{v}_{T/A}$ is measured from the translating x', y' axes attached to the automobile, Fig. 12–43*a*. It is determined from $\mathbf{v}_T = \mathbf{v}_A + \mathbf{v}_{T/A}$. Since $\mathbf{v}_T$ and $\mathbf{v}_A$ are known in *both* magnitude and direction, the unknowns become the x and y components of $\mathbf{v}_{T/A}$. Using the x, y axes in Fig. 12–43*a*, we have

$$\mathbf{v}_T = \mathbf{v}_A + \mathbf{v}_{T/A}$$

$$96\mathbf{i} = (72 \cos 45°\mathbf{i} + 72 \sin 45°\mathbf{j}) + \mathbf{v}_{T/A}$$

$$\mathbf{v}_{T/A} = \{45.1\mathbf{i} - 50.9\mathbf{j}\} \text{ km/h}$$

The magnitude of $\mathbf{v}_{T/A}$ is thus

$$v_{T/A} = \sqrt{(45.1)^2 + (-50.9)^2} = 68.0 \text{ km/h} \qquad \textit{Ans.}$$

From the direction of each component, Fig. 12–43*b*, the direction of $\mathbf{v}_{T/A}$ is

$$\tan\theta = \frac{(v_{T/A})_y}{(v_{T/A})_x} = \frac{50.9}{45.1}$$

$$\theta = 48.5° \;\measuredangle \qquad \textit{Ans.}$$

SOLUTION II

Scalar Analysis. The unknown components of $\mathbf{v}_{T/A}$ can also be determined by applying a scalar analysis. We will assume these components act in the *positive* x and y directions. Thus,

$$\mathbf{v}_T = \mathbf{v}_A + \mathbf{v}_{T/A}$$

$$\begin{bmatrix} 96 \text{ km/h} \\ \rightarrow \end{bmatrix} = \begin{bmatrix} 72 \text{ km/h} \\ \measuredangle 45° \end{bmatrix} + \begin{bmatrix} (v_{T/A})_x \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (v_{T/A})_y \\ \uparrow \end{bmatrix}$$

Resolving each vector into its x and y components yields

$$(\overset{+}{\rightarrow}) \qquad 96 = 72 \cos 45° + (v_{T/A})_x + 0$$

$$(+\uparrow) \qquad 0 = 72 \sin 45° + 0 + (v_{T/A})_y$$

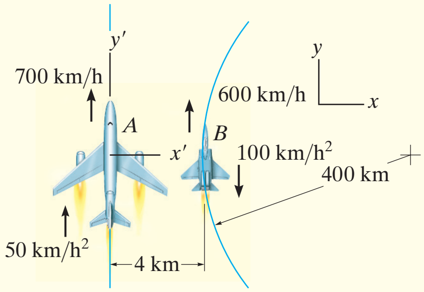
Solving, we obtain the previous results,

$$(v_{T/A})_x = 45.1 \text{ km/h} = 45.1 \text{ km/h} \rightarrow$$
$$(v_{T/A})_y = -50.9 \text{ km/h} = 50.9 \text{ km/h} \downarrow$$

12

EXAMPLE 12.26

Plane A in Fig. 12–44a is flying along a straight-line path, whereas plane B is flying along a circular path having a radius of curvature of $\rho_B = 400$ km. Determine the velocity and acceleration of B as measured by the pilot of A.



(a)

SOLUTION I

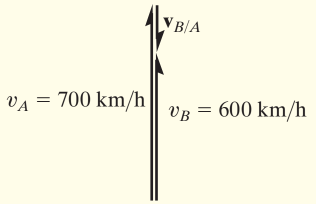
Velocity. Since the motion relative to plane A is to be determined, the *translating frame of reference* x', y' is attached to it, Fig. 12–44a. Applying the relative-velocity equation in scalar form, since the velocity vectors of both planes are parallel at the instant shown, we have

$$(+\uparrow) \qquad v_B = v_A + v_{B/A}$$

$$600 \text{ km/h} = 700 \text{ km/h} + v_{B/A}$$

$$v_{B/A} = -100 \text{ km/h} = 100 \text{ km/h} \downarrow \qquad \textit{Ans.}$$

The vector addition is shown in Fig. 12–44b.



(b)

Acceleration. Plane B has both tangential and normal components of acceleration since it is flying along a *curved path*. From Eq. 12–20, the magnitude of the normal component is

$$(a_B)_n = \frac{v_B^2}{\rho} = \frac{(600 \text{ km/h})^2}{400 \text{ km}} = 900 \text{ km/h}^2$$

Applying the relative-acceleration equation, we have

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

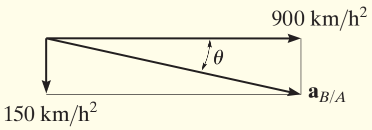
$$900\mathbf{i} - 100\mathbf{j} = 50\mathbf{j} + \mathbf{a}_{B/A}$$

Thus,

$$\mathbf{a}_{B/A} = \{900\mathbf{i} - 150\mathbf{j}\} \text{ km/h}^2$$

From Fig. 12–44c, the magnitude and direction of $\mathbf{a}_{B/A}$ are therefore

$$a_{B/A} = 912 \text{ km/h}^2 \quad \theta = \tan^{-1}\frac{150}{900} = 9.46^\circ \searrow \qquad \textit{Ans.}$$



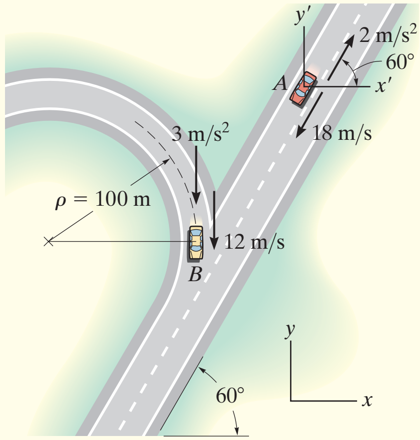
(c)

Fig. 12–44

NOTE: The solution to this problem was possible using a translating frame of reference, since the pilot in plane A is "translating." Observation of the motion of plane A with respect to the pilot of plane B, however, must be obtained using a *rotating* set of axes attached to plane B. (This assumes, of course, that the pilot of B is fixed in the rotating frame, so she does not turn her eyes to follow the motion of A.) The analysis for this case is more difficult and is given in Example 16.21.

EXAMPLE 12.27

At the instant shown in Fig. 12–45*a*, cars A and B are traveling with speeds of 18 m/s and 12 m/s, respectively. Also at this instant, A has a decrease in speed of 2 m/s^2, and B has an increase in speed of 3 m/s^2. Determine the velocity and acceleration of B with respect to A.



(a)

SOLUTION

Velocity. The fixed x, y axes are established at an arbitrary point on the ground, and the translating x', y' axes are attached to car A, Fig. 12–45*a*. Why? The relative velocity is determined from $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$. What are the two unknowns? Using a Cartesian vector analysis, we have

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

$$-12\mathbf{j} = (-18\cos 60^\circ\mathbf{i} - 18\sin 60^\circ\mathbf{j}) + \mathbf{v}_{B/A}$$

$$\mathbf{v}_{B/A} = \{9\mathbf{i} + 3.588\mathbf{j}\}\text{ m/s}$$

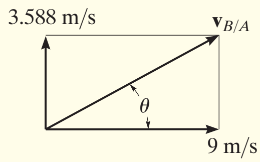
Thus,

$$v_{B/A} = \sqrt{(9)^2 + (3.588)^2} = 9.69\text{ m/s} \qquad \textit{Ans.}$$

Noting that $\mathbf{v}_{B/A}$ has $+\mathbf{i}$ and $+\mathbf{j}$ components, Fig. 12–45*b*, its direction is

$$\tan\theta = \frac{(v_{B/A})_y}{(v_{B/A})_x} = \frac{3.588}{9}$$

$$\theta = 21.7^\circ \qquad \textit{Ans.}$$



(b)

Acceleration. Car B has both tangential and normal components of acceleration. Why? The magnitude of the normal component is

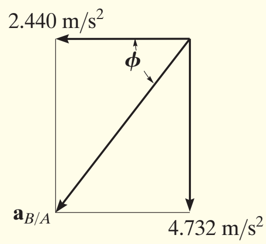
$$(a_B)_n = \frac{v_B^2}{\rho} = \frac{(12\text{ m/s})^2}{100\text{ m}} = 1.440\text{ m/s}^2$$

Applying the equation for relative acceleration yields

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

$$(-1.440\mathbf{i} - 3\mathbf{j}) = (2\cos 60^\circ\mathbf{i} + 2\sin 60^\circ\mathbf{j}) + \mathbf{a}_{B/A}$$

$$\mathbf{a}_{B/A} = \{-2.440\mathbf{i} - 4.732\mathbf{j}\}\text{ m/s}^2$$



(c)

Fig. 12–45

Here $\mathbf{a}_{B/A}$ has $-\mathbf{i}$ and $-\mathbf{j}$ components. Thus, from Fig. 12–45*c*,

$$a_{B/A} = \sqrt{(2.440)^2 + (4.732)^2} = 5.32\text{ m/s}^2 \qquad \textit{Ans.}$$

$$\tan\phi = \frac{(a_{B/A})_y}{(a_{B/A})_x} = \frac{4.732}{2.440}$$

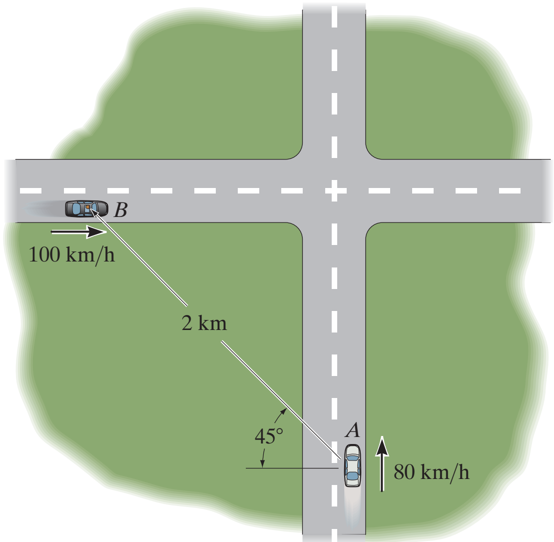
$$\phi = 62.7^\circ \qquad \textit{Ans.}$$

NOTE: Is it possible to obtain the relative acceleration of $\mathbf{a}_{A/B}$ using this method? Refer to the comment made at the end of Example 12–26.

Refer to the companion website for a self quiz of these Example problems.

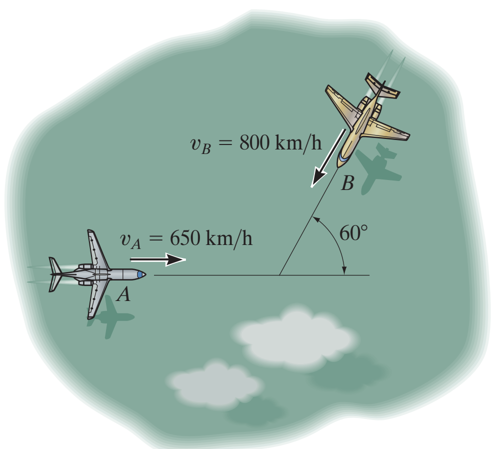
FUNDAMENTAL PROBLEMS

F12–45. Car A is traveling with a constant speed of 80 km/h due north, while car B is traveling with a constant speed of 100 km/h due east. Determine the velocity of car B relative to car A.



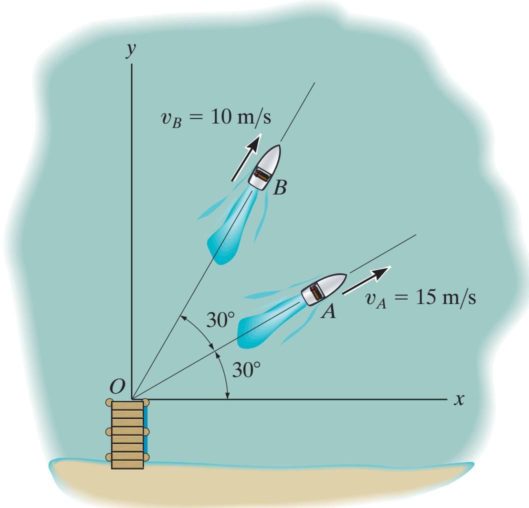
Prob. F12–45

F12–46. Two planes A and B are traveling with the constant velocities shown. Determine the magnitude and direction of the velocity of B relative to A.



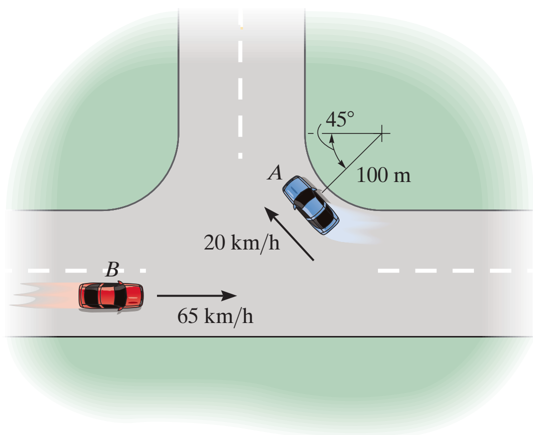
Prob. F12–46

F12–47. The boats A and B travel with constant speeds of $v_A = 15$ m/s and $v_B = 10$ m/s when they leave the pier at O at the same time. Determine the distance between them when $t = 4$ s.



Prob. F12–47

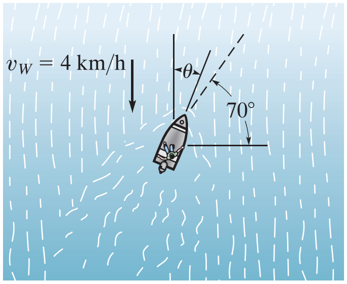
F12–48. Cars A and B are traveling at the speeds shown. If B is accelerating at 1200 km/h^2 while A maintains a constant speed, determine the velocity and acceleration of A with respect to B.



Prob. F12–48

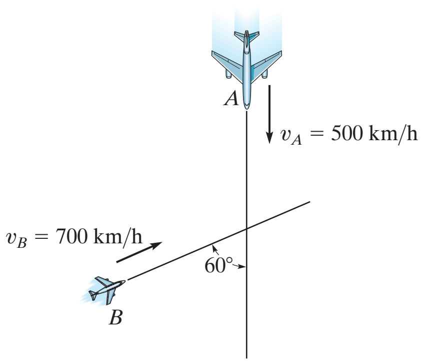
PROBLEMS

12–218. The boat can travel with a speed of 16 km/h in still water. The point of destination is located along the dashed line. If the water is moving at 4 km/h, determine the bearing angle θ at which the boat must travel to stay on course.



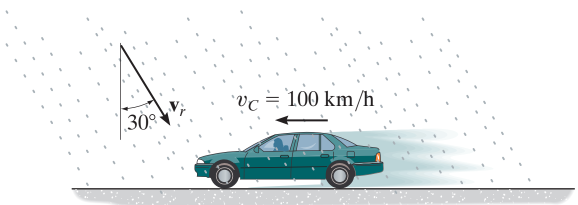
Prob. 12–218

12–219. Two planes, A and B, are flying at the same altitude. If their velocities are $v_A = 500$ km/h and $v_B = 700$ km/h such that the angle between their straight-line courses is $\theta = 60°$, determine the velocity of plane B with respect to plane A.



Prob. 12–219

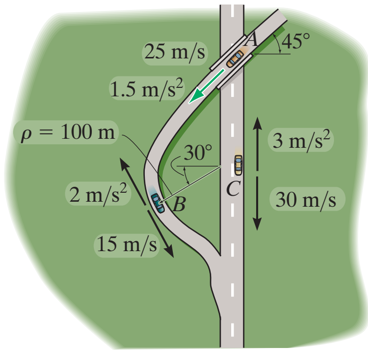
***12–220.** The car is traveling at a constant speed of 100 km/h. If the rain is falling at 6 m/s in the direction shown, determine the velocity of the rain as seen by the driver.



Prob. 12–220

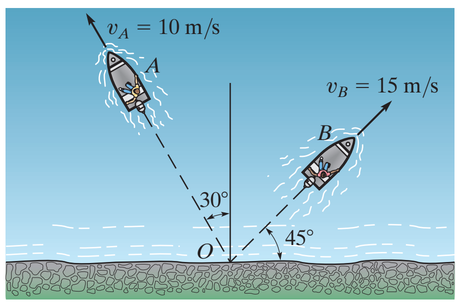
12–221. Car A travels along a straight road at a speed of 25 m/s while accelerating at 1.5 m/s^2. At this same instant car C is traveling along the straight road with a speed of 30 m/s while decelerating at 3 m/s^2. Determine the velocity and acceleration of car A relative to car C.

12–222. Car B is traveling along the curved road with a speed of 15 m/s while decreasing its speed at 2 m/s^2. At this same instant car C is traveling along the straight road with a speed of 30 m/s while decelerating at 3 m/s^2. Determine the velocity and acceleration of car B relative to car C.



Probs. 12–221/222

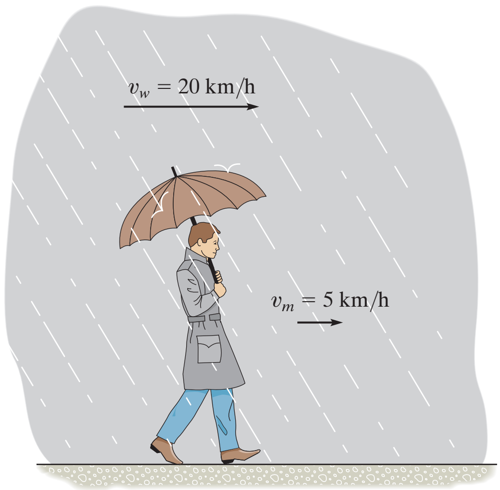
12–223. Two boats leave the shore at the same time and travel in the directions shown. If $v_A = 10$ m/s and $v_B = 15$ m/s, determine the velocity of boat A with respect to boat B. How long after leaving the shore will the boats be 600 m apart?



Prob. 12–223

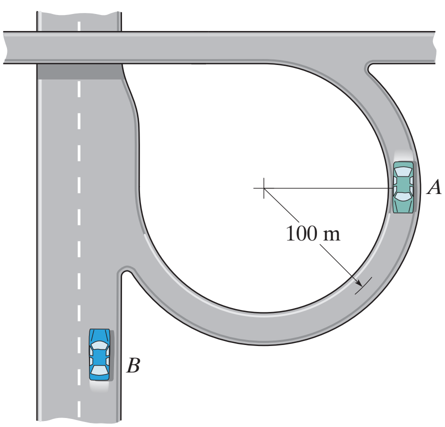
12

***12–224.** A man walks at 5 km/h in the direction of a 20 km/h wind. If raindrops fall vertically at 7 km/h in *still air*, determine direction in which the drops appear to fall with respect to the man.



Prob. 12–224

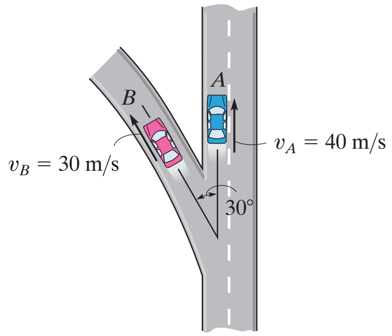
12–225. At the instant shown, car A has a speed of 20 km/h, which is being increased at the rate of 300 km/h^2 as the car enters the expressway. At the same instant, car B is decelerating at 250 km/h^2 while traveling forward at 100 km/h. Determine the velocity and acceleration of A with respect to B.



Prob. 12–225

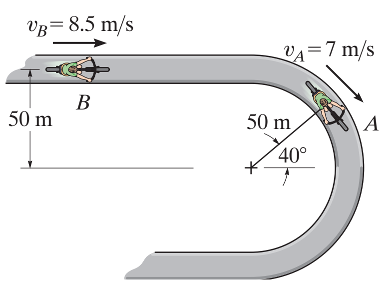
12–226. At the instant shown, cars A and B are traveling at velocities of 40 m/s and 30 m/s, respectively. If B is increasing its velocity by 2 m/s^2, while A maintains a constant velocity, determine the velocity and acceleration of B with respect to A. The radius of curvature at B is $\rho_B = 200$ m.

12–227. At the instant shown, cars A and B are traveling at velocities of 40 m/s and 30 m/s, respectively. If A is increasing its speed at 4 m/s^2, whereas the speed of B is decreasing at 3 m/s^2, determine the velocity and acceleration of B with respect to A. The radius of curvature at B is $\rho_B = 200$ m.



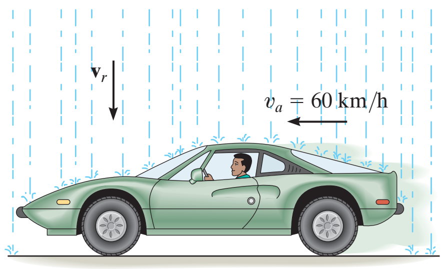
Probs. 12–226/227

***12–228.** At the instant shown, the bicyclist at A is traveling at 7 m/s around the curve on the race track while increasing the bicycle's speed at 0.5 m/s ^{2}. The bicyclist at B is traveling at 8.5 m/s along the straightaway and increasing the bicycle's speed at 0.7 m/s^2. Determine the relative velocity and relative acceleration of A with respect to B at this istant.



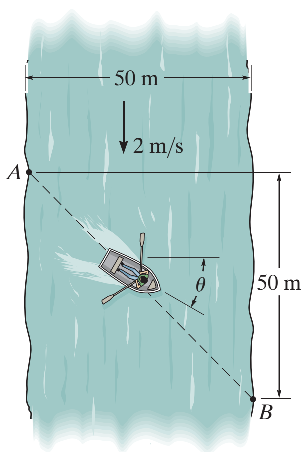
Prob. 12–228

12–229. A passenger in an automobile observes that raindrops make an angle of 30° with the horizontal as the auto travels forward with a speed of 60 km/h. Compute the terminal (constant) velocity $\mathbf{v}_r$ of the rain if it is assumed to fall vertically.



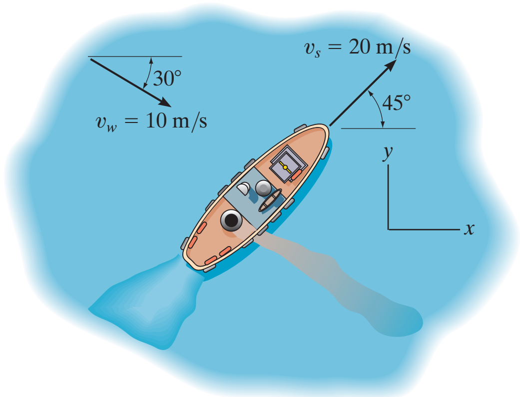
Prob. 12–229

12–230. A man can row a boat at 5 m/s in still water. He wishes to cross a 50-m-wide river to point B, 50 m downstream. If the river flows with a velocity of 2 m/s, determine the speed of the boat and the time needed to make the crossing.



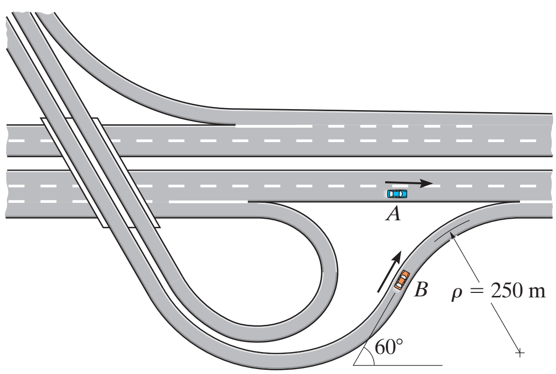
Prob. 12–230

12–231. The ship travels at a constant speed of $v_s = 20$ m/s and the wind is blowing at a speed of $v_w = 10$ m/s, as shown. Determine the magnitude and direction of the horizontal component of velocity of the smoke coming from the smoke stack as it appears to a passenger on the ship.



Prob. 12–231

***12–232.** At the instant shown car A is traveling with a velocity of 30 m/s and has an acceleration of 2 m/s^2 along the highway. At the same instant B is traveling on the trumpet interchange curve with a speed of 15 m/s, which is decreasing at 0.8 m/s^2. Determine the relative velocity and relative acceleration of B with respect to A at this instant.

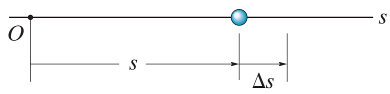


Prob. 12–232

CHAPTER REVIEW

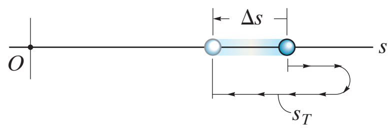
Rectilinear Kinematics

Rectilinear kinematics refers to motion along a straight line. A position coordinate s specifies the location of a particle on the line, and the displacement Δs is the change in its position.



The average velocity is a vector quantity, defined as the displacement divided by the time interval.

$$v_{avg} = \frac{\Delta s}{\Delta t}$$



The average speed is a scalar, and is the total distance traveled divided by the time of travel.

$$(v_{sp})_{avg} = \frac{s_T}{\Delta t}$$

The time, position, velocity, and acceleration are related by three differential equations.

$$a = \frac{dv}{dt}, \quad v = \frac{ds}{dt}, \quad a\,ds = v\,dv$$

If the acceleration is known to be *constant*, then the differential equations relating time, position, velocity, and acceleration can be integrated.

$$v = v_0 + a_c t$$

$$s = s_0 + v_0 t + \tfrac{1}{2} a_c t^2$$

$$v^2 = v_0^2 + 2a_c(s - s_0)$$

Graphical Solutions

If the motion is erratic, then it can be described by a graph. If one of these graphs is given, then the others can be established using the differential relations between a, v, s, and t.

$$a = \frac{dv}{dt},$$

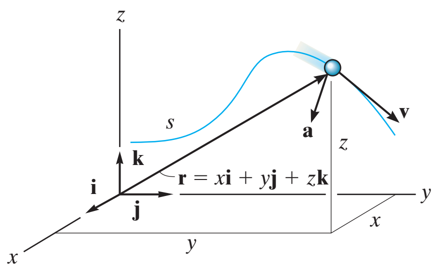
$$v = \frac{ds}{dt},$$

$$a\,ds = v\,dv$$

Curvilinear Motion, x, y, z

Curvilinear motion along the path can be resolved into rectilinear motion along the x, the y, and the z axes. The equation of the path is used to relate the motion along each axis.

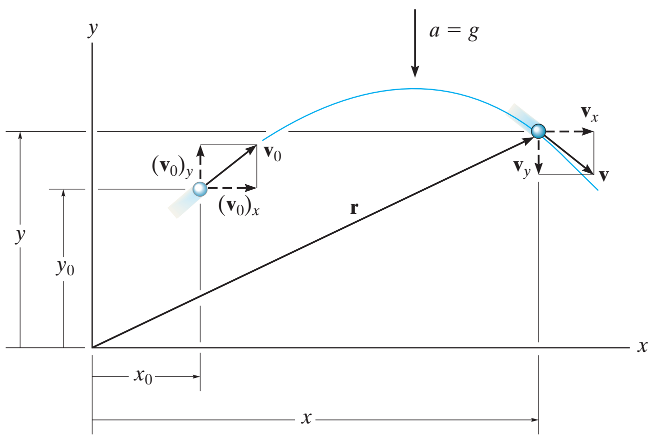
$$v_x = \dot{x} \quad a_x = \dot{v}_x$$
$$v_y = \dot{y} \quad a_y = \dot{v}_y$$
$$v_z = \dot{z} \quad a_z = \dot{v}_z$$



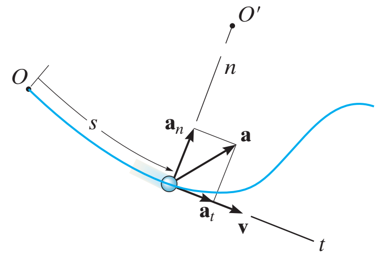
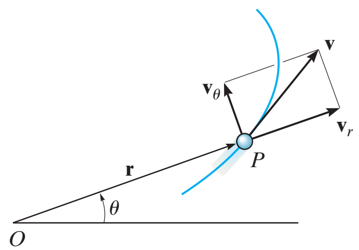
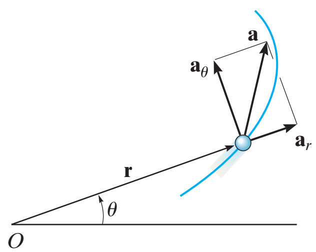
Projectile Motion

Free-flight motion of a projectile follows a parabolic path. It has a constant velocity in the horizontal direction, and a constant downward acceleration of $g = 9.81\ \text{m/s}^2$ in the vertical direction. Any two of the three equations for constant acceleration apply in the vertical direction, and in the horizontal direction only one equation applies.

$$(+\uparrow) \quad v_y = (v_0)_y + a_c t$$
$$(+\uparrow) \quad y = y_0 + (v_0)_y t + \tfrac{1}{2} a_c t^2$$
$$(+\uparrow) \quad v_y^2 = (v_0)_y^2 + 2a_c(y - y_0)$$
$$(\overset{+}{\rightarrow}) \quad x = x_0 + (v_0)_x t$$



12

Curvilinear Motion *n*, *t* If normal and tangential axes are used for the analysis, then **v** is always in the positive *t* direction. The acceleration has two components. The tangential component, $\mathbf{a}_t$, accounts for the change in the magnitude of the velocity; a slowing down is in the negative *t* direction, and a speeding up is in the positive *t* direction. The normal component $\mathbf{a}_n$ accounts for the change in the direction of the velocity. This component is always in the positive *n* direction.	$a_t = \dot{v}$ or $a_t\,ds = v\,dv$ $a_n = \dfrac{v^2}{\rho}$	
Curvilinear Motion *r*, *θ* If the path of motion is expressed in polar coordinates, then the velocity and acceleration components can be related to the time derivatives of *r* and θ. To determine the components, it is necessary to know r, $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, $\ddot{\theta}$ at the instant considered. If the path $r = f(\theta)$ is given, then the chain rule of calculus must be used to obtain these derivatives. (See Appendix C.)	$v_r = \dot{r}$ $v_\theta = r\dot{\theta}$ $a_r = \ddot{r} - r\dot{\theta}^2$ $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$	 Velocity  Acceleration

Absolute Dependent Motion of Two Particles

The dependent motion of blocks that are suspended from pulleys and cables can be related by the geometry of the system. This is done by first establishing position coordinates, measured from a fixed origin and directed along the line of motion of each block.

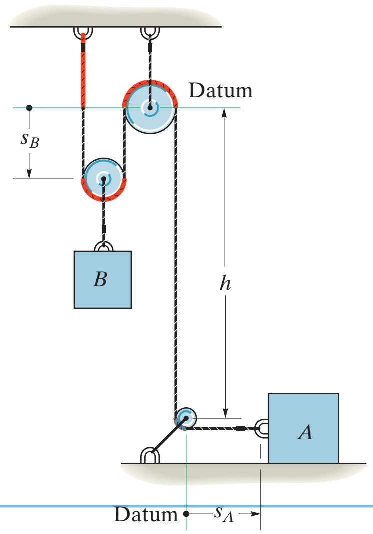
Using geometry and/or trigonometry, the coordinates are then related to the cable length in order to formulate a position coordinate equation.

The time derivatives of this equation give relationships between the velocities and the accelerations of the blocks.

$$2s_B + h + s_A = l$$

$$2v_B = -v_A$$

$$2a_B = -a_A$$



Relative-Motion Analysis Using Translating Axes

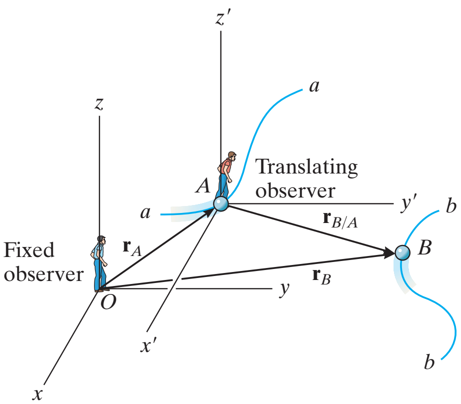
If two particles A and B undergo independent motions, then these motions can be related to their relative motion using a *translating set of axes* attached to one of the particles (A).

For planar motion, each vector equation produces two scalar component equations, one in the x, and the other in the y direction. For solution, the vectors can be expressed in Cartesian form, or the x and y scalar components can be written directly.

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$$

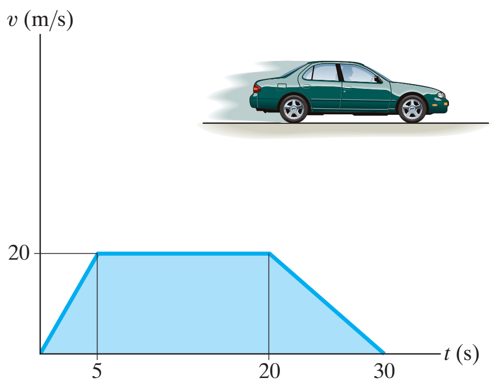
$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$



REVIEW PROBLEMS

R12–1. The v–t graph of a car while traveling along a road is shown. Determine the acceleration when $t = 2.5$ s, 10 s, and 25 s. Also if $s = 0$ when $t = 0$, find the position when $t = 5$ s, 20 s, and 30 s.



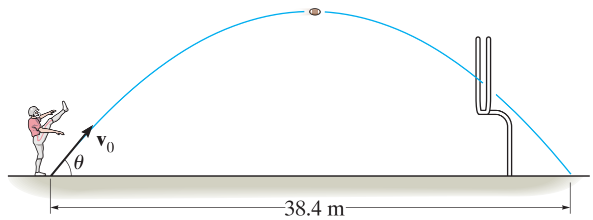
Prob. R12–1

R12–2. The position of a particle along a straight line is given by $s = (t^3 - 9t^2 + 15t)$ m, where t is in seconds. Determine its maximum acceleration and maximum velocity during the time interval $0 \le t \le 10$ s.

R12–3. If a particle has an initial velocity $v_0 = 12$ m/s to the right, and a constant acceleration of 2 m/s^2 to the left, determine the particle's displacement in 10 s. Originally $s_0 = 0$.

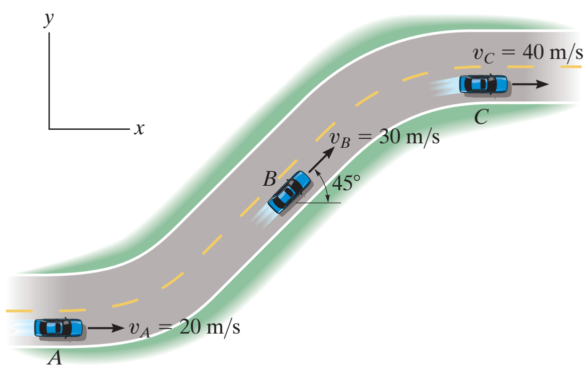
R12–4. A projectile, initially at the origin, moves along a straight-line path through a fluid medium such that its velocity is $v = 1800(1 - e^{-0.3t})$ mm/s where t is in seconds. Determine the displacement of the projectile during the first 3 s.

R12–5. From a videotape, it was observed that a player kicked a football 38.4 m during a measured time of 3.6 seconds. Determine the initial speed of the ball and the angle θ at which it was kicked.



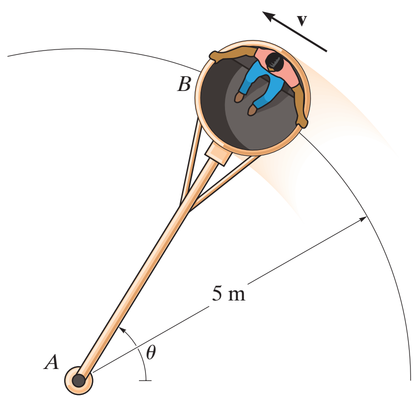
Prob. R12–5

R12–6. A car traveling along the straight portions of the road has the velocities indicated in the figure when it arrives at points A, B, and C. If it takes 3 s to go from A to B, and then 5 s to go from B to C, determine the average acceleration between points A and B and between points A and C.



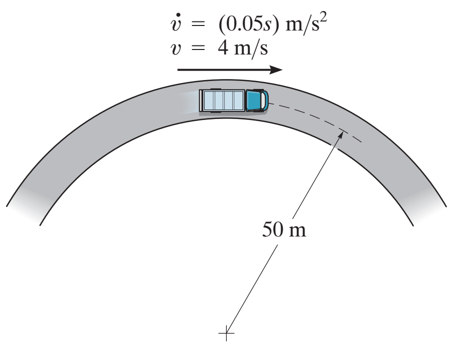
Prob. R12–6

R12–7. Car B turns such that its speed is increased by $(a_t)_B = (0.5e^t)\ \text{m/s}^2$, where t is in seconds. If the car starts from rest when $\theta = 0°$, determine the magnitudes of its velocity and acceleration when $t = 2$ s. Neglect the size of the car.



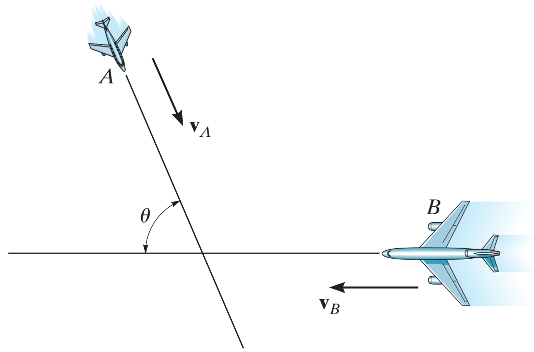
Prob. R12–7

R12–8. The truck travels in a circular path having a radius of 50 m at a speed of $v = 4$ m/s. For a short distance from $s = 0$, its speed is increased by $\dot{v} = (0.05s)\ \text{m/s}^2$, where s is in meters. Determine its speed and the magnitude of its acceleration when it has moved $s = 10$ m.



Prob. R12–8

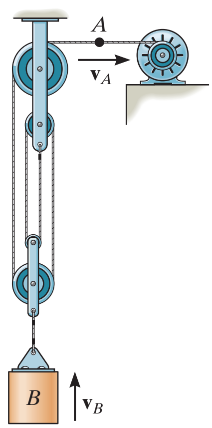
R12–9. Two planes, A and B, are flying at the same altitude. If their velocities are $v_A = 600$ km/h and $v_B = 500$ km/h such that the angle between their straight-line courses is $\theta = 75°$, determine the velocity of plane B with respect to plane A.



Prob. R12–9

R12–10. A particle is moving along a circular path of 2-m radius such that its position as a function of time is given by $\theta = (5t^2)$ rad, where t is in seconds. Determine the magnitude of the particle's acceleration when $\theta = 30°$. The particle starts from rest when $\theta = 0°$.

R12–11. Determine the time needed for the load at B to attain a speed of 8 m/s, starting from rest, if the cable is drawn into the motor with an acceleration of $0.2\ \text{m/s}^2$.



Prob. R12–11

CHAPTER 13

The design of conveyors for this bakery requires knowing the forces that act on these loaves of bread and their speed as they move along.

KINETICS OF A PARTICLE: FORCE AND ACCELERATION

Lecture Summary and Quiz, Example, and Problem-solving videos are available where this icon appears.

CHAPTER OBJECTIVES

- To state Newton's Second Law of Motion and to define mass and weight.
- To analyze the accelerated motion of a particle using the equation of motion with different coordinate systems.
- To investigate central-force motion and apply it to problems involving space mechanics.

13.1 NEWTON'S SECOND LAW OF MOTION

The jeep leans backward due to its inertia, which resists its forward acceleration.

Kinetics is a branch of dynamics that deals with the relationship between the change in motion of a body and the forces that cause this change. The basis for kinetics is Newton's second law, which states that when an *unbalanced force* acts on a particle, the particle will *accelerate* in the direction of the force with a magnitude that is proportional to the force.

We can determine the constant of proportionality by applying a known force **F** to a particle, and then measuring its acceleration **a**. Since the force and acceleration are directly proportional, then the constant of proportionality, m, is determined from the ratio $m = F/a$. This positive scalar m is called the ***mass*** of the particle. Since it is constant, m provides a quantitative measure of the resistance of the particle to a change in its velocity, that is, its ***inertia***.

For a particle of mass m, Newton's second law of motion may therefore be written in mathematical form as

$$\mathbf{F} = m\mathbf{a}$$

This equation is referred to as the ***equation of motion***, and it is one of the most important formulations in mechanics.* In 1905, however, Albert Einstein developed the theory of relativity and placed limitations on the use of Newton's second law, especially when the particle's speed approaches the speed of light (0.3 Gm/s). Developments of the theory of quantum mechanics by Erwin Schrödinger and others indicate further that conclusions drawn from using this equation are also invalid when particles approach the size of an atom and move close to one another. For the most part, however, these requirements regarding particle speed and size are not encountered in engineering problems, so their effects will not be considered in this book.

Newton's Law of Gravitational Attraction. Shortly after formulating his three laws of motion, Newton postulated a law governing the mutual attraction between any two particles. In mathematical form this law can be expressed as

$$F = G\frac{m_1 m_2}{r^2} \tag{13–1}$$

where

F = force of attraction between the two particles
G = universal constant of gravitation; according to experimental evidence $G = 66.73(10^{-12})\ \text{m}^3/(\text{kg}\cdot\text{s}^2)$
m_1, m_2 = mass of each of the two particles
r = distance between the centers of the two particles

* Since m is constant, we can also write the equation of motion in the form $\mathbf{F} = d(m\mathbf{v})/dt$, where $m\mathbf{v}$ is the particle's linear momentum. In other words, we can state that the unbalanced force acting on the particle is proportional to the time rate of change of the particle's linear momentum.

In the case of a particle located at or near the surface of the earth, the only gravitational force having any sizable magnitude is that between the earth and the particle. This force is termed the "weight" and, for our purpose, it will be the only gravitational force considered.

From Eq. 13–1, we can develop a general expression for finding the weight W of a particle having a mass $m_1 = m$. Let $m_2 = M_e$ be the mass of the earth and r the distance between the earth's center and the particle. Then, if we let $g = GM_e/r^2$, we have

$$W = mg$$

By comparison with $F = ma$, we see that g represents the acceleration due to gravity. For most engineering calculations we assume g is measured at a point on the surface of the earth at sea level, and at a latitude of 45°, which is considered the "standard location." Here $g = 9.81\ \text{m/s}^2$.

In the SI system the mass of the body is specified in kilograms, and so the weight must be calculated using the above equation, Fig. 13–1. Thus,

$$\boxed{W = mg\ (\text{N}) \quad (g = 9.81\ \text{m/s}^2)} \tag{13–2}$$

Thus, a 1-kg body weighs 9.81 N; a 2-kg body weighs 19.62 N; and so on.

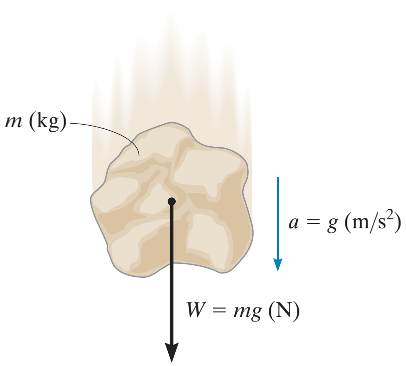
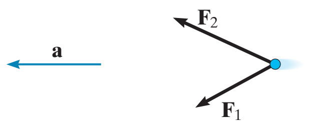


Fig. 13–1

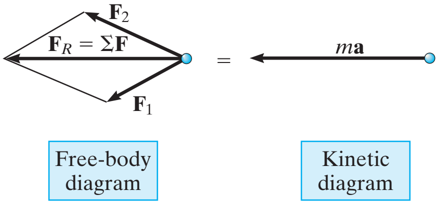
13.2 THE EQUATION OF MOTION

When more than one force acts on a particle, the resultant force is determined by a vector summation of all the forces; i.e., $\mathbf{F}_R = \Sigma\mathbf{F}$. For this more general case, the equation of motion may be written as

$$\Sigma\mathbf{F} = m\mathbf{a} \tag{13–3}$$



(a)



(b) (c)

Fig. 13–2

For example, consider the particle shown in Fig. 13–2*a*, which has a mass m and is subjected to the action of two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$. We can graphically account for the magnitude and direction of each force acting on the particle by drawing the particle's ***free-body diagram***, Fig. 13–2*b*. Since the *resultant* of these forces *produces* the vector $m\mathbf{a}$, its magnitude and direction can be represented graphically on the ***kinetic diagram***, shown in Fig. 13–2c. The equal sign written between the diagrams symbolizes the *graphical* equivalency between the free-body diagram and the kinetic diagram; i.e., $\Sigma\mathbf{F} = m\mathbf{a}$.*

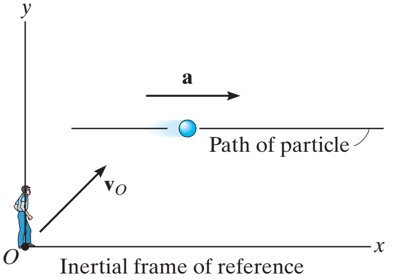


Fig. 13–3

Inertial Reference Frame. When applying the equation of motion, it is important that the acceleration of the particle be measured from a reference frame that is *either fixed or translates with a constant velocity*. In this way, the observer will not accelerate and measurements of the particle's acceleration will be the *same* from *any reference* of this type. Such a frame of reference is commonly known as a ***Newtonian*** or ***inertial reference frame***, Fig. 13–3.

When studying the motions of rockets and satellites, it is justifiable to consider the inertial reference frame as fixed to the stars, whereas dynamics problems concerned with motions on or near the surface of the earth may be solved by using an inertial frame which is assumed fixed to the earth. Even though the earth both rotates about its own axis and revolves about the sun, the accelerations created by these rotations are relatively small and so they can be neglected for most applications.

*The equation of motion can also be rewritten in the form $\Sigma\mathbf{F} - m\mathbf{a} = \mathbf{0}$. The vector $-m\mathbf{a}$ is referred to as the *inertia force vector*. If it is treated in the same way as a "force vector," then the state of "equilibrium" created is referred to as ***dynamic equilibrium***. This method of application, which will not be used in this book, is often referred to as the *D'Alembert principle*, named after the French mathematician Jean le Rond d'Alembert.

We are all familiar with the sensation one feels when sitting in a car that is subjected to a forward acceleration. Often people think this is caused by a "force" which acts on them and tends to push them back in their seats; however, this is not the case. Instead, this sensation occurs due to their inertia or the resistance of their mass to a change in velocity.

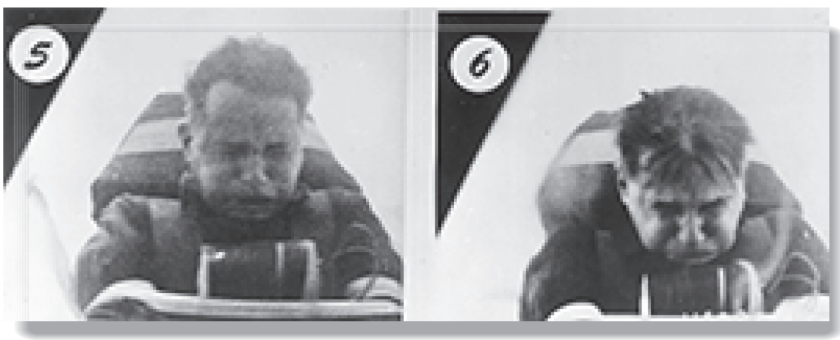
Consider the passenger who is strapped to the seat of a rocket sled. Provided the sled is at rest or is moving with constant velocity, then no force is exerted on his back as shown on his free-body diagram.

Keystone/Hulton Archive/
Getty Images

When the thrust of the rocket engine causes the sled to accelerate, then the seat upon which he is sitting exerts a force **F** on him which pushes him forward with the sled. In the photo, notice that the inertia of his head resists this change in motion (acceleration), and so his head moves back against the seat and his face, which is nonrigid, tends to distort backward.

Keystone/Hulton Archive/
Getty Images

Upon deceleration the force of the seatbelt $\mathbf{F}'$ tends to pull his body to a stop, but his head leaves contact with the back of the seat and his face distorts forward, again due to his inertia or tendency to continue to move forward. No force is pulling him forward, although this is the sensation he receives.



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13.3 EQUATION OF MOTION FOR A SYSTEM OF PARTICLES

The equation of motion will now be extended to include a system of particles enclosed within a region in space, as shown in Fig. 13–4*a*. At the instant considered, the arbitrary *i*-th particle, having a mass m_i, is subjected to both a resultant internal force, $\mathbf{f}_i$, and a resultant external force, $\mathbf{F}_i$. The *internal force* is the resultant of all the forces the other particles exert on the *i*th particle. The *resultant external force* represents, for example, the effect of gravitational, electrical, magnetic, or contact forces between the *i*th particle and adjacent bodies or particles not included within the system.

The free-body and kinetic diagrams for the *i*th particle are shown in Fig. 13–4*b*. Applying the equation of motion,

$$\Sigma \mathbf{F} = m\mathbf{a}; \qquad \mathbf{F}_i + \mathbf{f}_i = m_i \mathbf{a}_i$$

When the equation of motion is applied to each of the other particles of the system, similar equations will result, and if all these equations are added together, we obtain

$$\Sigma \mathbf{F}_i + \Sigma \mathbf{f}_i = \Sigma m_i \mathbf{a}_i$$

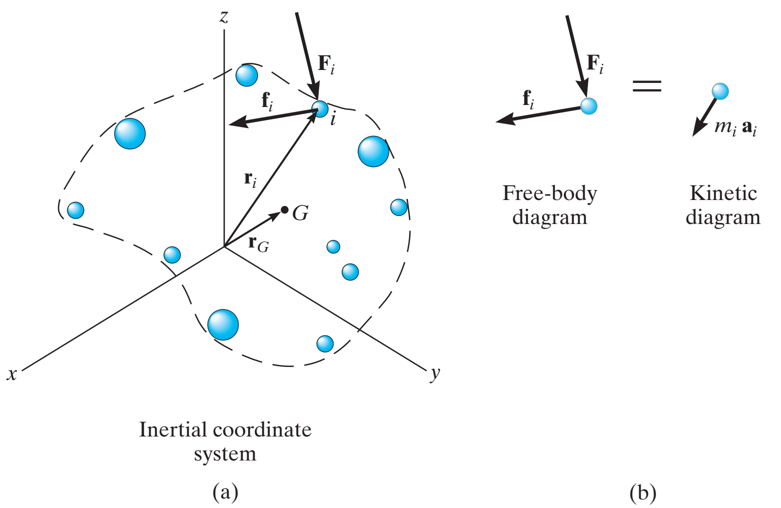

Fig. 13–4

The summation of the internal forces, if carried out, will equal zero, since internal forces between any two particles occur in equal but opposite collinear pairs. Therefore,

$$\Sigma \mathbf{F}_i = \Sigma m_i \mathbf{a}_i \qquad (13\text{–}4)$$

If $\mathbf{r}_G$ is the position vector which locates the center of mass G of the particles, Fig. 13–4a, then by definition of the center of mass, $m\mathbf{r}_G = \Sigma m_i \mathbf{r}_i$, where $m = \Sigma m_i$ is the total mass of all the particles. Differentiating this equation twice with respect to time, assuming that no mass is entering or leaving the system, we get $m\mathbf{a}_G = \Sigma m_i \mathbf{a}_i$. Substituting this result into Eq. 13–4, we obtain

$$\Sigma \mathbf{F} = m\mathbf{a}_G \qquad (13\text{–}5)$$

Considering a body composed of this system of particles, we can therefore state that the sum of the external forces acting on the body is equal to the mass of the body times the acceleration of its center of mass G.

IMPORTANT POINTS

- The equation of motion is based on experimental evidence and is valid only when applied within an inertial frame of reference.
- The equation of motion states that the *unbalanced force* on a particle causes it to *accelerate*.
- An inertial frame of reference does not rotate, rather its axes either translate with constant velocity or are at rest.
- Mass is a property of matter that provides a quantitative measure of its resistance to a change in velocity. It is an absolute quantity and so it does not change from one location to another.
- Weight is a force that is caused by the earth's gravitation. It is not absolute; rather it depends on the altitude of the mass from the earth's surface.

13.4 EQUATIONS OF MOTION: RECTANGULAR COORDINATES

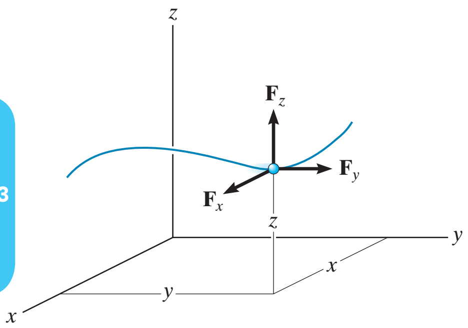

Fig. 13–5

When a particle moves relative to an inertial x, y, z frame of reference, the forces acting on the particle, as well as its acceleration, can be expressed in terms of their $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components, Fig. 13–5. Applying the equation of motion, we have

$$\Sigma \mathbf{F} = m\mathbf{a}; \quad \Sigma F_x\mathbf{i} + \Sigma F_y\mathbf{j} + \Sigma F_z\mathbf{k} = m(a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})$$

To satisfy this equation, the respective $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components on the left side must equal the corresponding components on the right side. As a result, we may write the following three scalar equations:

$$\begin{aligned} \Sigma F_x &= ma_x \\ \Sigma F_y &= ma_y \\ \Sigma F_z &= ma_z \end{aligned} \tag{13–6}$$

If the particle is constrained to move only in the x–y plane, then only the first two of these equations are used to specify the motion.

PROCEDURE FOR ANALYSIS

Application of the equation of motion requires the following steps.

Free-Body Diagram.

- Select the inertial coordinate system. Choose rectangular or x, y, z coordinates if the particle has *rectilinear motion*.
- Draw the particle's free-body diagram in order to account for *all the forces* ($\Sigma \mathbf{F}$) which act on the particle.
- The direction and sense of the particle's acceleration $\mathbf{a}$ should also be established. If the sense is unknown, for mathematical convenience assume its components act along the *positive* coordinate axes.

- The acceleration can be represented as the $m\mathbf{a}$ vector on the kinetic diagram.
- Identify the unknowns in the problem.

Equations of Motion.

- If the forces can be resolved directly from the free-body diagram, apply the equations of motion in their scalar component form.
- If the geometry of the problem appears complicated, which often occurs in three dimensions, Cartesian vector analysis can be used for the solution.
- *Friction.* If the particle slides on a rough surface, then use the *friction equation*, which relates the friction and normal forces with the coefficient of kinetic friction, i.e., $F_f = \mu_k N$.
- *Spring.* If the particle is connected to an *elastic spring* having negligible mass, the spring force F_s can be related to the stretch or compression s of the spring by the equation $F_s = ks$. Here k is the spring's stiffness measured as a force per unit length.

Kinematics.

- Once the particle's acceleration is known, its velocity or position can be found, using a kinematic equation.
- If *acceleration* is a function of time, integrate $a = dv/dt$ and $v = ds/dt$ to obtain $v = v(t)$ and $s = s(t)$.
- If *acceleration* is a function of displacement, integrate $a\,ds = v\,dv$ to obtain $v = f(s)$.
- If *acceleration is constant*, use $v = v_0 + a_c t$, $s = s_0 + v_0 t + \frac{1}{2}a_c t^2$, $v^2 = v_0^2 + 2a_c(s - s_0)$ to determine v and s.
- If the problem involves the dependent motion of several particles, use the method outlined in Sec. 12.9 to relate their accelerations.

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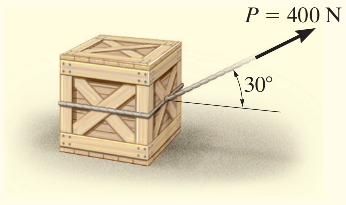
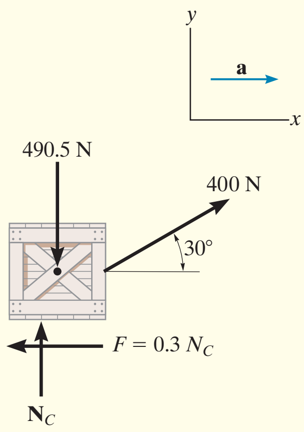
Refer to the companion website for Lecture Summary and Quiz videos.

EXAMPLE 13.1



(a)



(b)

The 50-kg crate shown in Fig. 13–6*a* rests on a horizontal surface for which the coefficient of kinetic friction is $\mu_k = 0.3$. If the crate is subjected to the 400-N towing force as shown, determine the velocity of the crate in 3 s starting from rest.

SOLUTION

Free-Body Diagram. The weight of the crate is $W = mg = 50\text{ kg}\,(9.81\text{ m/s}^2) = 490.5\text{ N}$. As shown in Fig. 13–6*b*, the frictional force has a magnitude $F = \mu_k N_C$ and acts to the left, since it opposes the motion of the crate. The acceleration **a** is assumed to act horizontally, in the positive *x* direction. There are two unknowns, namely, N_C and a.

Equations of Motion. From the data shown on the free-body diagram, we have

$$\xrightarrow{+} \Sigma F_x = ma_x; \quad 400\cos 30^\circ - 0.3N_C = 50a \qquad (1)$$

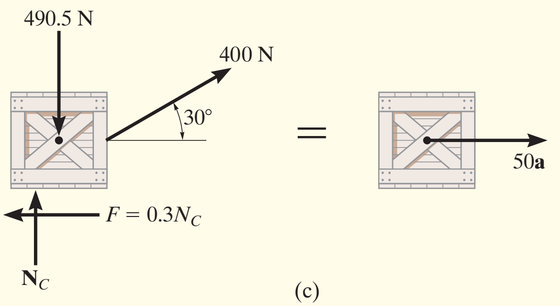
$$+\uparrow \Sigma F_y = ma_y; \quad N_C - 490.5 + 400\sin 30^\circ = 0 \qquad (2)$$

Solving Eq. 2 for N_C, substituting the result into Eq. 1, and solving for *a* yields

$$N_C = 290.5\text{ N}$$
$$a = 5.185\text{ m/s}^2$$

Kinematics. The acceleration is *constant*, since the applied force **P** is constant. Since the initial velocity is zero, the velocity of the crate in 3 s is

$$(\xrightarrow{+}) \qquad v = v_0 + a_c t$$
$$= 0 + 5.185(3)$$
$$= 15.6\text{ m/s} \rightarrow \qquad \textit{Ans.}$$



(c)

Fig. 13–6

NOTE: An alternative procedure would be to draw the crate's free-body *and* kinetic diagrams, Fig. 13–6*c*, prior to applying the equations of motion.

EXAMPLE 13.2

A 10-kg projectile is fired vertically upward from the ground, with an initial velocity of 50 m/s, Fig. 13–7*a*. Determine the maximum height to which it will travel if (a) atmospheric resistance is neglected; and (b) atmospheric resistance is measured as $F_D = (0.01v^2)$ N, where v is the speed of the projectile at any instant, measured in m/s.

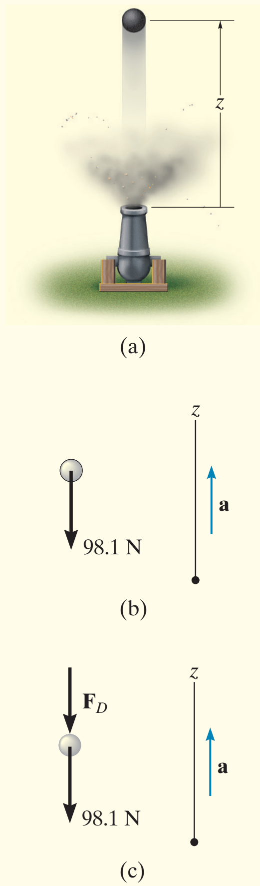


Fig. 13–7

SOLUTION

Part (a) Free-Body Diagram. As shown in Fig. 13–7*b*, the projectile's weight is $W = mg = 10(9.81) = 98.1$ N. We will assume the unknown acceleration **a** acts upward in the *positive* z direction.

Equation of Motion.

$$+\uparrow \Sigma F_z = ma_z; \qquad -98.1 = 10a, \qquad a = -9.81 \text{ m/s}^2$$

The result indicates that the projectile, like every object having free-flight motion near the earth's surface, is subjected to a *constant* downward acceleration of 9.81 m/s^2.

Kinematics. Initially, $z_0 = 0$ and $v_0 = 50$ m/s, and at the maximum height $z = h$, $v = 0$. Since the acceleration is *constant*, then

$$(+\uparrow) \qquad v^2 = v_0^2 + 2a_c(z - z_0)$$

$$0 = (50)^2 + 2(-9.81)(h - 0)$$

$$h = 127 \text{ m} \qquad \textit{Ans.}$$

Part (b) Free-Body Diagram. Since the force $F_D = (0.01v^2)$ N tends to retard the upward motion of the projectile, it acts downward as shown on the free-body diagram, Fig. 13–7*c*.

Equation of Motion.

$$+\uparrow \Sigma F_z = ma_z; \quad -0.01v^2 - 98.1 = 10a, \quad a = -(0.001v^2 + 9.81)$$

Kinematics. Here the acceleration is *not constant* since F_D depends on the velocity. Since the acceleration is a function of velocity, then the position can be determined from

$$(+\uparrow)\ a\,dz = v\,dv; \qquad -(0.001v^2 + 9.81)\,dz = v\,dv$$

Separating the variables and integrating, realizing that initially $z_0 = 0$, $v_0 = 50$ m/s (positive upward), and at $z = h$, $v = 0$, we have

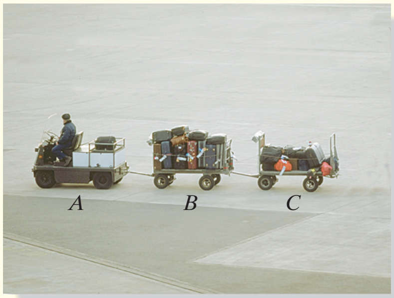
$$\int_0^h dz = -\int_{50\text{ m/s}}^{0} \frac{v\,dv}{0.001v^2 + 9.81}$$

$$h = -500 \ln(v^2 + 9810)\Big|_{50\text{ m/s}}^{0}$$

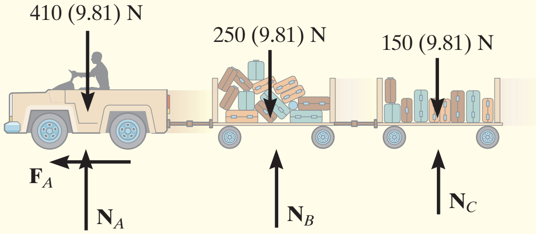
$$h = 114 \text{ m} \qquad \textit{Ans.}$$

NOTE: The answer indicates a lower elevation than that obtained in part (a) due to atmospheric resistance or drag.

EXAMPLE 13.3



The baggage truck A shown in the photo has a mass of 410 kg and tows a 250-kg cart B and a 150-kg cart C. For a short time the driving frictional force developed at the wheels of the truck is $F_A = (180t)$ N, where t is in seconds. If the truck starts from rest, determine its speed in 2 seconds. Also, what is the horizontal force acting on the coupling between the truck and cart B at this instant? Neglect the size of the truck and carts.



(a)

SOLUTION

Free-Body Diagram. As shown in Fig. 13–8*a*, it is the frictional driving force that gives both the truck and carts an acceleration. Here we have considered all three vehicles as a single system.

Equation of Motion. Only motion in the horizontal direction has to be considered.

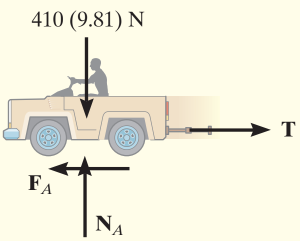
$$\overset{+}{\leftarrow} \Sigma F_x = ma_x; \qquad 180t = (410 + 250 + 150)a$$

$$a = 0.2222t$$

Kinematics Since the acceleration is a function of time, the velocity of the truck is obtained using $a = dv/dt$ with the initial condition that $v_0 = 0$ at $t = 0$. We have

$$\int_0^v dv = \int_0^{2\text{ s}} 0.2222t\,dt; \qquad v = 0.1111t^2 \Big|_0^{2\text{ s}} = 0.444 \text{ m/s} \qquad \textit{Ans.}$$

Free-Body Diagram. In order to determine the force between the truck and cart B, we will consider a free-body diagram of the truck so that we can "expose" the coupling force $\mathbf{T}$ as external to the free-body diagram, Fig. 13–8*b*.



(b)

Fig. 13–8

Equation of Motion. When $t = 2$ s, then

$$\overset{+}{\leftarrow} \Sigma F_x = ma_x; \qquad 180(2) - T = (410)[0.2222(2)]$$

$$T = 178 \text{ N} \qquad \textit{Ans.}$$

NOTE: Try and obtain this same result by considering a free-body diagram of carts B and C as a single system.

EXAMPLE 13.4

A smooth 2-kg collar, shown in Fig. 13–9*a*, is attached to a spring having a stiffness $k = 3\text{ N/m}$ and an unstretched length of 0.75 m. If the collar is released from rest at A, determine its acceleration and the normal force of the rod on the collar at the instant $y = 1\text{ m}$.

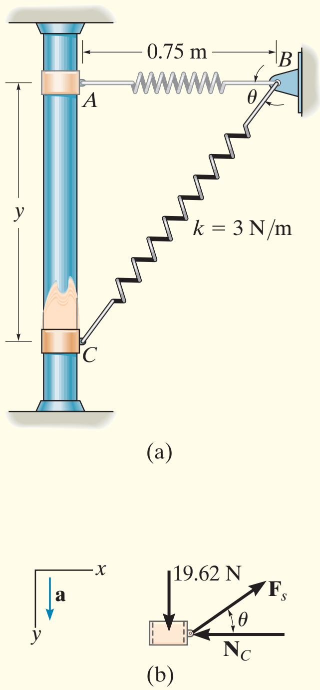


Fig. 13–9

SOLUTION

Free-Body Diagram. The free-body diagram of the collar when it is located at the arbitrary position y is shown in Fig. 13–9*b*. Furthermore, the collar is *assumed* to be accelerating so that "**a**" acts downward in the *positive* y direction. There are four unknowns, namely, N_C, F_s, a, and θ.

Equations of Motion.

$$\xrightarrow{+} \Sigma F_x = ma_x; \qquad -N_C + F_s \cos\theta = 0 \qquad (1)$$

$$+\downarrow \Sigma F_y = ma_y; \qquad 19.62 - F_s \sin\theta = 2a \qquad (2)$$

From Eq. 2 it is seen that the acceleration depends on the magnitude and direction of the spring force. Solution for N_C and a is possible once F_s and θ are known.

The magnitude of the spring force is a function of the stretch s of the spring; i.e., $F_s = ks$. Here the unstretched length is $AB = 0.75\text{ m}$, Fig. 13–9*a*; therefore, $s = CB - AB = \sqrt{y^2 + (0.75)^2} - 0.75$. Since $k = 3\text{ N/m}$, then

$$F_s = ks = 3\left(\sqrt{y^2 + (0.75)^2} - 0.75\right) \qquad (3)$$

From Fig. 13–9*a*, the angle θ is related to y by trigonometry.

$$\tan\theta = \frac{y}{0.75}$$

Substituting $y = 1\text{ m}$ into Eqs. 3 and 4 yields $F_s = 1.50\text{ N}$ and $\theta = 53.1°$. Substituting these results into Eqs. 1 and 2, we obtain

$$N_C = 0.900\text{ N} \qquad \textit{Ans.}$$

$$a = 9.21\text{ m/s}^2 \downarrow \qquad \textit{Ans.}$$

NOTE: This is not a case of constant acceleration, since the spring force changes both its magnitude and direction as the collar moves downward.

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EXAMPLE 13.5

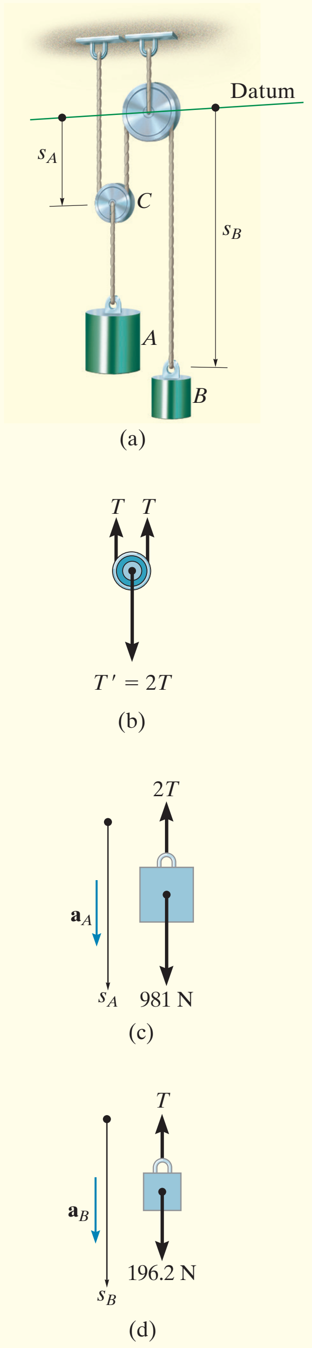


Fig. 13–10

The 100-kg block A shown in Fig. 13–10*a* is released from rest. If the masses of the pulleys and the cord are neglected, determine the acceleration of the 20-kg block B.

SOLUTION

Free-Body Diagrams. Since the mass of pulley C is *neglected*, ($m_C = 0$) then $T' = 2T$ as shown in Fig. 13–10*b*. The free-body diagrams for blocks A and B are shown in Figs. 13–10*c* and *d*, respectively. If A were to remain stationary, then $T = 490.5$ N; whereas for B to remain stationary, $T = 196.2$ N. Hence A will move down while B moves up. Although this is the case, we will *assume* both blocks accelerate downward, in the direction of $+s_A$ and $+s_B$. The three unknowns are T, a_A, and a_B.

Equations of Motion. Block A,

$$+\downarrow \Sigma F_y = ma_y; \qquad 981 - 2T = 100a_A \qquad (1)$$

Block B,

$$+\downarrow \Sigma F_y = ma_y; \qquad 196.2 - T = 20a_B \qquad (2)$$

Kinematics. The necessary third equation is obtained by relating a_A to a_B using a dependent motion analysis, discussed in Sec. 12.9. The coordinates s_A and s_B in Fig. 13–10*a* measure the positions of A and B from the fixed datum. It is seen that

$$2s_A + s_B = l$$

where l is constant and represents the total vertical length of cord. Differentiating this expression twice with respect to time yields

$$2a_A = -a_B \qquad (3)$$

Equations 1 to 3 have all been written so that the *positive direction was always assumed downward*. This is very important since we are seeking a simultaneous solution of equations. The results are

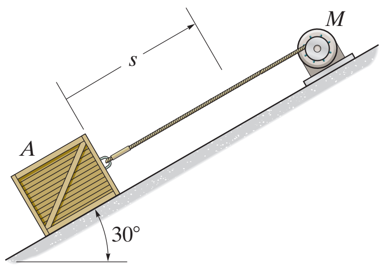
$$T = 327.0 \text{ N}$$
$$a_A = 3.27 \text{ m/s}^2$$
$$a_B = -6.54 \text{ m/s}^2 \qquad \textit{Ans.}$$

Hence when block A accelerates *downward*, block B accelerates *upward* as expected.

Refer to the companion website for a self quiz of these Example problems.

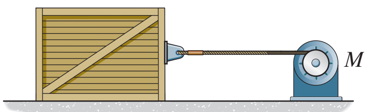
FUNDAMENTAL PROBLEMS

F13–1. The motor winds in the cable with a constant acceleration, such that the 20-kg crate moves a distance $s = 6$ m in 3 s, starting from rest. Determine the tension developed in the cable. The coefficient of kinetic friction between the crate and the plane is $\mu_k = 0.3$.



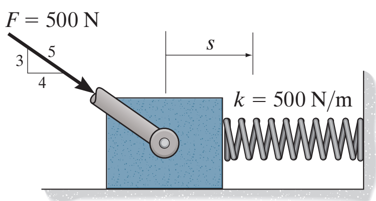
Prob. F13–1

F13–2. If motor M exerts a force of $F = (10t^2 + 100)$ N on the cable, where t is in seconds, determine the velocity of the 25-kg crate when $t = 4$ s. The coefficients of static and kinetic friction between the crate and the plane are $\mu_s = 0.3$ and $\mu_k = 0.25$, respectively. The crate is initially at rest.



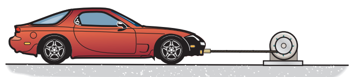
Prob. F13–2

F13–3. A spring of stiffness $k = 500$ N/m is mounted against the 10-kg block. If the block is subjected to the force of $F = 500$ N, determine its velocity at $s = 0.5$ m. When $s = 0$, the block is at rest and the spring is uncompressed. The contact surface is smooth.



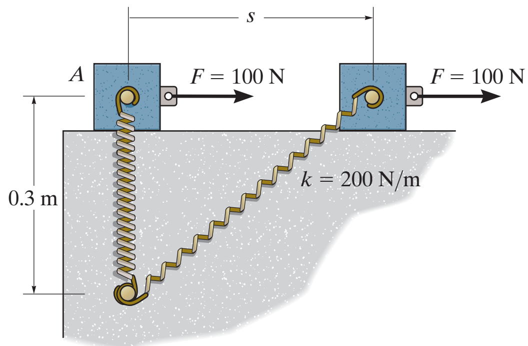
Prob. F13–3

F13–4. The 2-Mg car is being towed by a winch. If the winch exerts a force of $T = 100(s + 1)$ N on the cable, where s is the displacement of the car in meters, determine the speed of the car when $s = 10$ m, starting from rest. Neglect rolling resistance of the car.



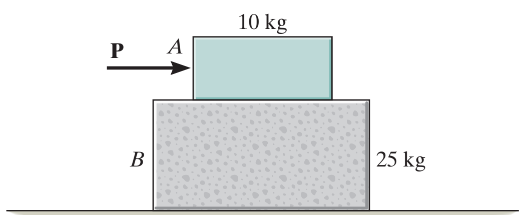
Prob. F13–4

F13–5. The spring has a stiffness $k = 200$ N/m and is unstretched when the 25-kg block is at A. Determine the acceleration of the block when $s = 0.4$ m. The contact surface between the block and the plane is smooth.



Prob. F13–5

F13–6. Block B rests upon a smooth surface. If the coefficients of static and kinetic friction between A and B are $\mu_s = 0.4$ and $\mu_k = 0.3$, respectively, determine the acceleration of each block if $P = 30$ N.



Prob. F13–6

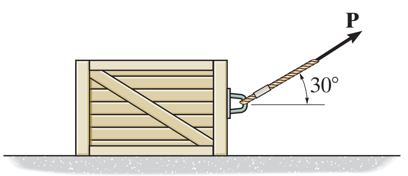
PROBLEMS

All solutions must include a free-body diagram.

13

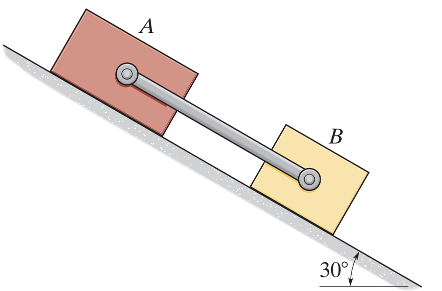
13–1. If the coefficient of kinetic friction between the 50-kg crate and the ground is $\mu_k = 0.3$, determine the distance the crate travels and its velocity when $t = 3$ s. The crate starts from rest and $P = 200$ N.

13–2. If the 50-kg crate starts from rest and achieves a velocity of $v = 4$ m/s when it travels a distance of 5 m to the right, determine the magnitude of force **P** acting on the crate. The coefficient of kinetic friction between the crate and the ground is $\mu_k = 0.3$.



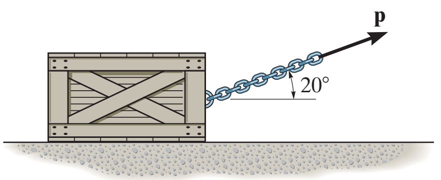
Probs. 13–1/2

***13–4.** If blocks A and B of mass 10 kg and 6 kg, respectively, are placed on the inclined plane and released, determine the force developed in the link. The coefficients of kinetic friction between the blocks and the inclined plane are $\mu_A = 0.1$ and $\mu_B = 0.3$. Neglect the mass of the link.



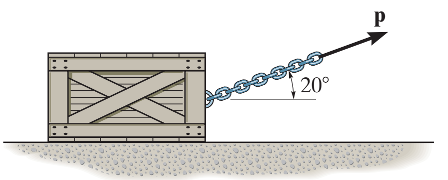
Prob. 13–4

13–3. The crate has a mass of 80 kg and is being towed by a chain which is always directed at 20° from the horizontal as shown. If the magnitude of **P** is increased until the crate begins to slide, determine the crate's initial acceleration if the coefficient of static friction is $\mu_s = 0.5$ and the coefficient of kinetic friction is $\mu_k = 0.3$.



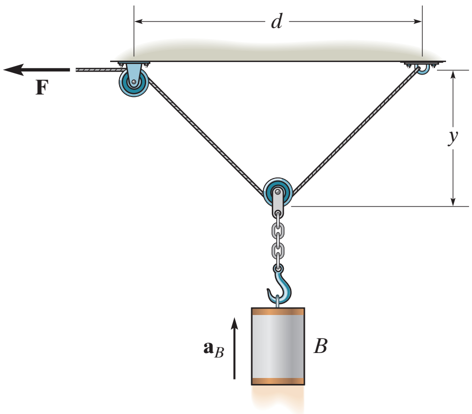
Prob. 13–3

13–5. The crate has a mass of 80 kg and is being towed by a chain which is always directed at 20° from the horizontal as shown. Determine the crate's acceleration in $t = 2$ s if the coefficient of static friction is $\mu_s = 0.4$, the coefficient of kinetic friction is $\mu_k = 0.3$, and the towing force is $P = (90t^2)$ N, where t is in seconds.



Prob. 13–5

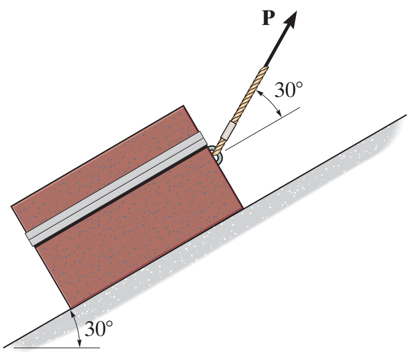
13–6. Cylinder B has a mass m and is hoisted using the cord and pulley system shown. Determine the magnitude of force $\mathbf{F}$ as a function of the block's vertical position y so that when $\mathbf{F}$ is applied the block rises with a constant acceleration $\mathbf{a}_B$. Neglect the mass of the cord and pulleys.



Prob. 13–6

13–7. If $P = 400$ N and the coefficient of kinetic friction between the 50-kg crate and the inclined plane is $\mu_k = 0.25$, determine the velocity of the crate after it travels 6 m up the plane. The crate starts from rest.

***13–8.** If the 50-kg crate starts from rest and travels a distance of 6 m up the plane in 4 s, determine the magnitude of force $\mathbf{P}$ acting on the crate. The coefficient of kinetic friction between the crate and the ground is $\mu_k = 0.25$.



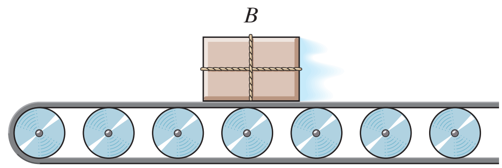
Probs. 13–7/8

13–9. If the force exerted on cable AB by the motor is $F = (100t^{3/2})$ N, where t is in seconds, determine the 50-kg crate's velocity when $t = 5$ s. The coefficients of static and kinetic friction between the crate and the ground are $\mu_s = 0.4$ and $\mu_k = 0.3$, respectively. Initially the crate is at rest.

Prob. 13–9

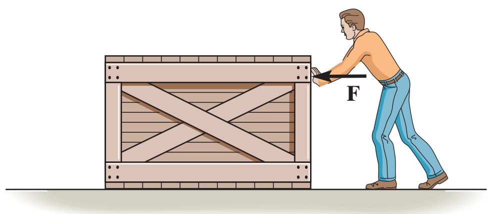
13–10. The conveyor belt is moving at 4 m/s. If the coefficient of static friction between the conveyor and the 10-kg package B is $\mu_s = 0.2$, determine the shortest time the belt can stop so that the package does not slide on the belt.

13–11. The conveyor belt is designed to transport packages of various weights. Each 10-kg package has a coefficient of kinetic friction $\mu_k = 0.15$. If the speed of the conveyor is 5 m/s, and then it suddenly stops, determine the distance the package will slide on the belt before coming to rest.



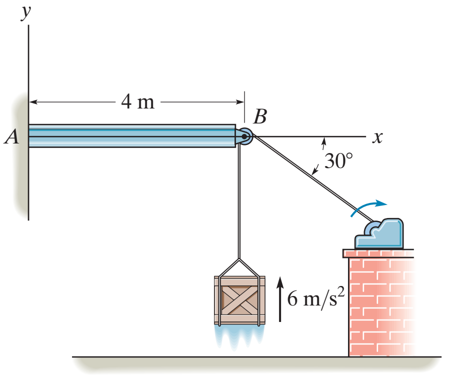
Probs. 13–10/11

***13–12.** The 75-kg man pushes on the 150-kg crate with a horizontal force $\mathbf{F}$. If the coefficients of static and kinetic friction between the crate and the surface are $\mu_s = 0.3$ and $\mu_k = 0.2$, and the coefficient of static friction between the man's shoes and the surface is $\mu_s = 0.8$, show that the man is able to move the crate. What is the greatest acceleration the man can give the crate?



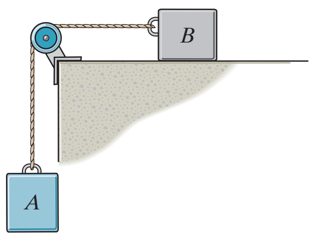
Prob. 13–12

13–13. The motor lifts the 50-kg crate with an acceleration of 6 m/s^2. Determine the components of force reaction and the couple moment at the fixed support A.



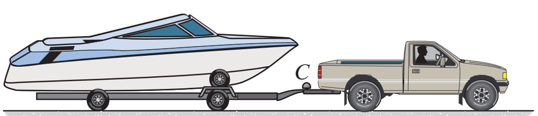
Prob. 13–13

13–14. Determine the acceleration of the blocks when the system is released. The coefficient of kinetic friction is μ_k, and the mass of each block is m. Neglect the mass of the pulleys and cord.



Prob. 13–14

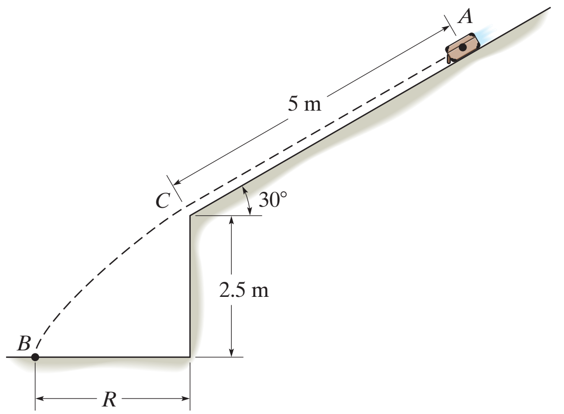
13–15. The 2-Mg truck is traveling at 15 m/s when the brakes on all its wheels are applied, causing it to skid for a distance of 10 m before coming to rest. Determine the constant horizontal force developed in the coupling C, and the frictional force developed between the tires of the truck and the road during this time. The total mass of the boat and trailer is 1 Mg.



Prob. 13–15

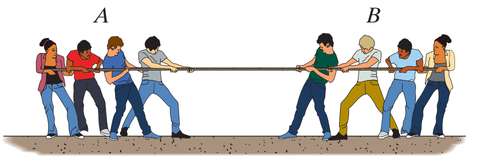
***13–16.** A 60-kg suitcase slides from rest 5 m down the smooth ramp. Determine the distance R where it strikes the ground at B. How long does it take to go from A to B?

13–17. Solve Prob. 13–16 if the suitcase has an initial velocity down the ramp of $v_A = 2 \text{ m/s}$, and the coefficient of kinetic friction along AC is $\mu_k = 0.2$.



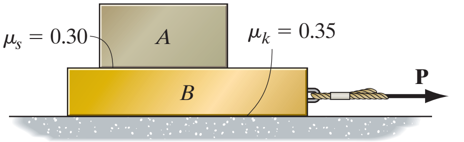
Probs. 13–16/17

13–18. Team A and team B consist of four members of total mass 320 kg and 330 kg, respectively. If the average coefficients of static and kinetic friction for team A are $\mu_s = 0.52$ and $\mu_k = 0.5$, and for team B, $\mu_s = 0.48$ and $\mu_k = 0.46$, which team will win the tug of war? Also, what is the acceleration of the losing team if the winning team develops its maximum static friction force?



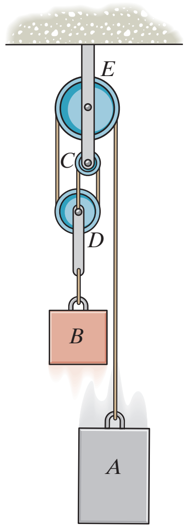
Prob. 13–18

13–19. Block A has a mass of 60 kg and rests on block B, which has a mass of 30 kg. If the coefficients of static and kinetic friction are indicated in the figure, determine the largest horizontal force **P** which can be applied to block B so that block A does not slip on block B while block B slides.



Prob. 13–19

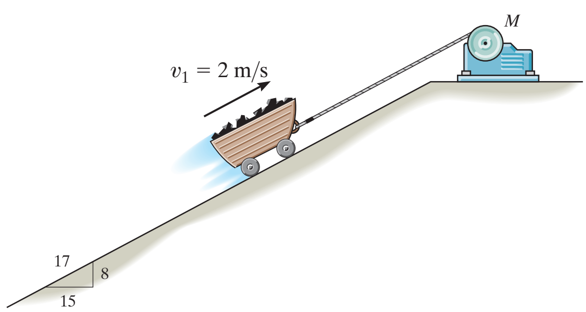
*13–20. The 50-kg block A is released from rest. Determine the velocity of the 15-kg block B in 2 s.



Prob. 13–20

13–21. The 400-kg mine car is hoisted up the incline using the cable and motor M. For a short time, the force in the cable is $F = (3200t^2)$ N, where t is in seconds. If the car has an initial velocity $v_1 = 2$ m/s when $t = 0$, determine its velocity when $t = 2$ s.

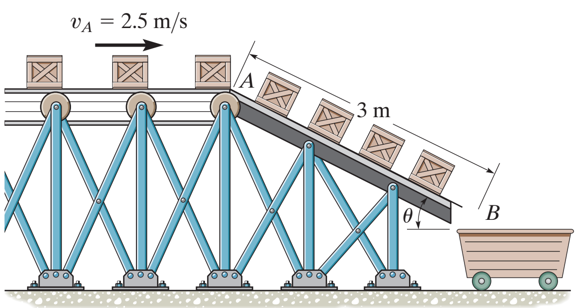
13–22. The 400-kg mine car is hoisted up the incline using the cable and motor M. For a short time, the force in the cable is $F = (3200t^2)$ N, where t is in seconds. If the car has an initial velocity $v_1 = 2$ m/s when $t = 0$, determine the distance it moves up the plane when $t = 2$ s.



Probs. 13–21/22

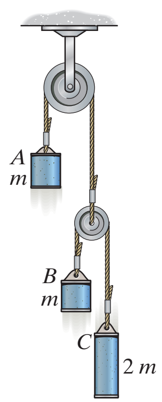
13–23. The conveyor belt delivers each 12-kg crate to the ramp at A such that the crate's speed is $v_A = 2.5$ m/s, directed down *along* the ramp. If the coefficient of kinetic friction between each crate and the ramp is $\mu_k = 0.3$, determine the speed at which each crate slides off the ramp at B. Assume that no tipping occurs. Take $\theta = 30°$.

*13–24. The conveyor belt delivers each 12-kg crate to the ramp at A such that the crate's speed is $v_A = 2.5$ m/s, directed down *along* the ramp. If the coefficient of kinetic friction between each crate and the ramp is $\mu_k = 0.3$, determine the smallest incline θ of the ramp so that the crates will slide off and fall into the cart.



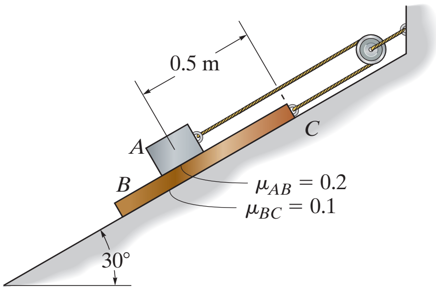
Probs. 13–23/24

13–25. Determine the acceleration of each block if A and B have a mass m and C has a mass $2m$. Neglect the mass of the cords and the pulleys.



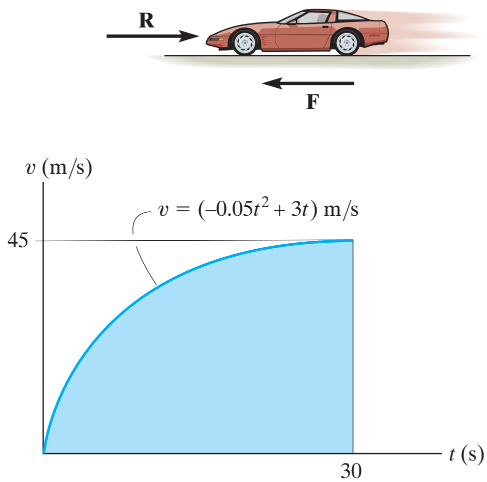
Prob. 13–25

13–26. The 20-kg block A rests on the 60-kg plate B in the position shown. Neglecting the mass of the rope and pulley, and using the coefficients of kinetic friction indicated, determine the time needed for block A to slide 0.5 m *on the plate* when the system is released from rest.



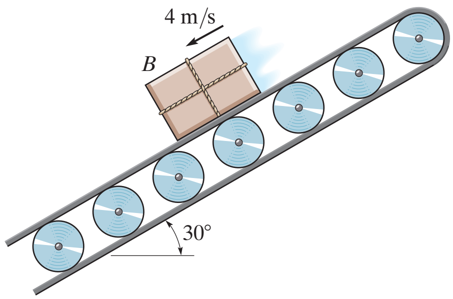
Prob. 13–26

13–27. The 1.5 Mg sports car has a tractive force of $F = 4.5$ kN. If it produces the velocity described by v-t graph shown, plot the air resistance R versus t for this time period.



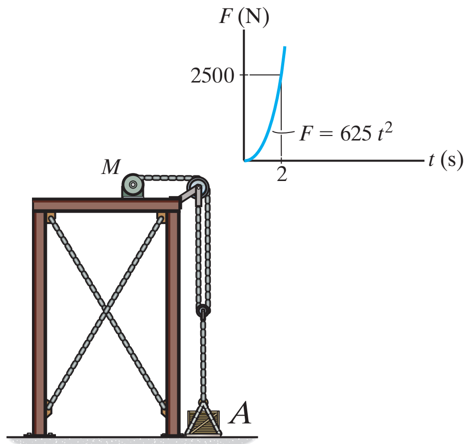
Prob. 13–27

***13–28.** The conveyor belt is moving downward at 4 m/s. If the coefficient of static friction between the conveyor and the 15-kg package B is $\mu_s = 0.8$, determine the shortest time the belt can stop so that the package does not slide on the belt.



Prob. 13–28

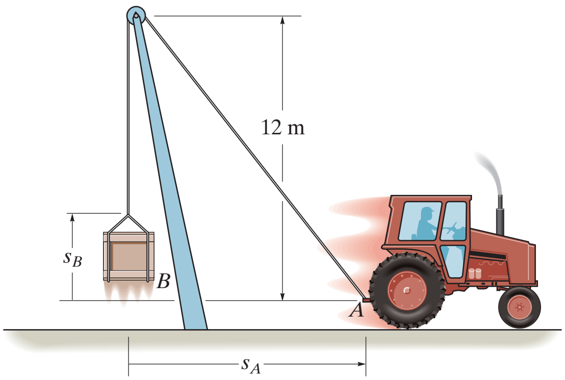
13–29. The force of the motor M on the cable is shown in the graph. Determine the velocity of the 400-kg crate A when $t = 2$ s.



Prob. 13–29

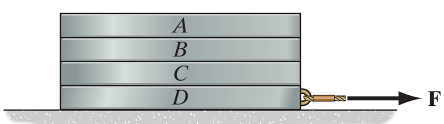
13–30. The tractor is used to lift the 150-kg load B with the 24-m-long rope, boom, and pulley system. If the tractor travels to the right at a constant speed of 4 m/s, determine the tension in the rope when $s_A = 5$ m. When $s_A = 0$, $s_B = 0$.

13–31. The tractor is used to lift the 150-kg load B with the 24-m-long rope, boom, and pulley system. If the tractor travels to the right with an acceleration of 3 m/s^2 and has a velocity of 4 m/s at the instant $s_A = 5$ m, determine the tension in the rope at this instant. When $s_A = 0$, $s_B = 0$.



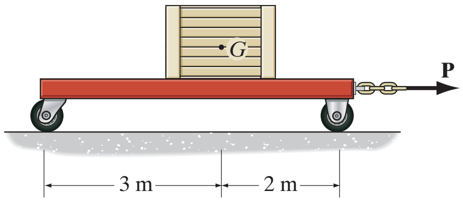
Probs. 13–30/31

*__13–32.__ The four 2-kg metal plates are stacked on top of one another. If the coefficient of static friction between any two plates is $\mu_s = 0.3$, determine the smallest horizontal force **F** that can be applied to the bottom plate which will cause any one of the plates to slip relative to another. The floor is smooth.



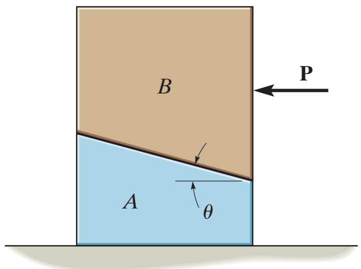
Prob. 13–32

13–33. The 3-kg crate rests on the 10-kg cart where the coefficients of static and kinetic friction are $\mu_s = 0.25$ and $\mu_k = 0.2$, respectively. Determine the smallest constant force **P** needed to cause the crate to slip. How much time does it take for the crate to slip off the cart?



Prob. 13–33

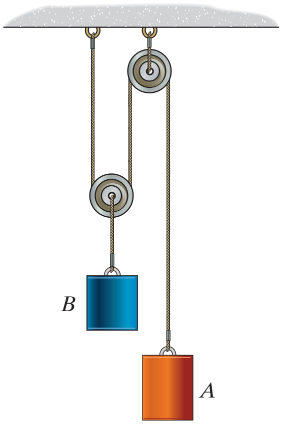
13–34. Block A and B each have a mass m. Determine the largest horizontal force **P** which can be applied to B so that it will not slide on A. Also, what is the corresponding acceleration? The coefficient of static friction between A and B is μ_s. Neglect any friction between A and the horizontal surface.



Prob. 13–34

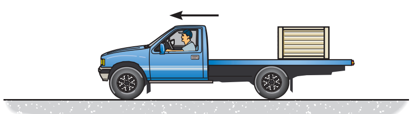
13

13–35. Determine the velocity of the cylinder A when $t = 4$ s if the system is released from rest. Cylinder A has a mass of 8 kg and cylinder B has a mass of 6 kg. Neglect the mass of the pulleys and cords.



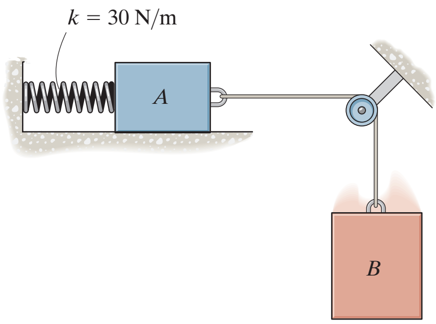
Prob. 13–35

***13–36.** The coefficient of static friction between the 200-kg crate and the flat bed of the truck is $\mu_s = 0.3$. Determine the shortest time for the truck to reach a speed of 60 km/h, starting from rest with constant acceleration, so that the crate does not slip.



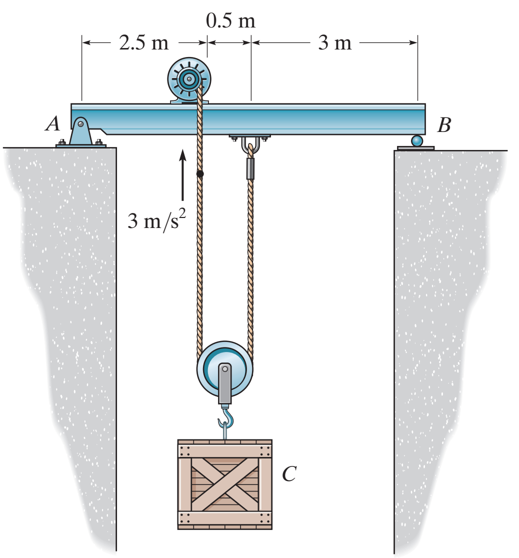
Prob. 13–36

13–37. Block A has a mass of 20 kg and block B has a mass of 50 kg. If the spring is stretched 0.2 m at the instant shown, determine the acceleration of block B at this instant if it is (a) originally at rest, (b) moving downward with a speed of 3 m/s. Neglect friction.



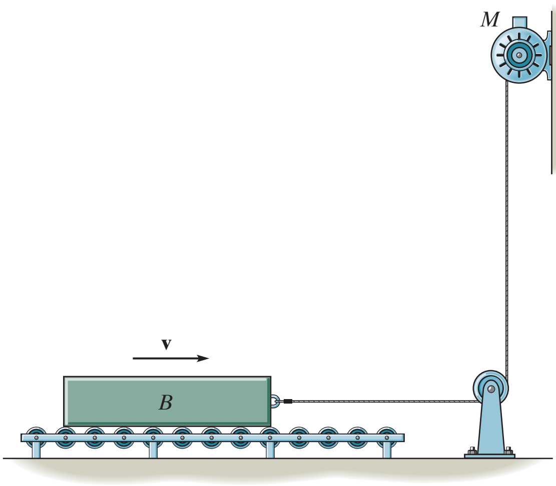
Prob. 13–37

13–38. If the motor draws in the cable with an acceleration of 3 m/s^2, determine the reactions at the supports A and B. The beam has a uniform mass of 30 kg/m, and the crate has a mass of 200 kg. Neglect the mass of the motor and pulleys.



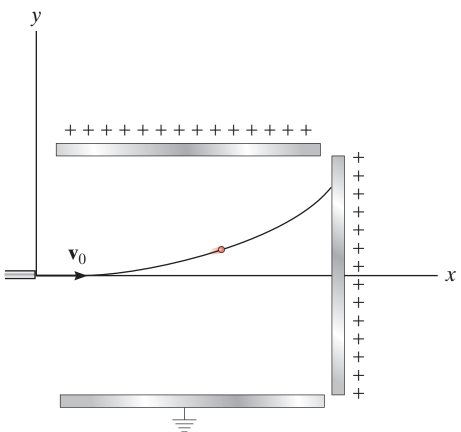
Prob. 13–38

13–39. The 300-kg bar B, originally at rest, is being towed over a series of small rollers. Determine the force in the cable when $t = 5$ s, if the motor M is drawing in the cable for a short time at a rate of $v = (0.4t^2)$ m/s, where t is in seconds $(0 \leq t \leq 6 \text{ s})$. How far does the bar move in 5 s? Neglect the mass of the cable, pulley, and the rollers.



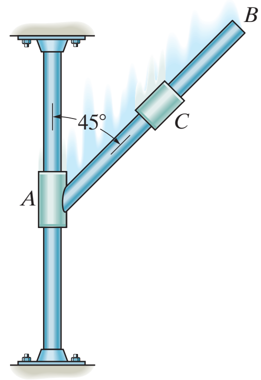
Prob. 13–39

***13–40.** An electron of mass m is discharged with an initial horizontal velocity of $\mathbf{v}_0$. If it is subjected to two fields of force for which $F_x = F_0$ and $F_y = 0.3F_0$, where F_0 is constant, determine the equation of the path, and the speed of the electron at any time t.



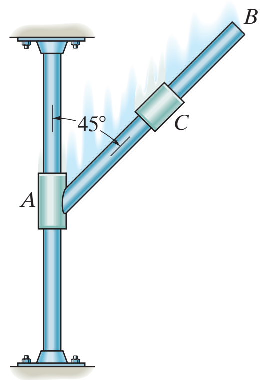
Prob. 13–40

13–41. The 2-kg collar C is free to slide along the smooth shaft AB. Determine the acceleration of collar C if collar A is subjected to an upward acceleration of 4 m/s^2.



Prob. 13–41

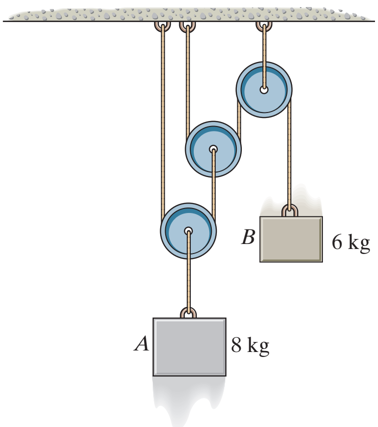
13–42. The 2-kg collar C is free to slide along the smooth shaft AB. Determine the acceleration of collar C if (a) the shaft is fixed from moving, (b) collar A, which is fixed to shaft AB, moves downward at constant velocity along the vertical rod, and (c) collar A is subjected to a downward acceleration of 2 m/s^2. In all cases, the collar moves in the plane.



Prob. 13–42

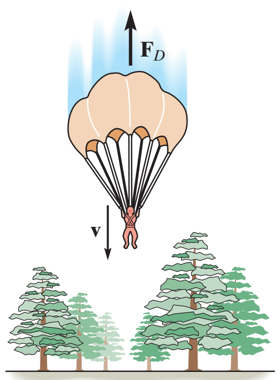
13

13–43. Determine the tension developed in the cords attached to each block and the accelerations of the blocks. Neglect the mass of the pulleys and cords.



Prob. 13–43

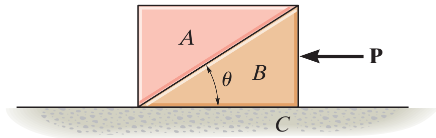
***13–44.** A parachutist having a mass m opens his parachute from an at-rest position at a very high altitude. If the atmospheric drag resistance is $F_D = kv^2$, where k is a constant, determine his velocity when he has fallen for a time t. What is his velocity when he lands on the ground? This velocity is referred to as the *terminal velocity*, which is found by letting the time of fall $t \rightarrow \infty$.



Prob. 13–44

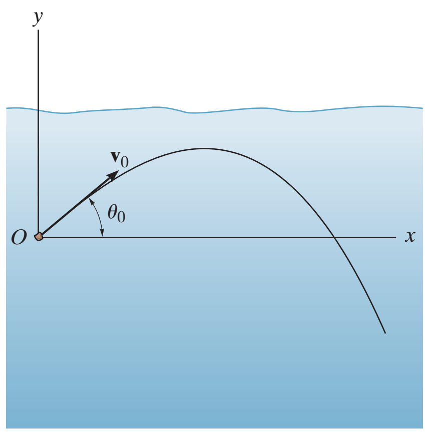
13–45. Blocks A and B each have a mass m. Determine the largest horizontal force P which can be applied to B so that A will not move relative to B. All surfaces are smooth.

13–46. Blocks A and B each have a mass m. Determine the largest horizontal force P which can be applied to B so that A will not slip on B. The coefficient of static friction between A and B is μ_s. Neglect any friction between B and C.



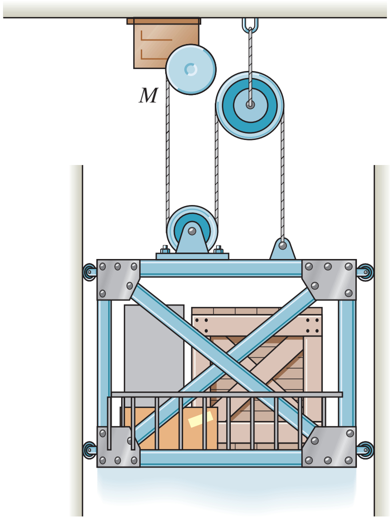
Probs. 13–45/46

13–47. A projectile of mass m is fired into a liquid at an angle θ_0 with an initial velocity $\mathbf{v}_0$ as shown. If the liquid develops a frictional or drag resistance on the projectile which is proportional to its velocity, i.e., $F = kv$, where k is a constant, determine the x and y components of its position at any instant. Also, what is the maximum distance x_{max} that it travels?



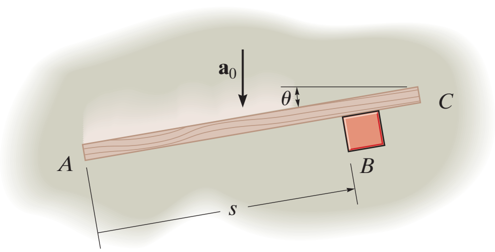
Prob. 13–47

*13–48. A freight elevator, including its load, has a mass of 1 Mg. It is prevented from rotating due to the track and wheels mounted along its sides. If the motor M develops a constant tension $T = 4$ kN in its attached cable, determine the velocity of the elevator when it has moved upward 6 m starting from rest. Neglect the mass of the pulleys and cables.



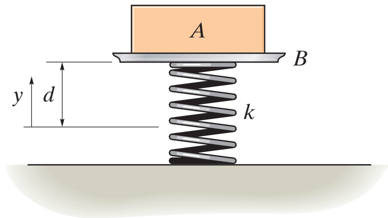
Prob. 13–48

13–49. The smooth block B of negligible size has a mass m and rests on the horizontal plane. If the board AC pushes on the block at an angle θ with a constant acceleration $\mathbf{a}_0$, determine the velocity of the block along the board and the distance s the block moves along the board as a function of time t. The block starts from rest when $s = 0, t = 0$.



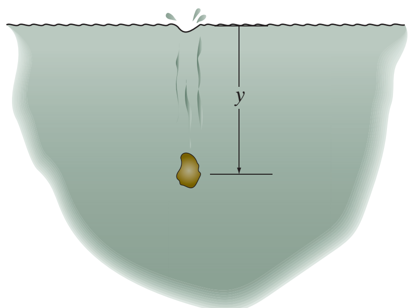
Prob. 13–49

13–50. The block A has a mass m_A and rests on the pan B, which has a mass m_B. Both are supported by a spring having a stiffness k that is attached to the bottom of the pan and to the ground. Determine the distance d the pan should be pushed down from the equilibrium position and then released from rest so that separation of the block will take place from the surface of the pan at the instant the spring becomes unstretched.



Prob. 13–50

13–51. The rock is dropped from rest within the thick liquid. If the drag force acting on it is $F = cv$, where c is a constant, determine the velocity and position of the rock as a function of time. Plot the velocity and position time graphs.

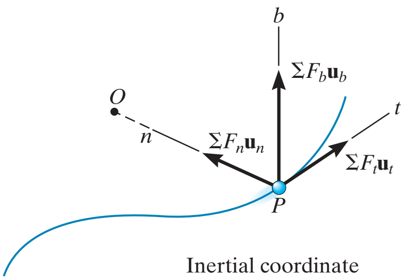


Prob. 13–51

13

13.5 EQUATIONS OF MOTION: NORMAL AND TANGENTIAL COORDINATES

When a particle moves along a curved path which is known, the equation of motion for the particle may be expressed in terms of components along the tangential, normal, and binormal directions, Fig. 13–11. There is no motion of the particle in the binormal direction, since the particle is constrained to move along the path. Therefore



Inertial coordinate system

Fig. 13–11

$$\Sigma \mathbf{F} = m\mathbf{a}$$

$$\Sigma F_t \mathbf{u}_t + \Sigma F_n \mathbf{u}_n + \Sigma F_b \mathbf{u}_b = m\mathbf{a}_t + m\mathbf{a}_n$$

This equation is satisfied provided

$$\Sigma F_t = ma_t$$
$$\Sigma F_n = ma_n \qquad (13\text{–}7)$$
$$\Sigma F_b = 0$$

Recall that $a_t \,(= dv/dt)$ represents the time rate of change in the magnitude of velocity. So if the resultant of $\Sigma \mathbf{F}_t$ acts in the direction of motion, the particle's speed will increase; whereas if it acts in the opposite direction, the particle will slow down. Likewise, $a_n \,(= v^2/\rho)$ represents the time rate of change in the velocity's direction. It is caused by the centripetal force or the resultant of $\Sigma \mathbf{F}_n$, which *always* acts in the positive n direction, i.e., toward the path's center of curvature.

As a roller coaster falls downward along the track, the cars have both normal and tangential components of acceleration.

PROCEDURE FOR ANALYSIS

When a problem involves the motion of a particle along a *known curved path*, normal and tangential coordinates should be considered for the analysis since the acceleration components can be readily formulated. The method for applying the equations of motion, which relate the forces to the acceleration, has been outlined in the procedure given in Sec. 13-4. Specifically, for *t, n, b* coordinates it may be stated as follows.

Free-Body Diagram.

- Establish the inertial *t, n, b* coordinate system at the particle and draw the particle's free-body diagram.
- The particle's normal acceleration $\mathbf{a}_n$ *always* acts in the positive n direction.
- If the tangential acceleration $\mathbf{a}_t$ is unknown, assume it acts in the positive t direction.
- There is no acceleration in the b direction.
- Identify the unknowns in the problem.

Equations of Motion.

- Apply the equations of motion, Eq. 13–7.

Kinematics.

- Formulate the tangential and normal components of acceleration; i.e., $a_t = dv/dt$ or $a_t = v\,dv/ds$ and $a_n = v^2/\rho$.
- If the path is defined as $y = f(x)$, the radius of curvature at the point where the particle is located can be obtained from $\rho = [1 + (dy/dx)^2]^{3/2}/|d^2y/dx^2|$.

Refer to the companion website for Lecture Summary and Quiz videos.

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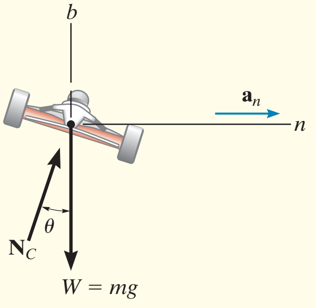
EXAMPLE 13.6

Determine the banking angle θ for the race track so that the wheels of the racing cars shown in Fig. 13–12a will not have to depend upon friction to prevent any car from sliding up or down the track. Assume the cars have negligible size, a mass m, and travel around the curve of radius ρ with a constant speed v.

(a)



(b)

Fig. 13–12

SOLUTION

Before looking at the following solution, give some thought as to why it should be solved using t, n, b coordinates.

Free-Body Diagram. As shown in Fig. 13–12b, and as stated in the problem, no frictional force acts on the car. Here $\mathbf{N}_C$ represents the *resultant* of the ground on all four wheels. Since a_n can be calculated, the unknowns are N_C and θ.

Equations of Motion. Using the n, b axes shown,

$$\overset{+}{\rightarrow}\Sigma F_n = ma_n; \qquad N_C \sin\theta = m\frac{v^2}{\rho} \tag{1}$$

$$+\uparrow\Sigma F_b = 0; \qquad N_C\cos\theta - mg = 0 \tag{2}$$

Eliminating N_C and m from these equations by dividing Eq. 1 by Eq. 2, we obtain

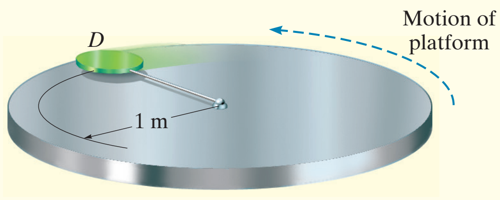
$$\tan\theta = \frac{v^2}{g\rho}$$

$$\theta = \tan^{-1}\left(\frac{v^2}{g\rho}\right) \qquad \textit{Ans.}$$

NOTE: The result is independent of the mass of the car. Also, a force summation in the tangential direction is of no consequence to the solution. If it were considered, then $a_t = dv/dt = 0$, since the car moves with *constant speed*. A further analysis of this problem is discussed in Prob. 21–52.

EXAMPLE 13.7

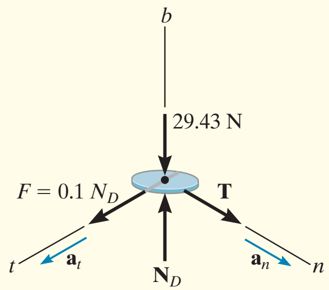
The 3-kg disk D is attached to the end of a cord as shown in Fig. 13–13*a*. The other end of the cord is attached to a ball-and-socket joint located at the center of a platform. If the platform rotates rapidly, and the disk is placed on it and released from rest, determine the time it takes for the disk to reach a speed great enough to break the cord. The maximum tension the cord can sustain is 100 N, and the coefficient of kinetic friction between the disk and the platform is $\mu_k = 0.1$.



(a)

SOLUTION

Free-Body Diagram. The frictional force has a magnitude $F = \mu_k N_D = 0.1N_D$ and a sense of direction that opposes the *relative motion* of the disk with respect to the platform. It is this force that gives the disk a tangential component of acceleration causing v to increase, thereby causing T to increase until it reaches 100 N. The weight of the disk is $W = 3(9.81) = 29.43$ N. Since a_n can be related to v, the unknowns are N_D, a_t, and v.



(b)

Fig. 13–13

Equations of Motion.

$$\Sigma F_n = ma_n; \qquad T = 3\left(\frac{v^2}{1}\right) \qquad (1)$$

$$\Sigma F_t = ma_t; \qquad 0.1N_D = 3a_t \qquad (2)$$

$$\Sigma F_b = 0; \qquad N_D - 29.43 = 0 \qquad (3)$$

Setting $T = 100$ N, Eq. 1 can be solved for the critical speed v_{cr} of the disk needed to break the cord. Solving all the equations, we obtain

$$N_D = 29.43 \text{ N}$$
$$a_t = 0.981 \text{ m/s}^2$$
$$v_{cr} = 5.77 \text{ m/s}$$

Kinematics. Since a_t is *constant*, the time needed to break the cord is

$$v_{cr} = v_0 + a_t t$$

$$5.77 = 0 + (0.981)t$$

$$t = 5.89 \text{ s} \qquad \textit{Ans.}$$

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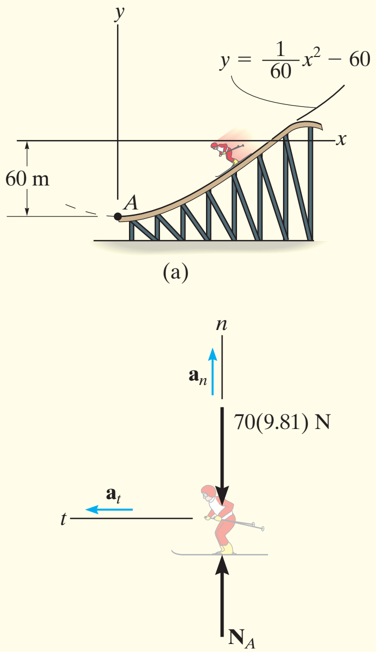
EXAMPLE 13.8



(a)

(b)

Fig. 13–14

If the ski jump can be approximated by the parabola shown in Fig. 13–14*a*, determine the normal force on the 70-kg skier the instant she arrives at the end of the jump, point *A*, where her velocity is 20 m/s. Also, what is her acceleration at this point?

SOLUTION

Why consider using *n, t* coordinates to solve this problem?

Free-Body Diagram. Since $dy/dx = x/30\big|_{x=0} = 0$, the slope at *A* is horizontal. Therefore, a free-body diagram of the skier when she is at *A* is shown in Fig. 13–14*b*. Since the path is *curved*, there are two components of acceleration, $\mathbf{a}_n$ and $\mathbf{a}_t$.

Here we can calculate a_n and so the unknowns are a_t and N_A.

Equations of Motion.

$$+\uparrow \Sigma F_n = ma_n; \qquad N_A - 70(9.81) = 70\left[\frac{(20)^2}{\rho}\right] \qquad (1)$$

$$\overset{+}{\leftarrow} \Sigma F_t = ma_t; \qquad 0 = 70a_t \qquad (2)$$

The radius of curvature ρ for the path must be determined at point $A(0, -60\text{ m})$. Here $y = \frac{1}{60}x^2 - 60$, $dy/dx = \frac{1}{30}x$, $d^2y/dx^2 = \frac{1}{30}$, so that at $x = 0$,

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|}\Bigg|_{x=0} = \frac{[1 + (0)^2]^{3/2}}{\left|\frac{1}{30}\right|} = 30\text{ m}$$

Substituting this into Eq. 1 and solving for N_A, we obtain

$$N_A = 1.62\text{ kN} \qquad \textit{Ans.}$$

Kinematics. From Eq. 2,

$$a_t = 0$$

Thus,

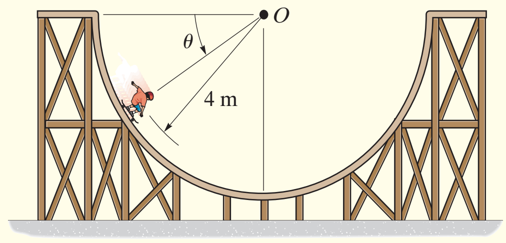
$$a_n = \frac{v^2}{\rho} = \frac{(20)^2}{30} = 13.33\text{ m/s}^2$$

$$a_A = a_n = 13.3\text{ m/s}^2 \uparrow \qquad \textit{Ans.}$$

NOTE: Apply the equation of motion in the *y* direction and show that when the skier is in mid air, her downward acceleration is 9.81 m/s^2.

EXAMPLE 13.9

The 60-kg skateboarder in Fig. 13–15*a* coasts down the circular track. If he starts from rest when $\theta = 0°$, determine the magnitude of the normal reaction the track exerts on him when $\theta = 60°$. Neglect his size for the calculation.



(a)

SOLUTION

Free-Body Diagram. The free-body diagram of the skateboarder when he is at an *arbitrary position* θ is shown in Fig. 13–15*b*.

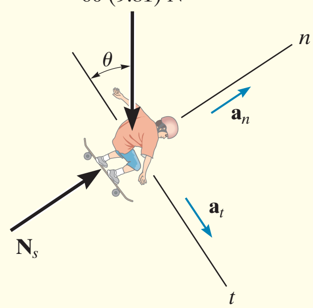
At $\theta = 60°$ there are three unknowns, N_s, a_t, and a_n (or v).

Equations of Motion.

$$+\nearrow \Sigma F_n = ma_n; \qquad N_s - [60(9.81)\text{ N}]\sin\theta = (60\text{ kg})\left(\frac{v^2}{4\text{ m}}\right) \qquad (1)$$

$$+\searrow \Sigma F_t = ma_t; \qquad [60(9.81)\text{ N}]\cos\theta = (60\text{ kg})\,a_t$$

$$a_t = 9.81\cos\theta$$



(b)

Kinematics. Since the increase in speed, a_t, is expressed in terms of θ, the equation $v\,dv = a_t\,ds$ must be used to determine the speed of the skateboarder when $\theta = 60°$. Using the geometric relation $s = \theta r$, where $ds = r\,d\theta = (4\text{ m})\,d\theta$, Fig. 13–15*c*, and the initial condition $v = 0$ at $\theta = 0°$, we have,

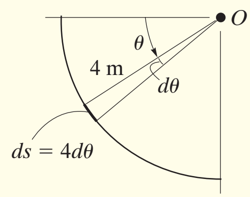
$$v\,dv = a_t\,ds$$

$$\int_0^v v\,dv = \int_0^{60°} 9.81\cos\theta(4\,d\theta)$$

$$\left.\frac{v^2}{2}\right|_0^v = 39.24\sin\theta\Big|_0^{60°}$$

$$\frac{v^2}{2} - 0 = 39.24(\sin 60° - 0)$$

$$v^2 = 67.97\text{ m}^2/\text{s}^2$$



(c)

Fig. 13–15

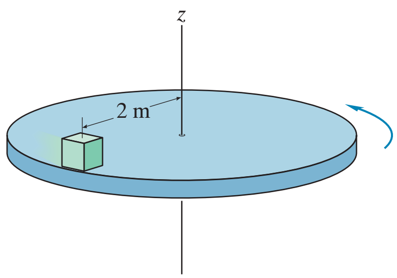
Substituting this result and $\theta = 60°$ into Eq. 1 yields

$$N_s = 1529.23\text{ N} = 1.53\text{ kN} \qquad \textit{Ans.}$$

Refer to the companion website for a self quiz of these Example problems.

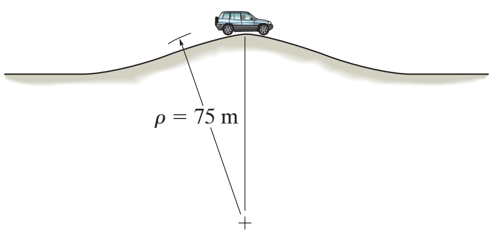
FUNDAMENTAL PROBLEMS

F13–7. The block rests at a distance of 2 m from the center of the platform. If the coefficient of static friction between the block and the platform is $\mu_s = 0.3$, determine the maximum speed which the block can attain before it begins to slip. Assume the angular motion of the disk is slowly increasing.



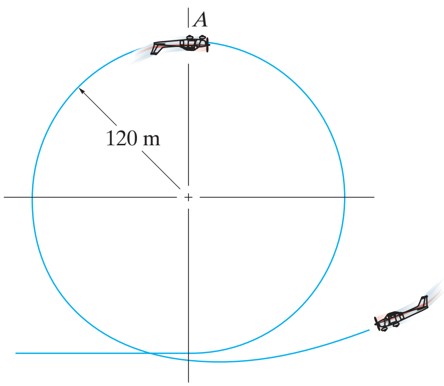
Prob. F13–7

F13–8. Determine the maximum speed that the jeep can travel over the crest of the hill and not lose contact with the road.



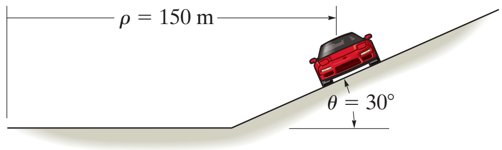
Prob. F13–8

F13–9. A pilot has a mass of 70 kg and is traveling at a constant speed of 36 m/s. Determine the normal force he exerts on the seat of the plane when he is upside down at A. The loop has a radius of curvature of 120 m.



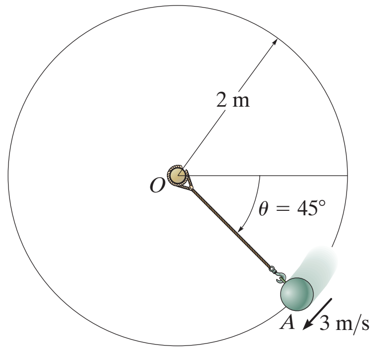
Prob. F13–9

F13–10. The sports car is traveling along a 30° banked road having a radius of curvature of $\rho = 150$ m. If the coefficient of static friction between the tires and the road is $\mu_s = 0.2$, determine the maximum speed so no slipping occurs. Neglect the size of the car.



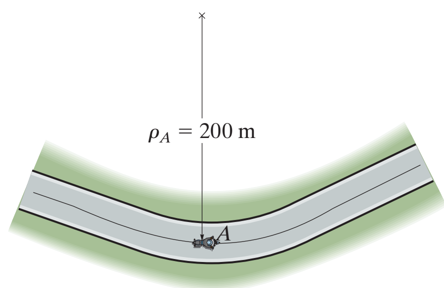
Prob. F13–10

F13–11. If the 10-kg ball has a velocity of 3 m/s when it is at the position A, along the vertical path, determine the tension in the cord and the increase in the speed of the ball at this position.



Prob. F13–11

F13–12. The motorcycle has a mass of 0.5 Mg and a negligible size. It passes point A traveling at a speed of 15 m/s, which is increasing at a constant rate of 1.5 m/s^2. Determine the resultant frictional force exerted by the road on the tires at this instant.



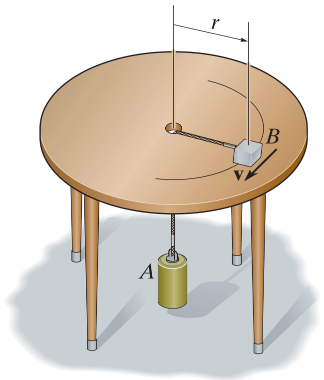
Prob. F13–12

PROBLEMS

All solutions must include a free-body diagram.

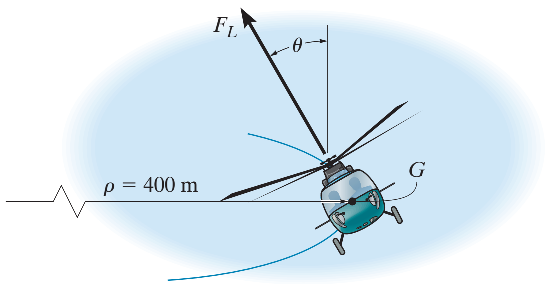
***13–52.** The 2-kg block B and 15-kg cylinder A are connected to a light cord that passes through a hole in the center of the smooth table. If the block is given a speed of $v = 10$ m/s, determine the radius r of the circular path along which it travels.

13–53. The 2-kg block B and 15-kg cylinder A are connected to a light cord that passes through a hole in the center of the smooth table. If the block travels along a circular path of radius $r = 1.5$ m, determine the speed of the block.



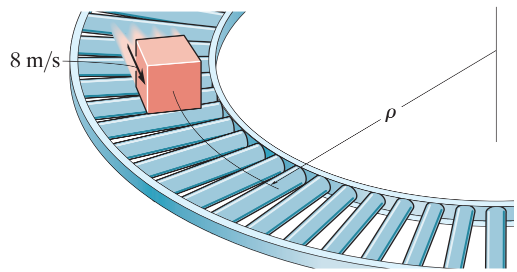
Probs. 13–52/53

13–54. The 4-Mg helicopter maneuvers a horizontal turn having a radius of curvature $\rho = 400$ m. Determine the lift force F_L required and the angle of the bank θ when it is flying horizontally with a constant speed of $v = 40$ m/s. Show that θ increases if v increases by also calculating θ when $v = 60$ m/s.



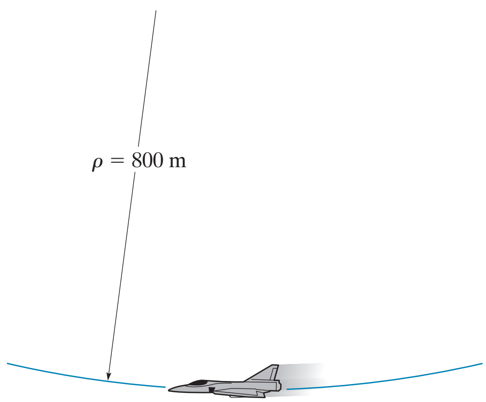
Prob. 13–54

13–55. Cartons having a mass of 5 kg are required to move along the assembly line at a constant speed of 8 m/s. Determine the smallest radius of curvature ρ for the conveyor so the cartons do not slip. The coefficients of static and kinetic friction between a carton and the conveyor are $\mu_s = 0.7$ and $\mu_k = 0.5$, respectively.



Prob. 13–55

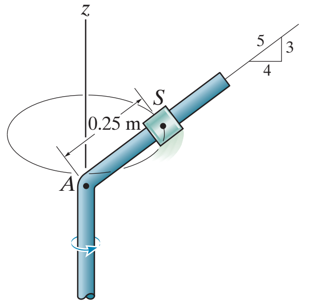
***13–56.** Determine the maximum constant speed at which the pilot can travel around the vertical curve having a radius of curvature $\rho = 800$ m, so that he experiences a maximum acceleration $a_n = 8g = 78.5$ m/s^2. If he has a mass of 70 kg, determine the normal force he exerts on the seat of the airplane when the plane is traveling at this speed and is at its lowest point.



Prob. 13–56

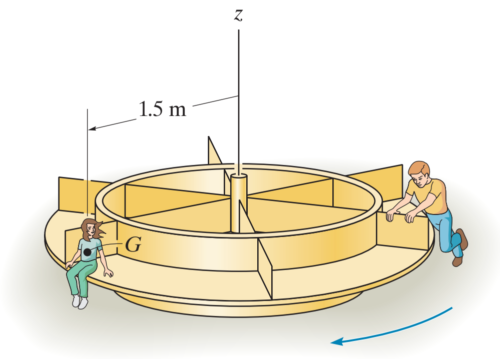
13–57. The 2-kg spool S fits loosely on the inclined rod for which the coefficient of static friction is $\mu_s = 0.2$. If the spool is located 0.25 m from A, determine the minimum constant speed the spool can have so that it does not slip down the rod.

13–58. The 2-kg spool S fits loosely on the inclined rod for which the coefficient of static friction is $\mu_s = 0.2$. If the spool is located 0.25 m from A, determine the maximum constant speed the spool can have so that it does not slip up the rod.



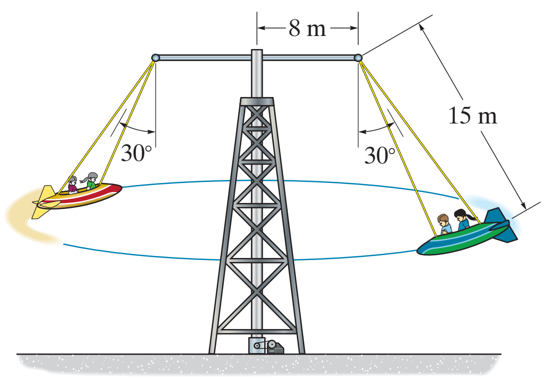
Probs. 13–57/58

***13–60.** A girl having a mass of 25 kg sits at the edge of the merry-go-round so her center of mass G is at a distance of 1.5 m from the axis of rotation. If the angular motion of the platform is *slowly* increased so that the girl's tangential component of acceleration can be neglected, determine the maximum speed which she can have before she begins to slip off the merry-go-round. The coefficient of static friction between the girl and the merry-go-round is $\mu_s = 0.3$.



Prob. 13–60

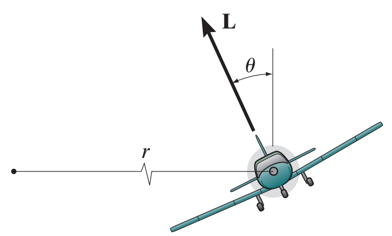
13–59. Calculate the constant speed of the cars on the amusement-park ride if it is observed that the cables are directed at 30° from the vertical. Each car, including its passengers, has a mass of 550 kg. Also, what are the components of force in the n, t, and z directions which a 60-kg passenger exerts on the car during the motion?



Prob. 13–59

13–61. A 5-Mg airplane is flying at a constant speed of 350 km/h along a horizontal circular path of radius $r = 3000$ m. Determine the uplift force $\mathbf{L}$ acting on the airplane and the banking angle θ. Neglect the size of the airplane.

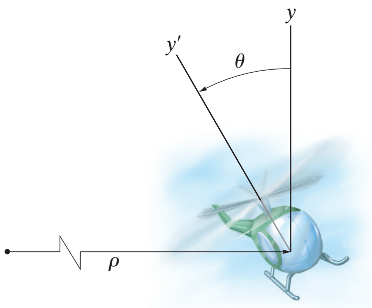
13–62. A 5-Mg airplane is flying at a constant speed of 350 km/h along a horizontal circular path. If the banking angle $\theta = 15°$, determine the uplift force $\mathbf{L}$ acting on the airplane and the radius r of the circular path. Neglect the size of the airplane.



Probs. 13–61/62

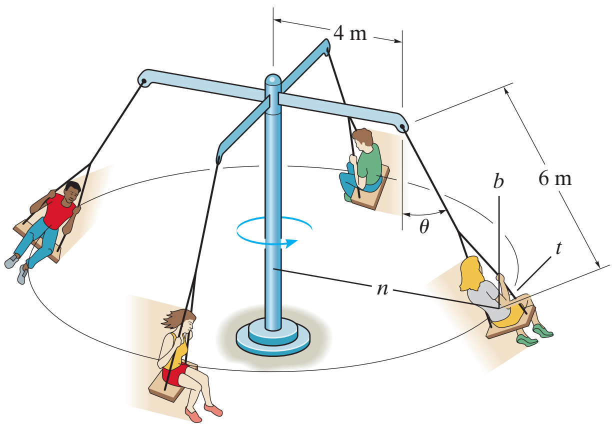
13–63. The 1.40-Mg helicopter is traveling at a constant speed of 40 m/s along the horizontal curved path while banking at $\theta = 30°$. Determine the force acting normal to the blade, i.e., in the y' direction, and the radius of curvature of the path.

***13–64.** The 1.40-Mg helicopter is traveling at a constant speed of 33 m/s along the horizontal curved path having a radius of curvature of $\rho = 300$ m. Determine the force the blade exerts on the frame and the bank angle θ.



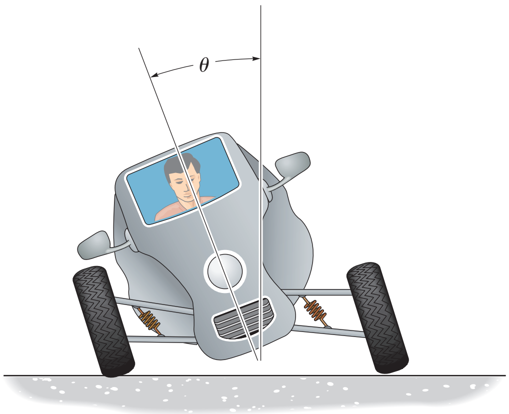
Probs. 13–63/64

13–65. Determine the constant speed of the passengers on the amusement-park ride if it is observed that the supporting cables are directed at $\theta = 30°$ from the vertical. Each chair including its passenger has a mass of 80 kg. Also, what are the components of force in the n, t, and b directions which the chair exerts on a 50-kg passenger during the motion?



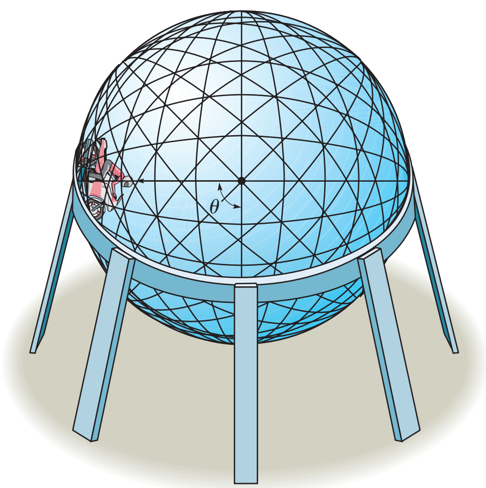
Prob. 13–65

13–66. The vehicle is designed to combine the feel of a motorcycle with the comfort and safety of an automobile. If the vehicle is traveling at a constant speed of 80 km/h along a circular curved road of radius 100 m, determine the tilt angle θ of the vehicle so that only a normal force from the seat acts on the driver. Neglect the size of the driver.



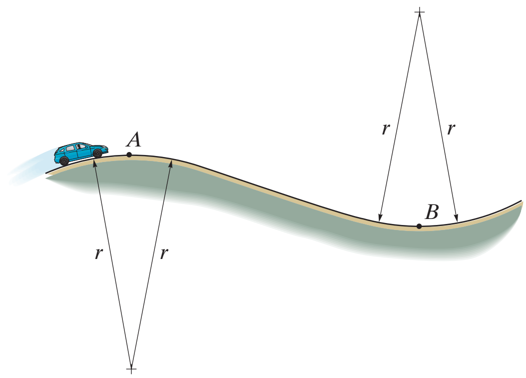
Prob. 13–66

13–67. A motorcyclist in a circus rides his motorcycle within the confines of the hollow sphere. If the coefficient of static friction between the wheels of the motorcycle and the sphere is $\mu_s = 0.4$, determine the minimum speed at which he must travel if he is to ride along the wall when $\theta = 90°$. The mass of the motorcycle and rider is 250 kg, and the radius of curvature to the center of gravity is $\rho = 20$ m. Neglect the size of the motorcycle for the calculation.



Prob. 13–67

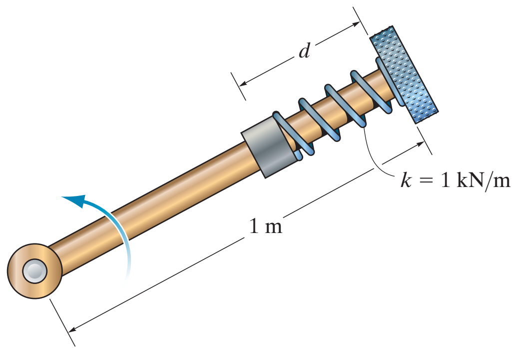
***13–68.** Determine the maximum speed at which the car with mass m can pass over the top point A of the vertical curved road and still maintain contact with the road. If the car maintains this speed, what is the normal reaction the road exerts on the car when it passes the lowest point B on the road?



Prob. 13–68

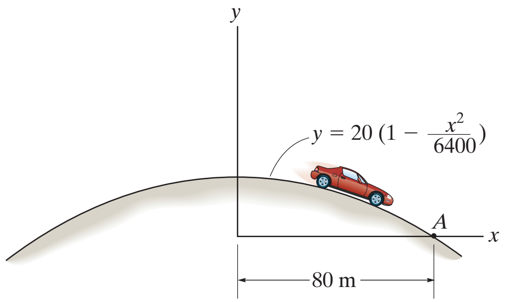
13–69. The collar has a mass of 2 kg and is free to slide along the smooth rod, which rotates in the *horizontal plane*. The attached spring has a stiffness $k = 1$ kN/m, and when no force is applied to the spring its unstretched length is $d_0 = 0.5$ m. Determine the force which the spring exerts on the collar when the collar is at rest with respect to the rod, yet due to the rod's rotation it is moving with a constant speed $v = 6$ m/s.

13–70. The collar has a mass of 2 kg and is free to slide along the smooth rod, which rotates in the *horizontal plane*. The attached spring has a stiffness $k = 1$ kN/m, and when no force is applied to the spring its unstretched length is $d_0 = 0.6$ m. Determine the length d if the collar is at rest with respect to the rod, yet due to the rod's rotation it is moving with a constant speed $v = 5$ m/s.



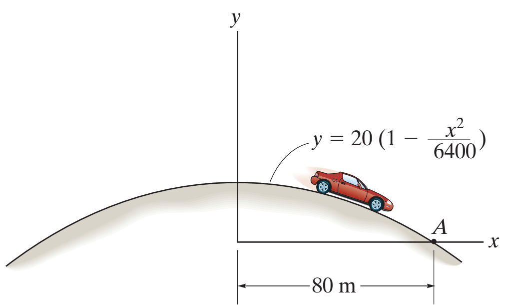
Probs. 13–69/70

13–71. The 0.8-Mg car travels over the hill having the shape of a parabola. If the driver maintains a constant speed of 9 m/s, determine both the resultant normal force and the resultant frictional force that all the wheels of the car exert on the road at the instant it reaches point A. Neglect the size of the car.



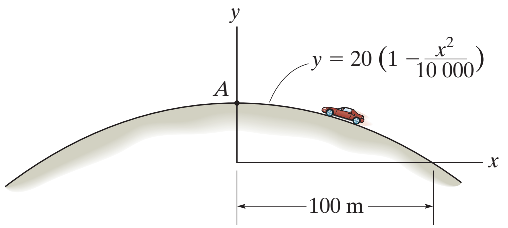
Prob. 13–71

***13–72.** The 0.8-Mg car travels over the hill having the shape of a parabola. When the car is at point A, it is traveling at 9 m/s and increasing its speed at 3 m/s^2. Determine both the resultant normal force and the resultant frictional force that all the wheels of the car exert on the road at this instant. Neglect the size of the car.



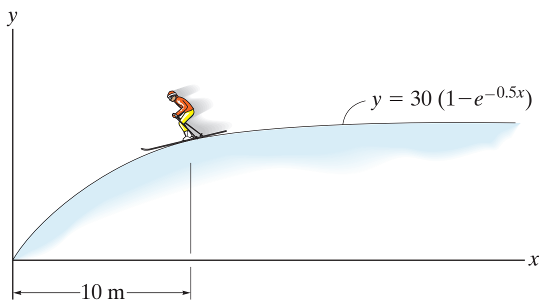
Prob. 13–72

13–73. Determine the maximum constant speed at which the 2-Mg car can travel over the crest of the hill at A without leaving the surface of the road. Neglect the size of the car in the calculation.



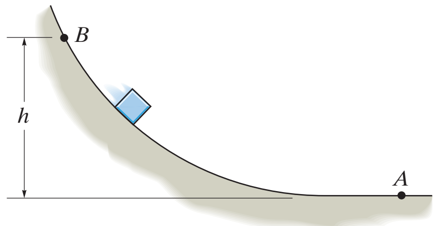
Prob. 13–73

13–74. The 52-kg skier is coasting freely down the hill with a speed of 4 m/s at $x = 10$ m. Determine the normal reaction on the ground and the rate of increase in speed at the instant shown. Neglect friction and the size of the skier.



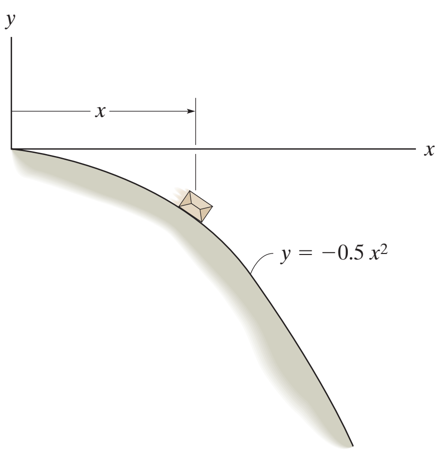
Prob. 13–74

13–75. Prove that if the block is released from rest at point B on a smooth path of *arbitrary shape*, the speed it attains when it reaches point A is equal to the speed it attains when it falls freely through a distance h; i.e., $v = \sqrt{2gh}$.



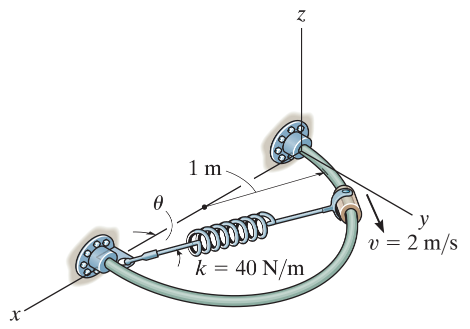
Prob. 13–75

***13–76.** The box has a mass m and slides down the smooth chute having the shape of a parabola. If it has an initial velocity of v_0 at the origin determine its velocity as a function of x. Also, what is the normal force on the box, and the tangential acceleration as a function of x?



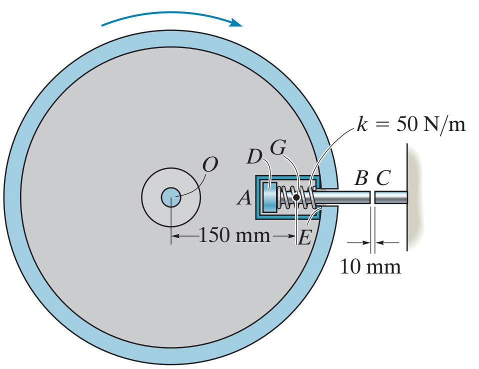
Prob. 13–76

13–77. The collar has a mass of 5 kg and is confined to move along the smooth circular rod which lies in the horizontal plane. The attached spring has an unstretched length of 200 mm. If at the instant $\theta = 30°$ the collar has a speed $v = 2$ m/s, determine the magnitudes of the normal force of the rod on the collar and the collar's acceleration.



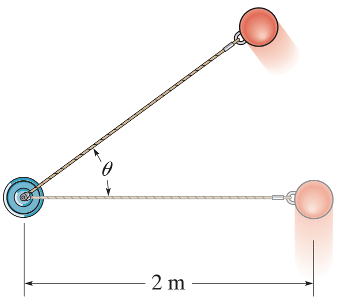
Prob. 13–77

13–78. The rotational speed of the disk is controlled by a 30-g smooth contact arm AB which is spring mounted on the disk. When the disk is *at rest*, the center of mass G of the arm is located 150 mm from the center O, and the preset compression in the spring is 20 mm. If the initial gap between B and the contact at C is 10 mm, determine the (controlling) speed v_G of the arm's mass center, G, which will close the gap. The disk rotates in the horizontal plane. The spring has a stiffness of $k = 50$ N/m, and its ends are attached to the contact arm at D and to the disk at E.



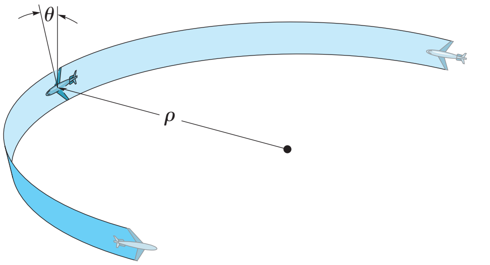
Prob. 13–78

13–79. The 2-kg pendulum bob moves in the vertical plane with a velocity of 8 m/s when $\theta = 0°$. Determine the initial tension in the cord and also at the instant the bob reaches $\theta = 30°$. Neglect the size of the bob.



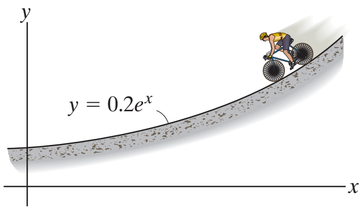
Prob. 13–79

***13–80.** The airplane, traveling at a constant speed of 50 m/s, is executing a horizontal turn. If the plane is banked at $\theta = 15°$, when the pilot experiences only a normal force on the seat of the plane, determine the radius of curvature ρ of the turn. Also, what is the normal force of the seat on the pilot if he has a mass of 70 kg?



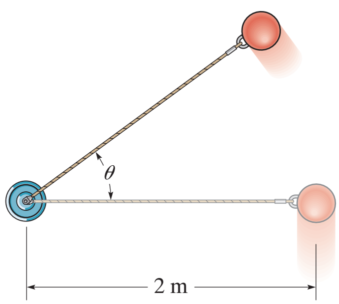
Prob. 13–80

13–81. The cyclist is coasting freely down the hill with a speed of 15 m/s at $y = 0.2$ m. Determine the resultant normal reaction on the bicycle and the rate of increase in speed at the instant shown. The bicycle and rider have a total mass of 80 kg. Neglect friction, the mass of the wheels, and the size of the bicycle.



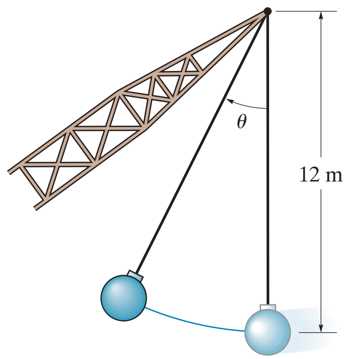
Prob. 13–81

13–82. The 2-kg pendulum bob moves in the vertical plane with a velocity of 6 m/s when $\theta = 0°$. Determine the angle θ where the tension in the cord becomes zero.



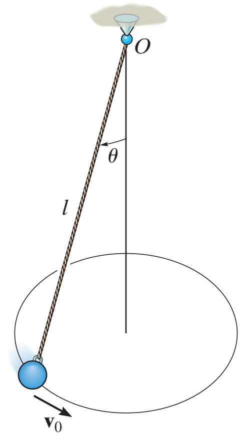
Prob. 13–82

13–83. The 600-kg wrecking ball is suspended from the crane by a cable having a negligible mass. If the ball has a speed $v = 8$ m/s at the instant it is at its lowest point, $\theta = 0°$, determine the tension in the cable at this instant. Also, determine the angle θ to which the ball swings before it stops.

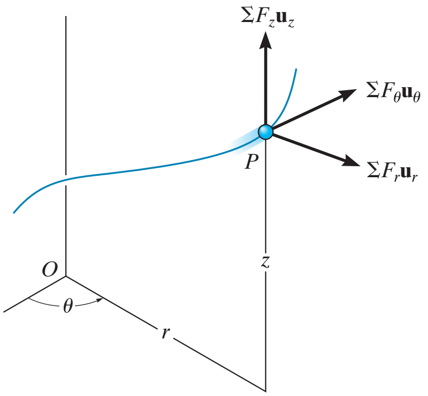


Prob. 13–83

***13–84.** The ball has a mass m and is attached to the cord of length l. The cord is tied at the top to a swivel and the ball is given a velocity $\mathbf{v}_0$. Show that the angle θ which the cord makes with the vertical as the ball travels around the circular path must satisfy the equation $\tan\theta \sin\theta = v_0^2/gl$. Neglect air resistance and the size of the ball.



Prob. 13–84



Inertial coordinate system

Fig. 13–16

*13.6 EQUATIONS OF MOTION: CYLINDRICAL COORDINATES

When all the forces acting on a particle are resolved into cylindrical components, i.e., along the unit-vector directions $\mathbf{u}_r$, $\mathbf{u}_\theta$, $\mathbf{u}_z$, Fig. 13–16, the equation of motion can be expressed as

$$\Sigma \mathbf{F} = m\mathbf{a}$$
$$\Sigma F_r \mathbf{u}_r + \Sigma F_\theta \mathbf{u}_\theta + \Sigma F_z \mathbf{u}_z = ma_r \mathbf{u}_r + ma_\theta \mathbf{u}_\theta + ma_z \mathbf{u}_z$$

To satisfy this equation, we require

$$\begin{aligned} \Sigma F_r &= ma_r \\ \Sigma F_\theta &= ma_\theta \\ \Sigma F_z &= ma_z \end{aligned} \qquad (13\text{–}8)$$

If the particle is constrained to move only in the r–θ plane, then only the first two of Eq. 13–8 are used to specify the motion.

Tangential and Normal Forces. When a particle is constrained to move along a path, as in Fig. 13–17*a*, then the free-body diagram for the particle must include a normal force **N** which the path exerts on the particle. Also, if the path is rough, then a friction force **F** must be included. As shown, these forces act normal to the tangent to the path and along the tangent in the opposite direction of motion. When the path is described as $r = f(\theta)$, then the *directions* of **N** and **F** can be specified relative to the radial coordinate by using the angle ψ (psi), Fig. 13–17*b*, which is defined between the *extended* radial line and the tangent to the curve.


Motion of the roller coaster along this spiral can be studied using cylindrical coordinates.

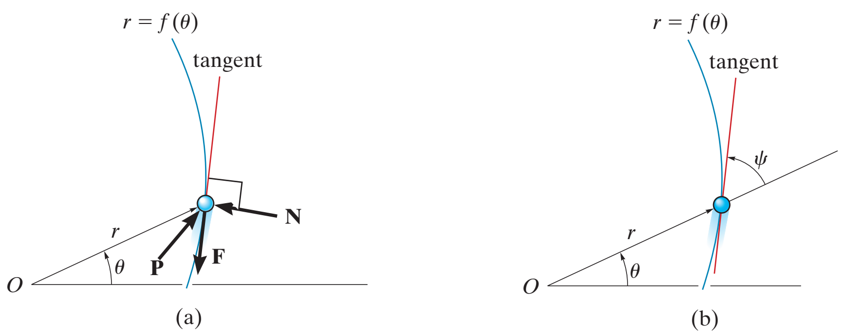


Fig. 13–17

This angle can be obtained by noting that when the particle is displaced a distance ds along the path, Fig. 13–17c, the component of displacement in the radial direction is dr and the component of displacement in the transverse direction is $r\,d\theta$. Since these two components are mutually perpendicular, the angle ψ can be determined from $\tan\psi = r\,d\theta/dr$, or

$$\tan\psi = \frac{r}{dr/d\theta} \tag{13–9}$$

If ψ is calculated as a positive quantity, it is measured from the *extended radial line* to the tangent in a counterclockwise sense or in the positive direction of θ, Fig.13–17c. If it is negative, it is measured in the opposite direction to positive θ. For example, consider the cardioid $r = a(1 + \cos\theta)$, shown in Fig. 13–18. Because $dr/d\theta = -a\sin\theta$, then when $\theta = 30°$, $\tan\psi = a(1 + \cos 30°)/(-a\sin 30°) = -3.732$, or $\psi = -75°$, measured clockwise, opposite to $+\theta$ as shown in the figure.

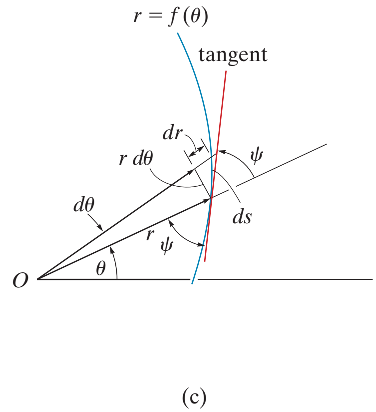


Fig. 13–17(cont.)

13

PROCEDURE FOR ANALYSIS

Cylindrical or polar coordinates are a suitable choice for the analysis of a problem for which data regarding the angular motion of the radial line r are given, or in cases where the path can be conveniently expressed in terms of these coordinates.

Free-Body Diagram.

- Establish the r, θ, z inertial coordinate system and draw the particle's free-body diagram.
- Assume that $\mathbf{a}_r$, $\mathbf{a}_\theta$, $\mathbf{a}_z$ act in the positive directions of r, θ, z if they are unknown.
- Identify the unknowns in the problem.

Equations of Motion.

- Apply the equations of motion, Eq. 13–8.

Kinematics.

- Use the methods of Sec. 12.8 to determine r and the time derivatives $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, $\ddot{\theta}$, $\ddot{z}$, and then evaluate the acceleration components $a_r = \ddot{r} - r\dot{\theta}^2$, $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$, $a_z = \ddot{z}$.
- When taking the time derivatives of $r = f(\theta)$, it is very important to use the chain rule, which is discussed in Appendix C.

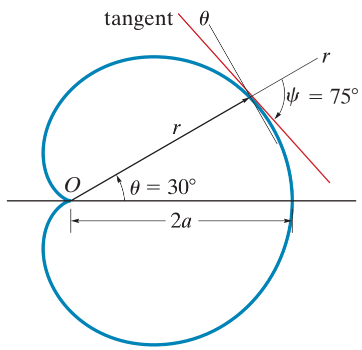
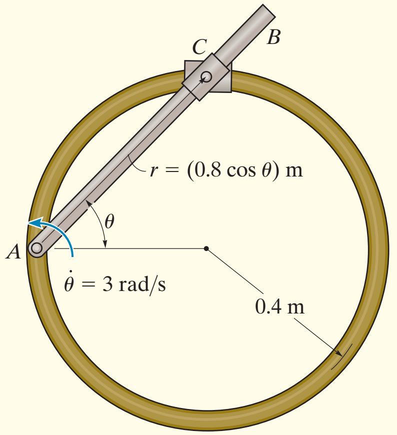


Fig. 13–18

13

EXAMPLE 13.10



(a)

The smooth 0.5-kg double-collar in Fig. 13–19*a* can freely slide on arm *AB* and the circular guide rod. If the arm rotates with a constant angular velocity of $\dot{\theta} = 3$ rad/s, determine the force the arm exerts on the collar at the instant $\theta = 45°$. Motion is in the horizontal plane.

SOLUTION

Free-Body Diagram. Only the normal reaction $\mathbf{N}_C$ of the guide rod and the force **F** of arm *AB* act on the collar, Fig. 13–19*b*. Here **F** acts perpendicular to arm *AB*, that is, in the direction of the θ axis, while $\mathbf{N}_C$ acts perpendicular to the tangent of the circular path at $\theta = 45°$. The acceleration components $\mathbf{a}_r$ and $\mathbf{a}_\theta$ are assumed to act in the positive r and θ directions. The four unknowns are N_C, F, a_r, a_θ.

Equations of Motion.

$$+\nearrow \Sigma F_r = ma_r; \qquad -N_C \cos 45° = (0.5 \text{ kg})\, a_r \qquad (1)$$

$$+\nwarrow \Sigma F_\theta = ma_\theta; \qquad F - N_C \sin 45° = (0.5 \text{ kg})\, a_\theta \qquad (2)$$

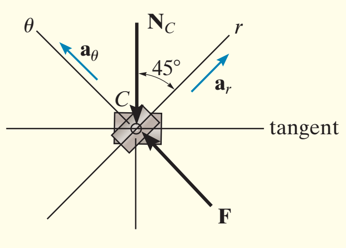
Kinematics. Using the chain rule (see Appendix C), the first and second time derivatives of r when $\theta = 45°$, $\dot{\theta} = 3$ rad/s, $\ddot{\theta} = 0$, are

$$r = 0.8 \cos \theta = 0.8 \cos 45° = 0.5657 \text{ m}$$

$$\dot{r} = -0.8 \sin \theta\, \dot{\theta} = -0.8 \sin 45°(3) = -1.6971 \text{ m/s}$$

$$\ddot{r} = -0.8\left[\sin \theta\, \ddot{\theta} + \cos \theta\, \dot{\theta}^2\right]$$

$$= -0.8[\sin 45°(0) + \cos 45°(3^2)] = -5.091 \text{ m/s}^2$$



(b)

Fig. 13–19

We have

$$a_r = \ddot{r} - r\dot{\theta}^2 = -5.091 \text{ m/s}^2 - (0.5657 \text{ m})(3 \text{ rad/s})^2 = -10.18 \text{ m/s}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = (0.5657 \text{ m})(0) + 2(-1.6971 \text{ m/s})(3 \text{ rad/s})$$

$$= -10.18 \text{ m/s}^2$$

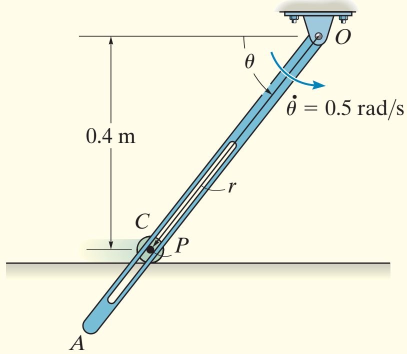
Substituting these results into Eqs. 1 and 2 and solving, we get

$$N_C = 7.20 \text{ N}$$

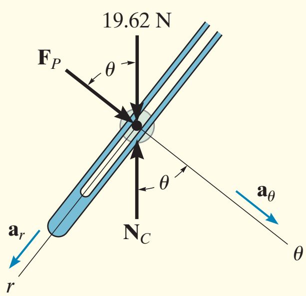
$$F = 0$$ *Ans.*

EXAMPLE 13.11

The smooth 2-kg cylinder C in Fig. 13–20*a* has a pin P through its center which passes through the slot in arm OA. If the arm is forced to rotate in the *vertical plane* at a constant rate $\dot{\theta} = 0.5$ rad/s, determine the force that the arm exerts on the peg at the instant $\theta = 60°$.



(a)



(b)

Fig. 13–20

SOLUTION

Why is it a good idea to use polar coordinates to solve this problem?

Free-Body Diagram. The free-body diagram for the cylinder is shown in Fig. 13–20*b*. The force on the peg, $\mathbf{F}_P$, acts perpendicular to the slot in the arm. As usual, $\mathbf{a}_r$ and $\mathbf{a}_\theta$ are assumed to act in the directions of *positive* r and θ, respectively. Identify the four unknowns.

Equations of Motion. Using the data in Fig. 13–20*b*, we have

$$+\swarrow \Sigma F_r = ma_r; \quad 19.62 \sin\theta - N_C \sin\theta = 2a_r \qquad (1)$$

$$+\searrow \Sigma F_\theta = ma_\theta; \quad 19.62 \cos\theta + F_P - N_C \cos\theta = 2a_\theta \qquad (2)$$

Kinematics. From Fig. 13–20a, r can be related to θ by the equation

$$r = \frac{0.4}{\sin\theta} = 0.4 \csc\theta$$

Since $d(\csc\theta) = -(\csc\theta \cot\theta)\, d\theta$ and $d(\cot\theta) = -(\csc^2\theta)\, d\theta$, then r and the necessary time derivatives become

$$\dot{\theta} = 0.5 \qquad r = 0.4 \csc\theta$$

$$\ddot{\theta} = 0 \qquad \dot{r} = -0.4(\csc\theta \cot\theta)\dot{\theta}$$

$$= -0.2 \csc\theta \cot\theta$$

$$\ddot{r} = -0.2(-\csc\theta \cot\theta)(\dot{\theta}) \cot\theta - 0.2 \csc\theta(-\csc^2\theta)\dot{\theta}$$

$$= 0.1 \csc\theta(\cot^2\theta + \csc^2\theta)$$

Evaluating these formulas at $\theta = 60°$, we get

$$\dot{\theta} = 0.5 \qquad r = 0.462$$

$$\ddot{\theta} = 0 \qquad \dot{r} = -0.133$$

$$\ddot{r} = 0.192$$

$$a_r = \ddot{r} - r\dot{\theta}^2 = 0.192 - 0.462(0.5)^2 = 0.0770$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 + 2(-0.133)(0.5) = -0.133$$

Substituting these results into Eqs. 1 and 2 with $\theta = 60°$ and solving yields

$$N_C = 19.4 \text{ N} \qquad F_P = -0.356 \text{ N} \qquad \textit{Ans.}$$

The negative sign indicates that $\mathbf{F}_P$ acts opposite to the direction shown in Fig. 13–20*b*.

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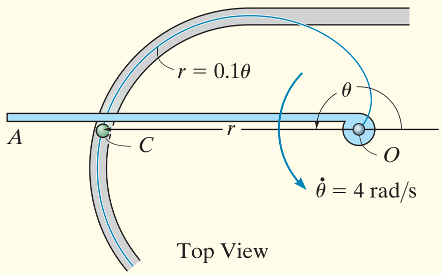
EXAMPLE 13.12

A can C, having a mass of 0.5 kg, moves along a grooved horizontal slot shown in Fig. 13–21*a*. The slot is in the form of a spiral, which is defined by the equation $r = (0.1\theta)$ m, where θ is in radians. If the arm OA rotates at a constant rate $\dot{\theta} = 4$ rad/s, determine the force it exerts on the can at the instant $\theta = \pi$ rad. Neglect friction and the size of the can.



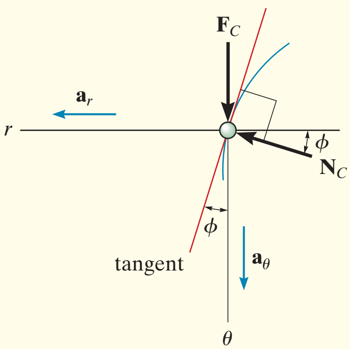
(a)

SOLUTION

Free-Body Diagram. The driving force $\mathbf{F}_C$ acts perpendicular to the arm OA, whereas the normal force of the slot on the can, $\mathbf{N}_C$, acts perpendicular to the tangent to the curve at $\theta = \pi$ rad, Fig. 13–21*b*. As usual, $\mathbf{a}_r$ and $\mathbf{a}_\theta$ are assumed to act in the *positive directions* of r and θ. Since the path is specified, the angle ψ which the extended radial line r makes with the tangent, Fig. 13–21*c*, can be determined from Eq. 13–9. We have $r = 0.1\theta$, so that $dr/d\theta = 0.1$, and therefore

$$\tan\psi = \frac{r}{dr/d\theta} = \frac{0.1\theta}{0.1} = \theta$$

When $\theta = \pi$, $\psi = \tan^{-1}\pi = 72.3^\circ$, so that $\phi = 90^\circ - \psi = 17.7^\circ$, as shown in Fig. 13–21*c*. Identify the four unknowns in Fig. 13–21*b*.



(b)

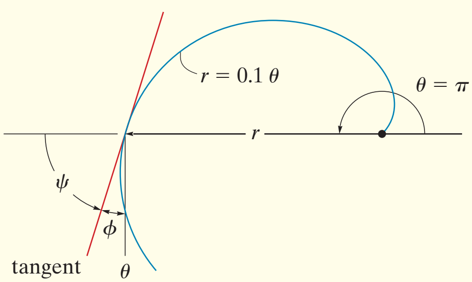
Equations of Motion. Using $\phi = 17.7^\circ$ and the data shown in Fig. 13–21*b*, we have

$$\xrightarrow{+} \Sigma F_r = ma_r; \qquad N_C \cos 17.7^\circ = 0.5a_r \qquad (1)$$

$$+\downarrow \Sigma F_\theta = ma_\theta; \qquad F_C - N_C \sin 17.7^\circ = 0.5a_\theta \qquad (2)$$

Kinematics. The time derivatives of r and θ are

$$\dot{\theta} = 4 \text{ rad/s} \qquad r = 0.1\theta$$
$$\ddot{\theta} = 0 \qquad \dot{r} = 0.1\dot{\theta} = 0.1(4) = 0.4 \text{ m/s}$$
$$\ddot{r} = 0.1\ddot{\theta} = 0$$



(c)

Fig. 13–21

At the instant $\theta = \pi$ rad,

$$a_r = \ddot{r} - r\dot{\theta}^2 = 0 - 0.1(\pi)(4)^2 = -5.03 \text{ m/s}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 + 2(0.4)(4) = 3.20 \text{ m/s}^2$$

Substituting these results into Eqs. 1 and 2 and solving yields

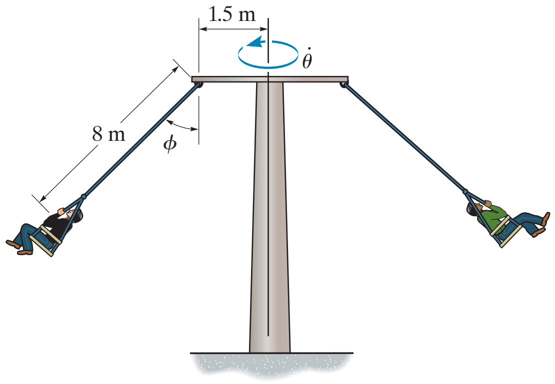
$$N_C = -2.64 \text{ N}$$

$$F_C = 0.800 \text{ N} \qquad \textit{Ans.}$$

What does the negative sign for N_C indicate?

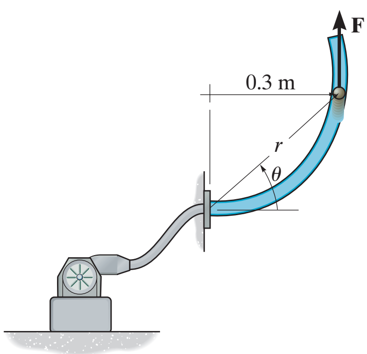
FUNDAMENTAL PROBLEMS

F13–13. Determine the constant angular velocity $\dot{\theta}$ of the vertical shaft of the amusement ride if $\phi = 45^\circ$. Neglect the mass of the cables and the size of the passengers.



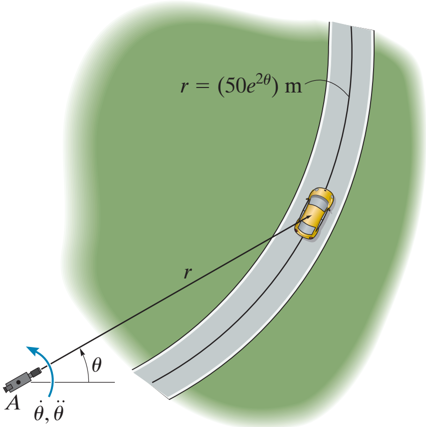
Prob. F13–13

F13–14. The 0.2-kg ball is blown through the smooth vertical circular tube whose shape is defined by $r = (0.6 \sin \theta)$ m, where θ is in radians. If $\theta = (\pi t^2)$ rad, where t is in seconds, determine the magnitude of force **F** exerted by the blower on the ball when $t = 0.5$ s.



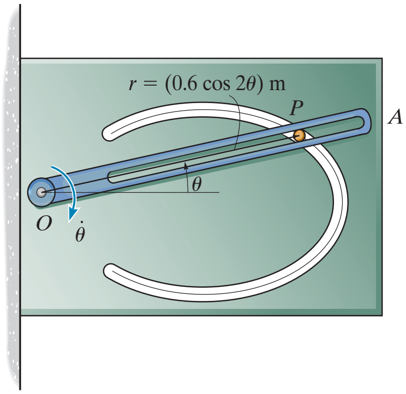
Prob. F13–14

F13–15. The 2-Mg car is traveling along the curved road described by $r = (50e^{2\theta})$ m, where θ is in radians. If a camera is located at A and it rotates with an angular velocity of $\dot{\theta} = 0.05$ rad/s and an angular acceleration of $\ddot{\theta} = 0.01$ rad/s^2 at the instant $\theta = \frac{\pi}{6}$ rad, determine the resultant friction force developed between the tires and the road at this instant.



Prob. F13–15

F13–16. The 0.2-kg pin P is constrained to move in the smooth curved slot, which is defined by the lemniscate $r = (0.6 \cos 2\theta)$ m. Its motion is controlled by the rotation of the slotted arm OA, which has a constant clockwise angular velocity of $\dot{\theta} = -3$ rad/s. Determine the force arm OA exerts on the pin P when $\theta = 0^\circ$. Motion is in the vertical plane.



Prob. F13–16

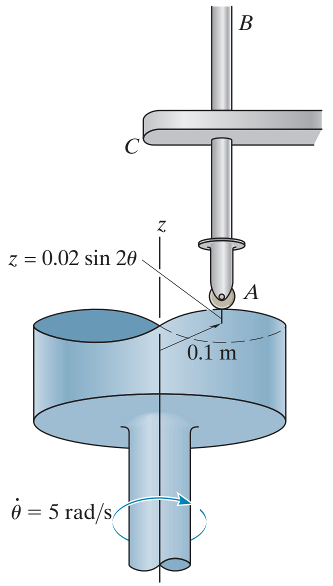
13

PROBLEMS

All solutions must include a free-body diagram.

13–85. The path of motion of a 2.5-kg particle in the horizontal plane is described in terms of polar coordinates as $r = (0.6t + 0.3)$ m and $\theta = (0.5t^2 - t)$ rad, where t is in seconds. Determine the magnitude of the unbalanced force acting on the particle when $t = 2$ s.

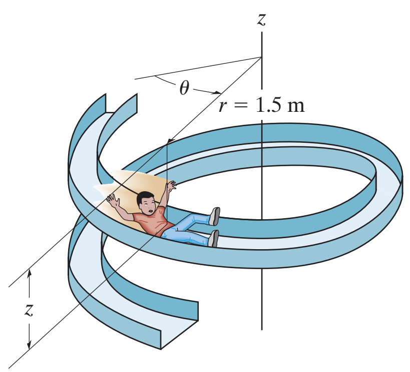
13–86. The 2-kg rod AB moves up and down as its end slides on the smooth contoured surface of the cam, where $r = 0.1$ m and $z = (0.02 \sin 2\theta)$ m. If the cam is rotating at a constant rate of 5 rad/s, determine the maximum and minimum force the cam exerts on the rod.



Prob. 13–86

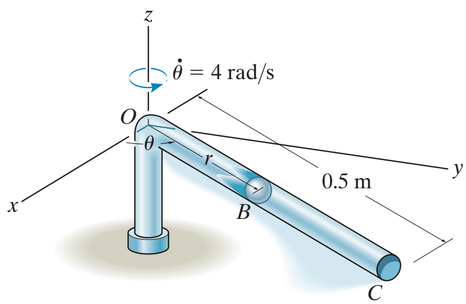
13–87. Determine the magnitude of the resultant force acting on a 5-kg particle at the instant $t = 2$ s, if the particle is moving along a horizontal path defined by the equations $r = (2t + 10)$ m and $\theta = (1.5t^2 - 6t)$ rad, where t is in seconds.

***13–88.** The boy of mass 40 kg is sliding down the spiral slide at a constant speed such that his position, measured from the top of the chute, has components $r = 1.5$ m, $\theta = (0.7t)$ rad, and $z = (-0.5t)$ m, where t is in seconds. Determine the components of force $\mathbf{F}_r$, $\mathbf{F}_\theta$, and $\mathbf{F}_z$ which the slide exerts on him at the instant $t = 2$ s. Neglect the size of the boy.



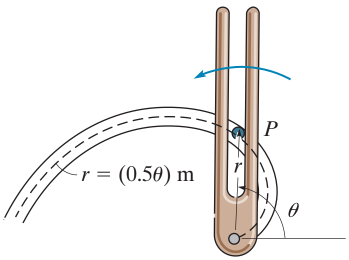
Prob. 13–88

13–89. The tube rotates in the horizontal plane at a constant rate of $\dot{\theta} = 4$ rad/s. If a 0.2-kg ball B starts at the origin O with an initial radial velocity of $\dot{r} = 1.5$ m/s and moves outward through the tube, determine the radial and transverse components of the ball's velocity at the instant it leaves the outer end at C, $r = 0.5$ m. *Hint*: Show that the equation of motion in the r direction is $\ddot{r} - 16r = 0$. The solution is of the form $r = Ae^{-4t} + Be^{4t}$. Evaluate the integration constants A and B, and determine the time t when $r = 0.5$ m. Proceed to obtain v_r and v_θ.



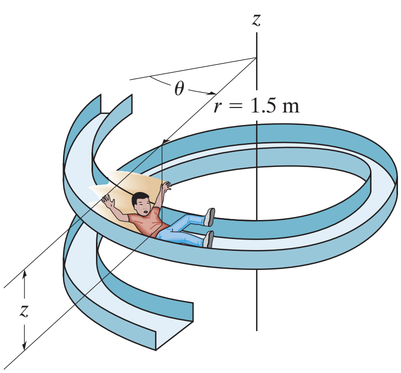
Prob. 13–89

13–90. Using a forked rod, a 0.5-kg smooth peg P is forced to move along the *vertical slotted* path $r = (0.5\theta)$ m, where θ is in radians. If the angular position of the arm is $\theta = (\frac{\pi}{8}t^2)$ rad, where t is in seconds, determine the force of the rod on the peg and the normal force of the slot on the peg at the instant $t = 2$ s. The peg is in contact with only *one edge* of the rod and slot at any instant.



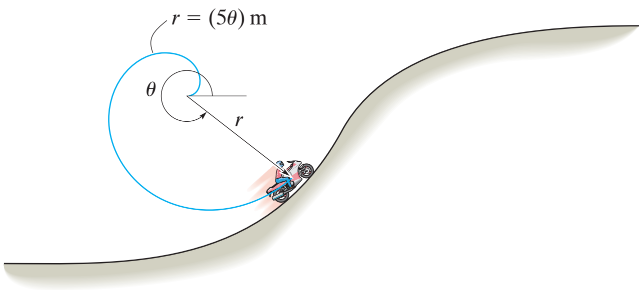
Prob. 13–90

13–91. The 40-kg boy is sliding down the smooth spiral slide such that $z = -2$ m when he rotates $\theta = 360°$ and his speed is 2 m/s. Determine the r, θ, z components of force the slide exerts on him at this instant. Neglect the size of the boy.



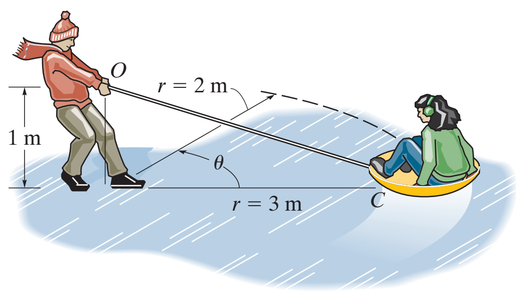
Prob. 13–91

***13–92.** Determine the normal and frictional driving forces that the partial spiral track exerts on the 200-kg motorcycle at the instant $\theta = \frac{5}{3}\pi$ rad, $\dot{\theta} = 0.4$ rad/s, $\ddot{\theta} = 0.8$ rad/s^2. Neglect the size of the motorcycle.



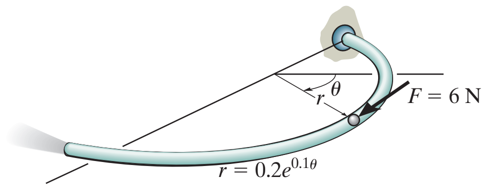
Prob. 13–92

13–93. A boy standing firmly spins the girl sitting on a circular "dish" or sled in a circular path of radius $r_0 = 3$ m such that her angular velocity is $\dot{\theta}_0 = 0.1$ rad/s. If the attached cable OC is drawn inward such that the radial coordinate r changes with a constant speed of $\dot{r} = -0.5$ m/s, determine the tension it exerts on the sled at the instant $r = 2$ m. The sled and girl have a total mass of 50 kg. Neglect the size of the girl and sled and the effects of friction between the sled and ice. *Hint*: First show that the equation of motion in the θ direction yields $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = (1/r)\,d/dt\,(r^2\dot{\theta}) = 0$. When integrated, $r^2\dot{\theta} = C$, where the constant C is determined from the problem data.



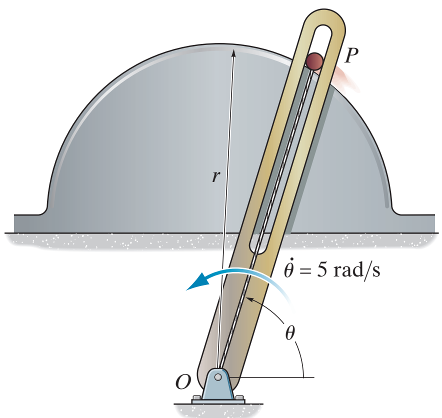
Prob. 13–93

13–94. Using air pressure, the 0.5-kg ball is forced to move through the tube lying in the horizontal plane and having the shape of a logarithmic spiral. If the tangential force exerted on the ball due to air pressure is 6 N, determine the rate of increase in the ball's speed at the instant $\theta = \pi/2$. Also, what is the angle ψ from the extended radial coordinate r to the line of action of the 6-N force?



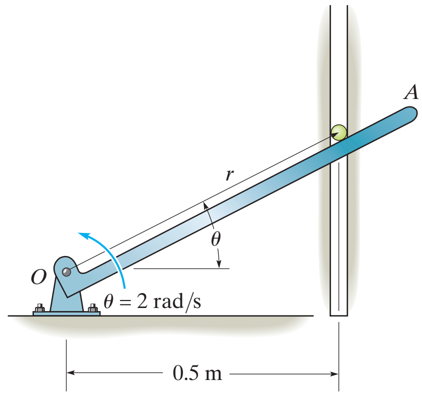
Prob. 13–94

13–95. The smooth pin P has a mass of 80 g. It is attached to an elastic cord extending from O to P and, due to the slotted arm guide, moves along the *horizontal* circular path $r = (0.8 \sin \theta)$ m. If the cord has a stiffness $k = 30$ kN/m and an unstretched length of 0.25 m, determine the force of the guide and the normal force of the circular path on the pin when $\theta = 60°$. The guide has a constant angular velocity $\dot{\theta} = 5$ rad/s.



Prob. 13–95

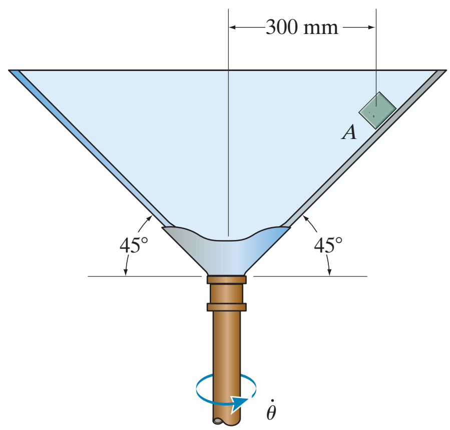
***13–96.** The particle has a mass of 0.5 kg and is confined to move along the smooth vertical slot due to the rotation of the arm OA. Determine the force of the rod on the particle and the normal force of the slot on the particle when $\theta = 30°$. The rod is rotating with a constant angular velocity $\dot{\theta} = 2$ rad/s. Assume the particle contacts only one side of the slot at any instant.



Prob. 13–96

13–97. If the coefficient of static friction between the conical surface and the block of mass m is $\mu_s = 0.2$, determine the minimum constant angular velocity $\dot{\theta}$ so that the block does not slide downwards.

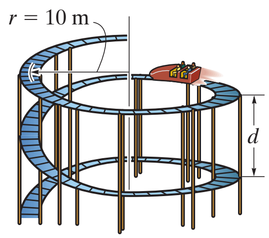
13–98. If the coefficient of static friction between the conical surface and the block is $\mu_s = 0.2$, determine the maximum constant angular velocity $\dot{\theta}$ without causing the block to slide upwards.



Probs. 13–97/98

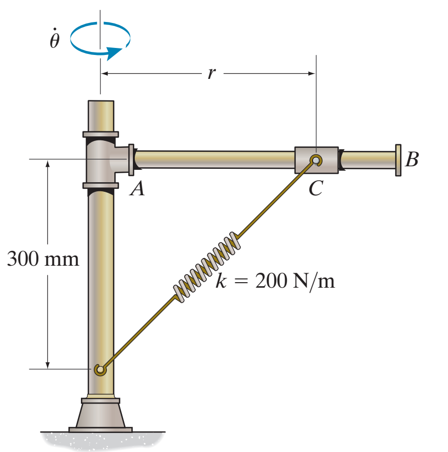
13–99. For a short time, the 250-kg roller-coaster car with passengers is traveling along the spiral track at a constant speed of $v = 8$ m/s. If the track descends $d = 12$ m for every full revolution, $\theta = 2\pi$ rad, determine the magnitudes of the components of force which the track exerts on the car in the r, θ, and z directions. Neglect the size of the car.

***13–100.** For a short time, the 250-kg roller-coaster car with passengers is traveling along the spiral track at a constant speed such that its position measured from the top of the track has components $r = 10$ m, $\theta = (0.2t)$ rad, and $z = (-0.3t)$ m, where t is in seconds. Determine the magnitudes of the components of force which the track exerts on the car in the r, θ, and z directions at the instant $t = 2$ s. Neglect the size of the car.



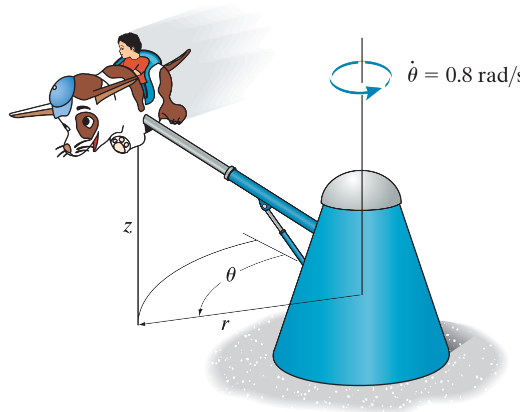
Probs. 13–99/100

13–101. If the position of the 3-kg collar C on the smooth rod AB is held at $r = 720$ mm, determine the constant angular velocity $\dot{\theta}$ at which the mechanism is rotating about the vertical axis. The spring has an unstretched length of 400 mm. Neglect the mass of the rod and the size of the collar.



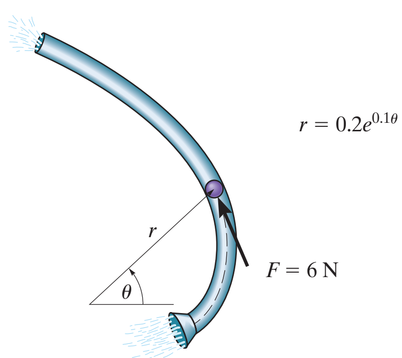
Prob. 13–101

13–102. The amusement park ride rotates with a constant angular velocity of $\dot{\theta} = 0.8$ rad/s. If the path of the ride is defined by $r = (3 \sin \theta + 5)$ m and $z = (3 \cos \theta)$ m, determine the r, θ, and z components of force exerted by the seat on the 20-kg boy when $\theta = 120°$.



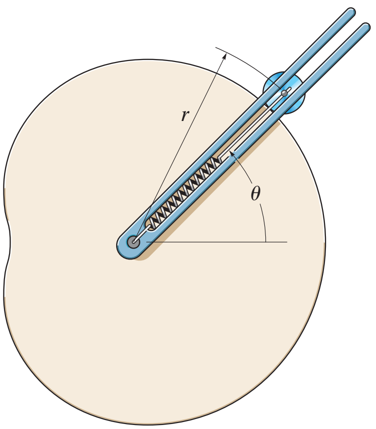
Prob. 13–102

13–103. Using air pressure, the 0.5-kg ball is forced to move through the tube lying in the *horizontal plane* and having the shape of a logarithmic spiral. If the tangential force exerted on the ball due to the air is 6 N, determine the rate of increase in the ball's speed at the instant $\theta = \pi/2$. What direction does the tangential force act in?



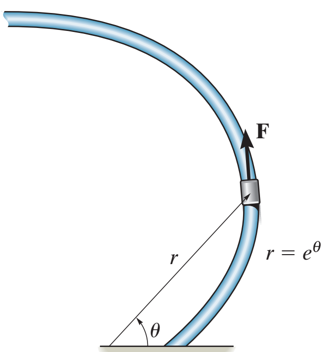
Prob. 13–103

***13–104.** The smooth surface of the vertical cam is defined in part by the curve $r = (0.2 \cos \theta + 0.3)$ m. The forked rod is rotating with an angular acceleration of $\ddot{\theta} = 2 \text{ rad/s}^2$, and when $\theta = 45^\circ$, the angular velocity is $\dot{\theta} = 6$ rad/s. Determine the force the cam and the rod exert on the 2-kg roller at this instant. The attached spring has a stiffnesss $k = 100$ N/m and an unstretched length of 0.1 m.



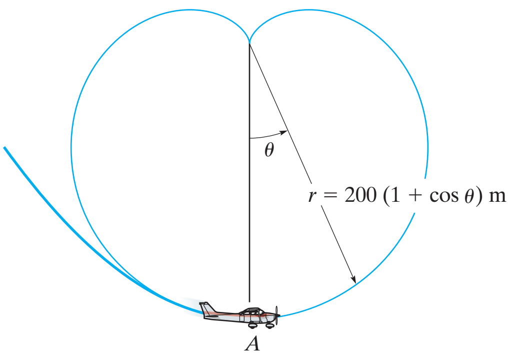
Prob. 13–104

13–105. The collar has a mass of 2 kg and travels along the smooth horizontal rod defined by the equiangular spiral $r = (e^{\theta})$ m, where θ is in radians. Determine the tangential force F and the normal force N acting on the collar when $\theta = 45^\circ$, if the force F maintains a constant angular motion $\dot{\theta} = 2$ rad/s.



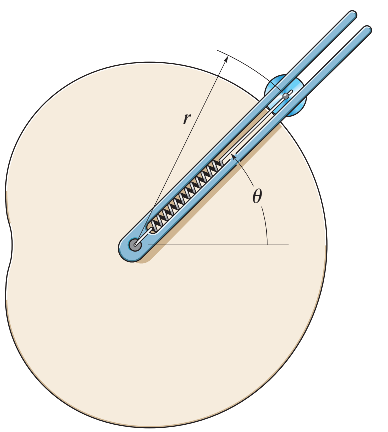
Prob. 13–105

13–106. The pilot of the airplane executes a vertical loop which in part follows the path of a cardioid, $r = 200(1 + \cos \theta)$ m, where θ is in radians. If his speed at A is a constant $v_p = 85$ m/s, determine the vertical reaction the seat of the plane exerts on the pilot when the plane is at A. He has a mass of 80 kg. *Hint:* To determine the time derivatives necessary to calculate the acceleration components a_r and a_θ, take the first and second time derivatives of $r = 200(1 + \cos \theta)$. Then, for further information, use Eq. 12–26 to determine $\dot{\theta}$.



Prob. 13–106

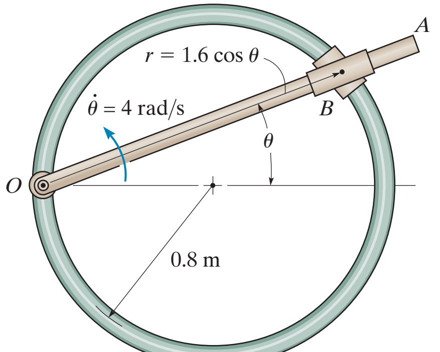
13–107. The smooth surface of the vertical cam is defined in part by the curve $r = (0.2 \cos \theta + 0.3)$ m. If the forked rod is rotating with a constant angular velocity of $\dot{\theta} = 4$ rad/s, determine the force the cam and the rod exert on the 2-kg roller when $\theta = 30^\circ$. The attached spring has a stiffnesss $k = 30$ N/m and an unstretched length of 0.1 m.



Prob. 13–107

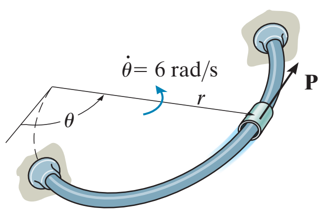
***13–108.** Rod OA rotates counterclockwise at a constant angular rate $\dot{\theta} = 4$ rad/s. The double collar B is pin connected together such that one collar slides over the rotating rod and the other collar slides over the circular rod described by the equation $r = (1.6 \cos \theta)$ m. If *both* collars have a mass of 0.5 kg, determine the force which the circular rod exerts on one of the collars and the force that OA exerts on the other collar at the instant $\theta = 45°$. Motion is in the horizontal plane.

13–109. Solve Prob. 13–108 if motion is in the vertical plane.



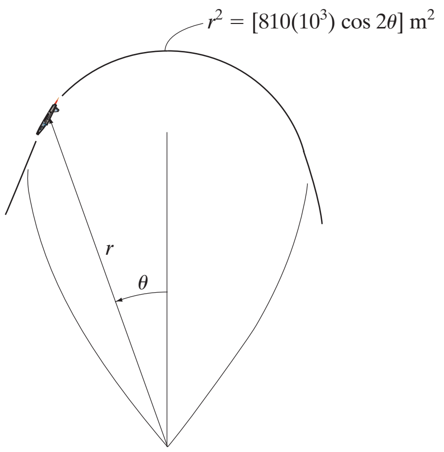
Probs. 13–108/109

13–110. The spool, which has a mass of 2 kg, slides along the smooth *horizontal* spiral rod, $r = (0.4\theta)$ m, where θ is in radians. If its angular rate of rotation is constant and equals $\dot{\theta} = 6$ rad/s, determine the horizontal tangential force P needed to cause the motion, and the horizontal normal force component that the spool exerts on the rod at the instant $\theta = 45°$.



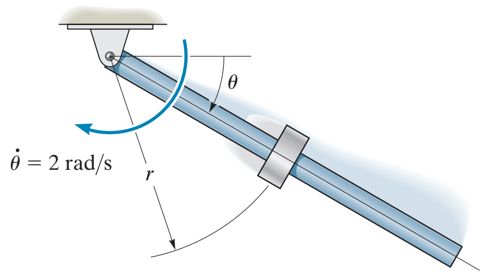
Prob. 13–110

13–111. The airplane executes the vertical loop defined by $r^2 = [810(10^3)\cos 2\theta]$ m^2. If the pilot maintains a constant speed $v = 120$ m/s along the path, determine the normal force the seat exerts on him at the instant $\theta = 0°$. The pilot has a mass of 75 kg.

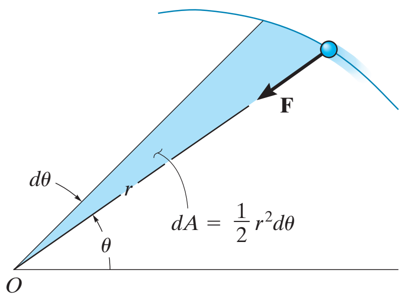


Prob. 13–111

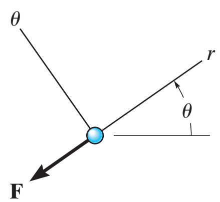
***13–112.** A 0.2-kg spool slides down along a smooth rod. If the rod has a constant angular rate of rotation $\dot{\theta} = 2$ rad/s in the vertical plane, show that the equations of motion for the spool are $\ddot{r} - 4r - 9.81 \sin \theta = 0$ and $0.8\dot{r} + N_s - 1.962 \cos \theta = 0$, where N_s is the magnitude of the normal force of the rod on the spool. Using the methods of differential equations, it can be shown that the solution of the first of these equations is $r = C_1 e^{-2t} + C_2 e^{2t} - (9.81/8) \sin 2t$. If r, $\dot{r}$, and θ are zero when $t = 0$, evaluate the constants C_1 and C_2 and determine r at the instant $\theta = \pi/4$ rad.



Prob. 13–112



(a)



(b)

Fig. 13–22

*13.7 CENTRAL-FORCE MOTION AND SPACE MECHANICS

If a particle is moving only under the influence of a force **F** which is always directed toward a fixed point, as in Fig. 13–22*a*, the motion is called ***central-force motion***. This type of motion is commonly caused by electrostatic and gravitational forces.

For this case, the free-body diagram for the particle is shown in Fig. 13–22*b*. Using polar coordinates (r, θ), the equations of motion, Eq. 13–8, become

$$\Sigma F_r = ma_r; \qquad -F = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] \qquad (13\text{–}10)$$

$$\Sigma F_\theta = ma_\theta; \qquad 0 = m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)$$

The second of these equations may be written in the form

$$\frac{1}{r}\left[\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right] = 0$$

so that integrating yields

$$r^2\frac{d\theta}{dt} = h \qquad (13\text{–}11)$$

Here h is the constant of integration.

From Fig. 13–22*a* notice that the shaded area described by the radius r, as r moves through an angle $d\theta$, is a triangle having two sides r, and, neglecting higher order derivatives, a base of $r\,d\theta$. The area of this triangle is therefore $dA = \frac{1}{2}r^2\,d\theta$. If the ***areal velocity*** is defined as

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{h}{2} \qquad (13\text{–}12)$$

then it is seen that this term is *constant*. In other words, the particle will sweep out equal segments of area per unit of time as it travels along the path.

To obtain the *path of motion*, $r = f(\theta)$, the independent variable t must be eliminated from Eq. 13–10. Using the chain rule, the time derivative d^2r/dt^2 may be replaced by

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{dr}{d\theta}$$

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right) = \frac{d}{d\theta}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right)\frac{d\theta}{dt} = \left[\frac{d}{d\theta}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right)\right]\frac{h}{r^2}$$

Substituting a new dependent variable (xi) $\xi = 1/r$ into this equation, we have

$$\frac{d^2r}{dt^2} = -h^2\xi^2\frac{d^2\xi}{d\theta^2}$$

Also, the square of Eq. 13–11 becomes

$$\left(\frac{d\theta}{dt}\right)^2 = h^2\xi^4$$

Finally, substituting these two equations into Eq. 13–10 we get

$$-h^2\xi^2\frac{d^2\xi}{d\theta^2} - h^2\xi^3 = -\frac{F}{m}$$

or

$$\frac{d^2\xi}{d\theta^2} + \xi = \frac{F}{mh^2\xi^2} \qquad (13\text{–}13)$$

This differential equation defines the path over which the particle travels when it is subjected to the central force **F**.

To illustrate a common application, consider the trajectory of a space satellite or space vehicle launched into free-flight orbit with an initial velocity $\mathbf{v}_0$, Fig. 13–23. Here, it will be assumed that this velocity is initially *parallel* to the tangent at the surface of the earth, as shown in the figure.* Also, just after the satellite is released, the only gravitational force of any sizable magnitude will be the gravitational force of the earth. According to Newton's law of gravitation, this force will always act between the mass centers of the earth and the satellite, Fig. 13–23, and from Eq. 13–1, this force has a magnitude of

$$F = G\frac{M_e m}{r^2}$$

where M_e and m represent the mass of the earth and the satellite, respectively, G is the gravitational constant, and r is the distance between

This satellite is subjected to a central force and its orbital motion can be closely predicted using the equations developed in this section. (Universal ImagesGroup/ Getty Images)

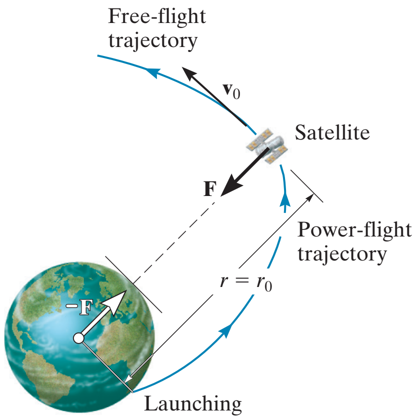


Fig. 13–23

* The case where $\mathbf{v}_0$ acts at some initial angle θ to the tangent is best described using the conservation of angular momentum. See Prob. 15–112.

the mass centers. To obtain the orbital path, we set $\xi = 1/r$ in the foregoing equation and substitute the result into Eq. 13–13. This yields

$$\frac{d^2\xi}{d\theta^2} + \xi = \frac{GM_e}{h^2} \tag{13–14}$$

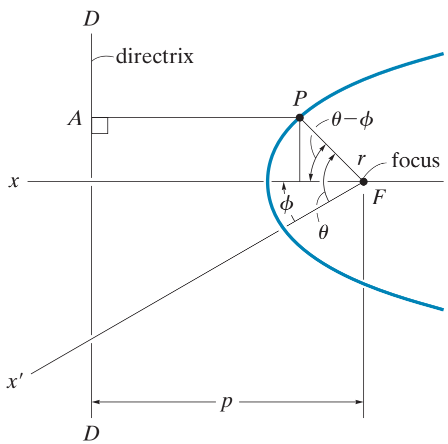


Fig. 13–24

The solution of this second-order differential equation is the sum of its complementary and particular solutions given by

$$\xi = \frac{1}{r} = C\cos(\theta - \phi) + \frac{GM_e}{h^2} \tag{13–15}$$

where C and ϕ are constants of integration.

When graphed it represents the equation of a conic section, expressed in terms of polar coordinates. As shown in Fig. 13–24, a conic section is defined as the locus of a point P that moves in such a way that the ratio of its distance FP from a ***focus***, or fixed point F, to its perpendicular distance PA measured to the fixed line DD called the ***directrix***, is constant. This constant ratio will be denoted as e and is called the ***eccentricity***. Therefore,

$$e = \frac{FP}{PA}$$

From Fig. 13–24,

$$FP = r = e(PA) = e[\,p - r\cos(\theta - \phi)]$$

or

$$\frac{1}{r} = \frac{1}{p}\cos(\theta - \phi) + \frac{1}{ep}$$

Comparing this equation with Eq. 13–15, it is seen that the fixed distance from the focus to the directrix is

$$p = \frac{1}{C} \tag{13–16}$$

And the eccentricity of the conic section is

$$\boxed{e = \frac{Ch^2}{GM_e}} \tag{13–17}$$

When the polar angle θ in Fig. 13–24 is measured from the x axis rather than the x' axis, then the angle ϕ is zero, and therefore Eq. 13–15 reduces to

$$\frac{1}{r} = C\cos\theta + \frac{GM_e}{h^2} \tag{13–18}$$

The constants h and C can be determined from the initial values of the position and velocity of the satellite just after the end of its *power-flight trajectory*. For example, if the initial height or distance to the space vehicle is r_0, measured from the center of the earth, and its initial speed is v_0 at the beginning of its free flight, Fig. 13–25, then the constant h may be obtained from Eq. 13–11. When $\theta = \phi = 0°$, the velocity $\mathbf{v}_0$ has no radial component; therefore, from Eq. 12–25, $v_0 = r_0(d\theta/dt)$, so that

$$h = r_0^2\frac{d\theta}{dt}$$

or

$$\boxed{h = r_0 v_0} \tag{13–19}$$

To determine C, use Eq. 13–18 with $\theta = 0°$, $r = r_0$, and substitute Eq. 13–19 for h. We get

$$\boxed{C = \frac{1}{r_0}\left(1 - \frac{GM_e}{r_0 v_0^2}\right)} \tag{13–20}$$

The equation for the free-flight trajectory therefore becomes

$$\boxed{\frac{1}{r} = \frac{1}{r_0}\left(1 - \frac{GM_e}{r_0 v_0^2}\right)\cos\theta + \frac{GM_e}{r_0^2 v_0^2}} \tag{13–21}$$

The type of path traveled by the satellite is determined from the value of the eccentricity of the conic section as given by Eq. 13–17. If

$$\boxed{\begin{array}{ll} e = 0 & \text{free-flight trajectory is a circle} \\ e = 1 & \text{free-flight trajectory is a parabola} \\ e < 1 & \text{free-flight trajectory is an ellipse} \\ e > 1 & \text{free-flight trajectory is a hyperbola} \end{array}} \tag{13–22}$$

Each of these trajectories is shown in Fig. 13–25.

13

Parabolic Path. It is seen that when the satellite follows a hyperbolic orbit, $e > 1$, it will never return to its starting point. When $e = 1$, the orbit is parabolic, and it is "on the border" of never returning. This launch velocity, $\mathbf{v}_0$, is called the ***escape velocity***, v_e. It can be determined by using the second of Eqs. 13–22, $e = 1$, with Eqs. 13–17, 13–19, and 13–20. It is left as an exercise to show that

$$v_e = \sqrt{\frac{2GM_e}{r_0}} \tag{13–23}$$

Circular Orbit. The speed v_c required to launch a satellite into a circular orbit can be found using the first of Eqs. 13–22, $e = 0$. Since e is related to h and C, Eq. 13–17, C must be zero to satisfy this equation (from Eq. 13–19, h cannot be zero); and therefore, using Eq. 13–20, we have

$$v_c = \sqrt{\frac{GM_e}{r_0}} \tag{13–24}$$

Any speed at launch which is less than v_c will cause the satellite to reenter the earth's atmosphere and either burn up or crash, Fig. 13–25.

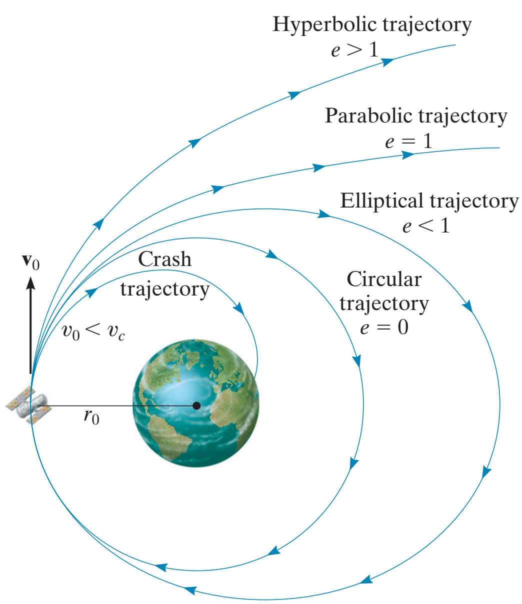


Fig. 13–25

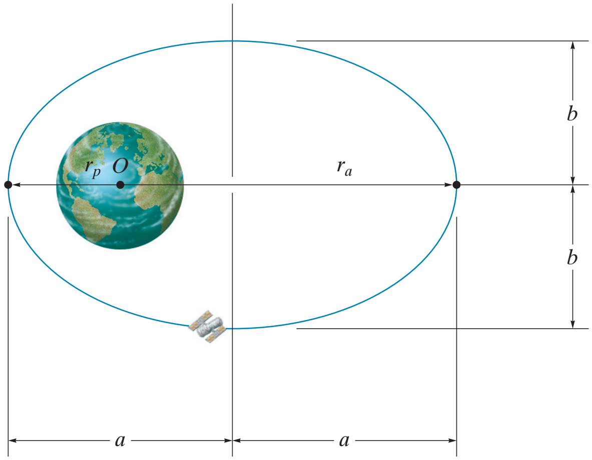


Fig. 13–26

Elliptical Orbit. All the trajectories attained by planets and most satellites are elliptical, Fig. 13–26. For a satellite, the *minimum distance* from the orbit to the center of the earth O (which is located at one of the foci of the ellipse) is r_p and can be found using Eq. 13–21 with $\theta = 0°$. Therefore

$$r_p = r_0 \tag{13–25}$$

This distance is called the ***perigee*** of the orbit. The ***apogee*** or maximum distance r_a can be found using Eq. 13–21 with $\theta = 180°$.* Thus,

$$r_a = \frac{r_0}{(2GM_e/r_0 v_0^2) - 1} \tag{13–26}$$

From Fig. 13–26, the half-length of the major axis of the ellipse is

$$a = \frac{r_p + r_a}{2} \tag{13–27}$$

Using analytical geometry, it can be shown that the half-length of the minor axis is determined from the equation

$$b = \sqrt{r_p r_a} \tag{13–28}$$

* Actually, the terminology perigee and apogee pertains only to orbits about the *earth*. If any other heavenly body is located at the focus of its elliptical orbit, the minimum and maximum distances are referred to respectively as the ***periapsis*** and ***apoapsis*** of the orbit.

Finally, by direct integration, the area of an ellipse is

$$A = \pi ab = \frac{\pi}{2}(r_p + r_a)\sqrt{r_p r_a} \tag{13–29}$$

and since the areal velocity has been defined by Eq. 13–12, $dA/dt = h/2$, then integrating yields $A = hT/2$, where T is the ***period*** of time required to make one orbital revolution. From Eq. 13–29, the period is therefore

$$T = \frac{\pi}{h}(r_p + r_a)\sqrt{r_p r_a} \tag{13–30}$$

In addition to predicting the orbital trajectory of earth satellites, the theory developed in this section is also valid, to a surprisingly close approximation, at predicting the actual motion of the planets orbiting the sun. In this case the mass of the sun, M_s, should be substituted for M_e when the appropriate formulas are used.

The fact that the planets do indeed follow elliptic orbits about the sun was discovered by the German astronomer Johannes Kepler in the early seventeenth century. Since his discovery was made *before* Newton had developed the laws of motion and the law of gravitation, it provided important proof as to the validity of these laws.

Kepler's three laws, developed after 20 years of planetary observation, are summarized as follows:

1. Every planet travels in its orbit such that the radial line measured to it from the center of the sun sweeps over equal areas in equal intervals of time.

2. The orbit of every planet is an ellipse with the sun placed at one of its foci.

3. The square of the period of any planet is directly proportional to the cube of the major axis of its orbit.

A mathematical statement of the first and second laws is given by Eqs. 13–12 and 13–21, respectively. The third law can be shown from Eq. 13–30 using Eqs. 13–18, 13–27, and 13–28. (See Prob. 13–113.)

EXAMPLE 13.13

A satellite is launched 600 km from the surface of the earth, with an initial velocity of 30 Mm/h acting parallel to the tangent at the surface of the earth, Fig. 13–27. Assuming that the radius of the earth is 6378 km and that its mass is 5.976(10^{24}) kg, determine (a) the eccentricity of the orbital path, and (b) the velocity of the satellite at apogee.

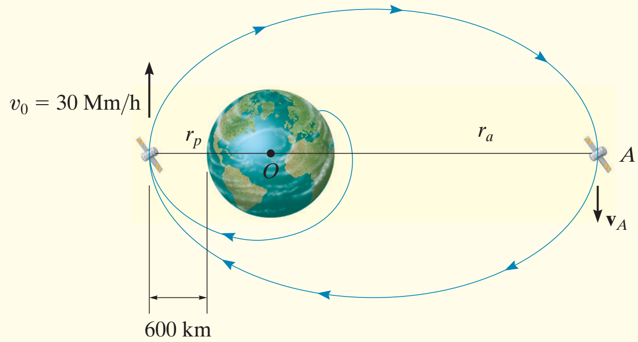


Fig. 13–27

SOLUTION

Part (a). The eccentricity of the orbit is obtained using Eq. 13–17. The constants h and C are first determined from Eq. 13–19 and 13–20. Since

$$r_p = r_0 = 6378 \text{ km} + 600 \text{ km} = 6.978(10^6) \text{ m}$$

$$v_0 = 30 \text{ Mm/h} = 8333.3 \text{ m/s}$$

then

$$h = r_p v_0 = 6.978(10^6)(8333.3) = 58.15(10^9) \text{ m}^2/\text{s}$$

$$C = \frac{1}{r_p}\left(1 - \frac{GM_e}{r_p v_0^2}\right)$$

$$= \frac{1}{6.978(10^6)}\left[1 - \frac{66.73(10^{-12})[5.976(10^{24})]}{6.978(10^6)(8333.3)^2}\right] = 25.4(10^{-19}) \text{ m}^{-1}$$

Hence,

$$e = \frac{Ch^2}{GM_e} = \frac{2.54(10^{-8})[58.15(10^9)]^2}{66.73(10^{-12})[5.976(10^{24})]} = 0.125 < 1 \quad \textit{Ans.}$$

From Eqs. 13-22, observe that the orbit is an *ellipse*.

Part (b). If the satellite were launched at the apogee A shown in Fig. 13–27, with a velocity $\mathbf{v}_A$, the same orbit would be maintained provided

$$h = r_p v_0 = r_a v_A = 58.15(10^9) \text{ m}^2/\text{s}$$

Using Eq. 13–26, we have

$$r_a = \frac{r_p}{\dfrac{2GM_e}{r_p v_0^2} - 1} = \frac{6.978(10^6)}{\dfrac{2[66.73(10^{-12})][5.976(10^{24})]}{6.978(10^6)(8333.3)^2} - 1} = 10.804(10^6) \text{ m}$$

Thus,

$$v_A = \frac{58.15(10^9)}{10.804(10^6)} = 5382.2 \text{ m/s} = 19.4 \text{ Mm/h} \qquad \textit{Ans.}$$

NOTE: The farther the satellite is from the earth, the slower it moves, which is to be expected since h is constant.

PROBLEMS

In the following problems, except where otherwise indicated, assume that the radius of the earth is 6378 km, the earth's mass is $5.976(10^{24})$ kg, the mass of the sun is $1.99(10^{30})$ kg, and the gravitational constant is $G = 66.73(10^{-12})\ \text{m}^3/(\text{kg}\cdot\text{s}^2)$.

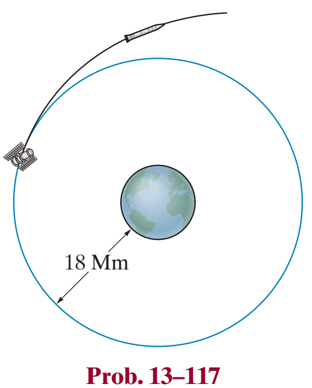
13–113. Prove Kepler's third law of motion. *Hint*: Use Eqs. 13–18, 13–27, 13–28, and 13–30.

13–114. The earth has an orbit with eccentricity 0.0167 around the sun. Knowing that the earth's minimum distance from the sun is $146(10^6)$ km, find the speed at which the earth travels when it is at this distance. Determine the equation in polar coordinates which describes the earth's orbit about the sun.

13–115. A communications satellite is in a circular orbit above the earth such that it always remains directly over a point on the earth's surface. As a result, the period of the satellite must equal the rotation of the earth, which is approximately 24 hours. Determine the satellite's altitude h above the earth's surface and its orbital speed.

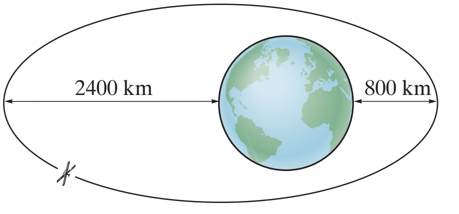
***13–116.** The speed of a satellite launched into a circular orbit about the earth is given by Eq. 13–24. Determine the speed of a satellite launched parallel to the surface of the earth so that it travels in a circular orbit 800 km from the earth's surface.

13–117. The rocket is docked next to a satellite located 18 Mm above the earth's surface. If the satellite is traveling in a circular orbit, determine the speed tangent to the earth's surface which must suddenly be given to the rocket, relative to the satellite, such that it travels in free flight away from the satellite along a parabolic trajectory as shown.



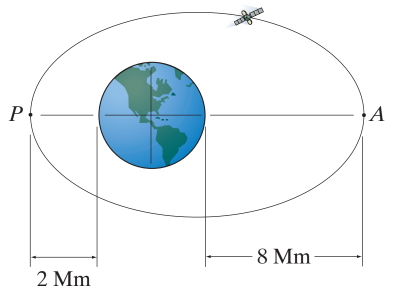
Prob. 13–117

13–118. A satellite is to be placed into an elliptical orbit about the earth such that at the perigee of its orbit it has an *altitude* of 800 km, and at apogee its *altitude* is 2400 km. Determine its required launch velocity tangent to the earth's surface at perigee, and the period of its orbit.



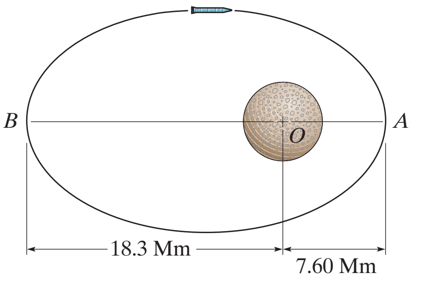
Prob. 13–118

13–119. The satellite is in an elliptic orbit around the earth as shown. Determine its velocity at perigee P and apogee A, and the period of the satellite.



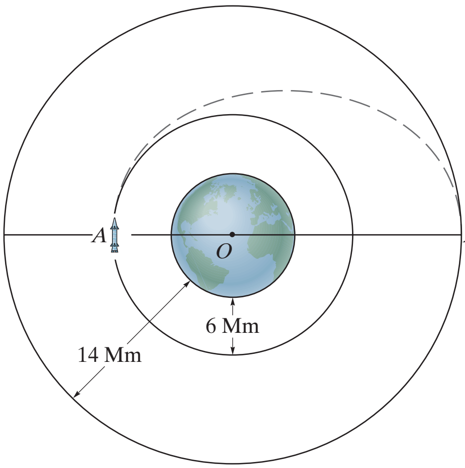
Prob. 13–119

***13–120.** The rocket is traveling in free flight along the elliptical orbit. The planet has no atmosphere, and its mass is 0.60 times that of the earth. If the rocket has the orbit shown, determine the rocket's speed when it is at A and at B.



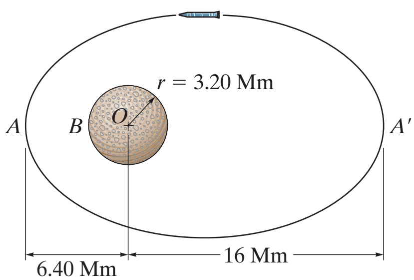
Prob. 13–120

13–121. The rocket shown is originally in a circular orbit 6 Mm above the surface of the earth. It is required that it travel in another circular orbit having an altitude of 14 Mm. To do this, the rocket is given a short pulse of power at A so that it travels in free flight along the dashed elliptical path from the first orbit to the second orbit. Determine the necessary speed it must have at A just after the power pulse, and the time required to get to the outer orbit along the path AA'. What adjustment in speed must be made at A' to maintain the second circular orbit?



Prob. 13–121

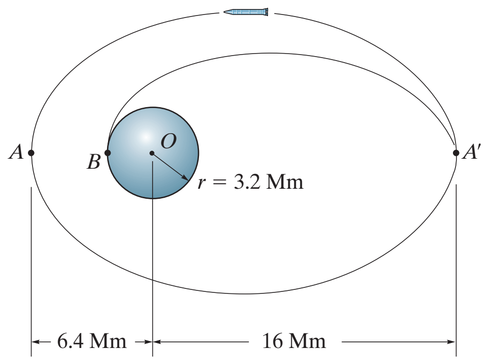
13–122. The rocket is traveling in free flight along an elliptical trajectory $A'A$. The planet has a mass 0.60 times that of the earth. If the rocket has an apoapsis and periapsis as shown in the figure, determine the speed of the rocket when it is at point A.



Prob. 13–122

13–123. The rocket is in free flight along an elliptical trajectory $A'A$. The planet has no atmosphere, and its mass is 0.60 times that of the earth. If the orbit has the apoapsis and periapsis shown, determine the rocket's velocity when it is at point A.

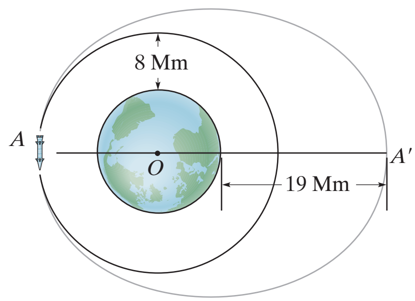
***13–124.** If the rocket is to land on the surface of the planet, determine the required free-flight speed it must have at A' so that the landing occurs at B. How long does it take for the rocket to land, in going from A' to B? The planet has no atmosphere, and its mass is 0.6 times that of the earth.



Probs. 13–123/124

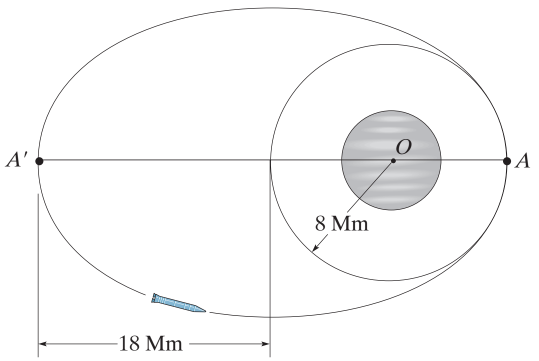
13–125. The rocket is initially in free-flight circular orbit around the earth. Determine the speed of the rocket at A. What change in the speed at A is required so that it can move in an elliptical orbit to reach point A'?

13–126. The rocket is in free-flight circular orbit around the earth. Determine the time needed for the rocket to travel from the inner orbit at A to the outer orbit at A'.



Probs. 13–125/126

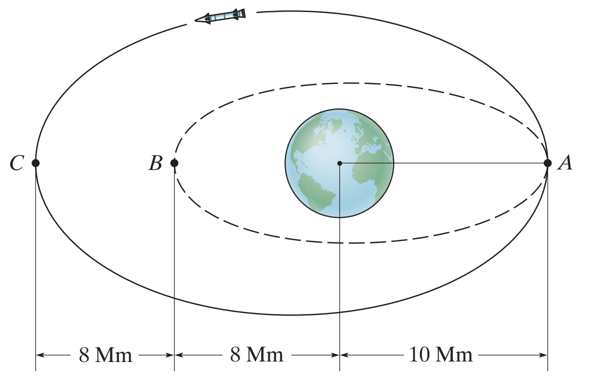
13–127. A rocket is in free-flight elliptical orbit around the planet Venus. Knowing that the periapsis and apoapsis of the orbit are 8 Mm and 26 Mm, respectively, determine (a) the speed of the rocket at point A', (b) the required speed it must attain at A just after braking so that it undergoes an 8-Mm free-flight circular orbit around Venus, and (c) the periods of both the circular and elliptical orbits. The mass of Venus is 0.816 times the mass of the earth.



Prob. 13–127

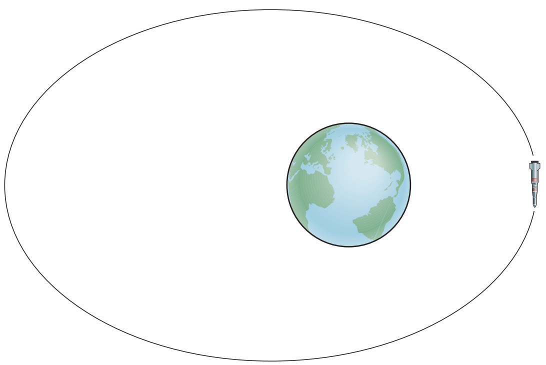
13–129. The rocket is traveling around the earth in free flight along an elliptical orbit AC. If the rocket has the orbit shown, determine the rocket's velocity when it is at point A.

13–130. The rocket is traveling around the earth in free flight along the elliptical orbit AC. Determine its change in speed when it reaches A so that it travels along the elliptical orbit AB.



Probs. 13–129/130

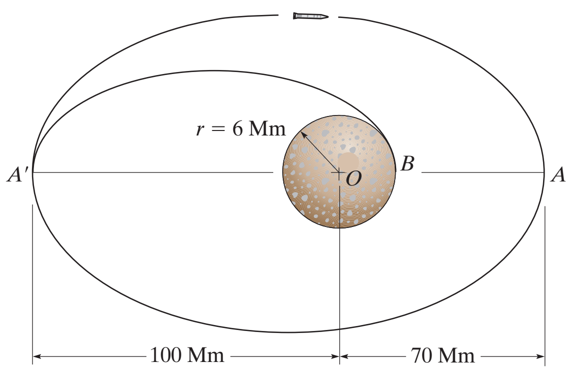
***13–128.** The rocket is in a free-flight elliptical orbit about the earth such that the eccentricity of its orbit is e and its perigee is r_0. Determine the minimum increment of speed it should have in order to escape the earth's gravitational field when it is at this point along its orbit.



Prob. 13–128

13–131. The rocket is traveling in free flight along an elliptical trajectory $A'A$. The planet has no atmosphere, and its mass is 0.60 times that of the earth. If the rocket has the orbit shown, determine the rocket's velocity when it is at point A.

***13–132.** If the rocket is to land on the surface of the planet, determine the required free-flight speed it must have at A' so that the landing occurs at B. How long does it take for the rocket to land, going from A' to B? The planet has no atmosphere, and its mass is 0.6 times that of the earth.



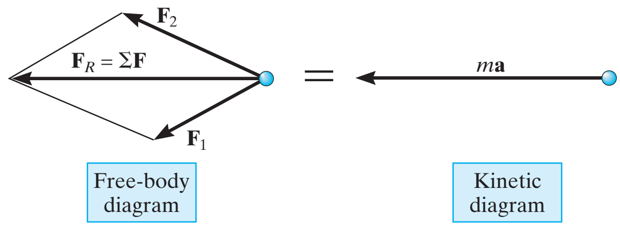
Probs. 13–131/132

CHAPTER REVIEW

Kinetics

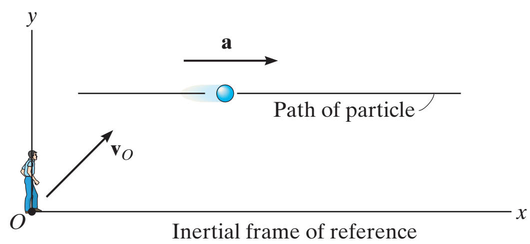
Kinetics is the study of the relationship between forces and the acceleration they cause. This relationship is based on Newton's second law of motion, expressed mathematically as $\Sigma \mathbf{F} = m\mathbf{a}$.

Before applying the equation of motion, it is important to first draw the particle's *free-body diagram* in order to account for all of the forces that act on the particle. Graphically, this diagram is equal to the *kinetic diagram*, which shows the result of the forces, that is, the $m\mathbf{a}$ vector.



Inertial Coordinate System

When applying the equation of motion, it is important to measure the acceleration from an inertial coordinate system. This system has axes that do not rotate but are either fixed or translate with a constant velocity. Various types of inertial coordinate systems can be used to apply $\Sigma \mathbf{F} = m\mathbf{a}$ in component form.



Rectangular x, y, z axes are used to describe the motion along each of the straight-line axes.

$$\Sigma F_x = ma_x \qquad \Sigma F_y = ma_y \qquad \Sigma F_z = ma_z$$

Normal, tangential, and binormal axes n, t, b are often used when the path is known. The normal component of acceleration, $\mathbf{a}_n$, is always directed in the $+n$ direction. It indicates the change in the velocity direction. The tangential component, $\mathbf{a}_t$, is tangent to the path. It indicates the change in the velocity magnitude.

$$\Sigma F_t = ma_t \qquad \Sigma F_n = ma_n \qquad \Sigma F_b = 0$$

$$a_t = dv/dt \quad \text{or} \quad a_t = v\,dv/ds$$

$$a_n = v^2/\rho \quad \text{where} \quad \rho = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|}$$

Cylindrical coordinates are useful when angular motion of the radial line r is known or when the path can conveniently be described with these coordinates.

$$\Sigma F_r = m(\ddot{r} - r\dot{\theta}^2)$$
$$\Sigma F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$
$$\Sigma F_z = m\ddot{z}$$

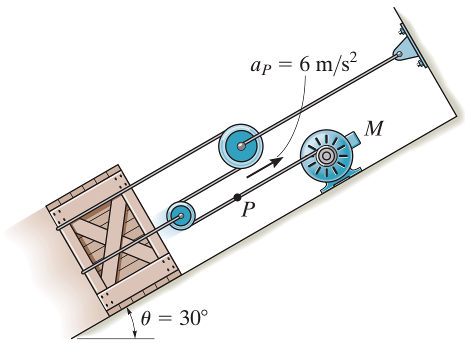
Central-Force Motion

When a single force acts upon a particle, such as during the free-flight trajectory of a satellite in a gravitational field, then the motion is referred to as central-force motion. The orbit depends upon its eccentricity e; and as a result, the trajectory can either be circular, parabolic, elliptical, or hyperbolic.

13

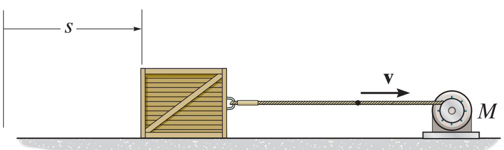
REVIEW PROBLEMS

R13–1. The motor M pulls in its attached rope with an acceleration $a_p = 6\ \text{m/s}^2$. Determine the towing force exerted by M on the rope in order to move the 50-kg crate up the inclined plane. The coefficient of kinetic friction between the crate and the plane is $\mu_k = 0.3$. Neglect the mass of the pulleys and rope.



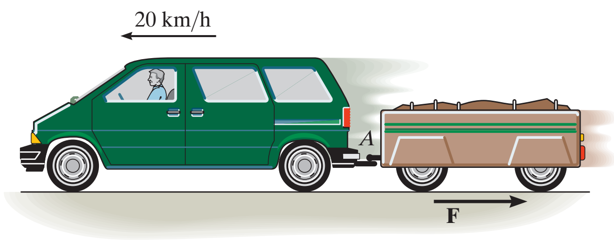
Prob. R13–1

R13–2. If the motor draws in the cable at a rate of $v = (0.05s^{3/2})\ \text{m/s}$, where s is in meters, determine the tension developed in the cable when $s = 10$ m. The crate has a mass of 20 kg, and the coefficient of kinetic friction between the crate and the ground is $\mu_k = 0.2$.



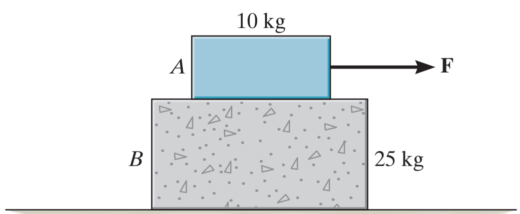
Prob. R13–2

R13–3. The van is traveling at 20 km/h when the coupling of the trailer at A fails. If the trailer has a mass of 250 kg and coasts 45 m before coming to rest, determine the constant horizontal force F created by rolling friction which causes the trailer to stop.



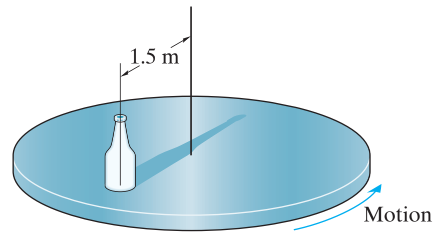
Prob. R13–3

R13–4. Block B rests on a smooth surface. If the coefficients of friction between A and B are $\mu_s = 0.4$ and $\mu_k = 0.3$, determine the acceleration of each block if $F = 250$ N.



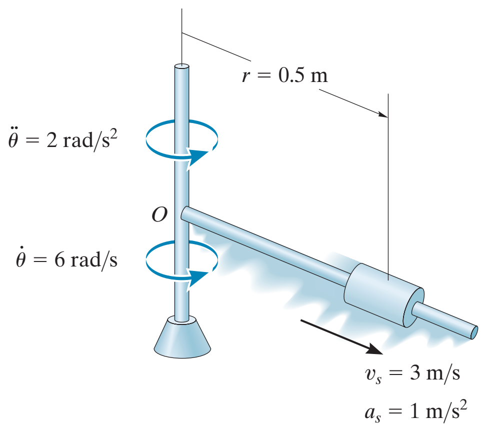
Prob. R13–4

R13–5. The bottle rests at a distance of 1.5 m from the center of the horizontal platform. If the coefficient of static friction between the bottle and the platform is $\mu_s = 0.3$, determine the maximum speed that the bottle can attain before slipping. Assume the angular motion of the platform is slowly increasing.



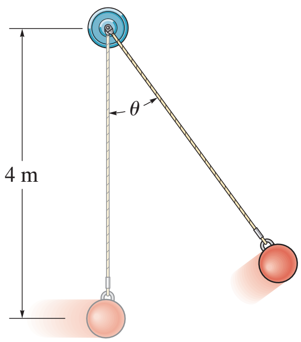
Prob. R13–5

R13–6. The spool, which has a mass of 4 kg, slides along the rotating rod. At the instant shown, the angular rate of rotation of the rod is $\dot{\theta} = 6$ rad/s and this rotation is increasing at $\ddot{\theta} = 2$ rad/s^2. At this same instant, the spool has a velocity of 3 m/s and an acceleration of 1 m/s^2, both measured relative to the rod and directed away from the center O when $r = 0.5$ m. Determine the radial frictional force and the normal force, both exerted by the rod on the spool at this instant.



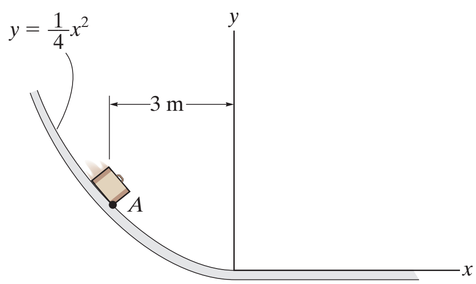
Prob. R13–6

R13–7. The ball has a mass of 30 kg and a speed $v = 4$ m/s at the instant it is at its lowest point, $\theta = 0°$. Determine the tension in the cord and the rate at which the ball's speed is decreasing at the instant $\theta = 20°$. Neglect the size of the ball.



Prob. R13–7

R13–8. The 5-kg suitcase slides down the curved ramp for which the coefficient of kinetic friction is $\mu_k = 0.2$. If at the instant it reaches point A it has a speed of 2.5 m/s, determine the normal force on the suitcase and the rate of increase of its speed.



Prob. R13–8

CHAPTER 14

The power requirements of this crane and the motion of this hook involve work and energy principles that can be analyzed using the methods outlined in this chapter.

KINETICS OF A PARTICLE: WORK AND ENERGY

Lecture Summary and Quiz, Example, and Problem-solving videos are available where this icon appears.

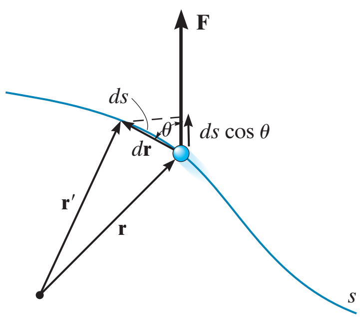
CHAPTER OBJECTIVES

- To develop the principle of work and energy and apply it to solve kinetic problems.
- To study problems that involve power and efficiency.
- To introduce the concept of a conservative force and apply the theorem of conservation of energy to solve kinetic problems.

14.1 THE WORK OF A FORCE

In this chapter, we will analyze the motion of a particle using the concepts of work and energy. Before we do this, however, we must first define the work of a force. Specifically, a force will do *work* on a particle only when the particle undergoes a *displacement in the direction of the force*. For example, if the force **F** in Fig. 14–1*a* causes the particle to move along the path s from position $\mathbf{r}$ to a new position $\mathbf{r}'$, the displacement is then $d\mathbf{r} = \mathbf{r}' - \mathbf{r}$. The magnitude of $d\mathbf{r}$ becomes ds, the length of the differential segment along the path. If the angle between the tails of $d\mathbf{r}$ and **F** is θ, Fig. 14–1*a*, then the work dU done by **F** is a *scalar quantity*, defined by the product of the force magnitude F times the component of displacement in the direction of the force, $ds \cos\theta$, i.e.,

$$dU = F(ds\cos\theta)$$



$dU = F(ds\cos\theta)$

(a)

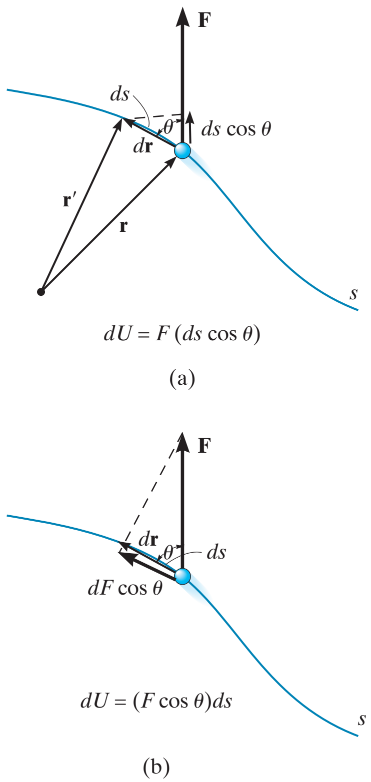


Fig. 14–1

We can also interpret the work as the product of displacement ds and the component of force, $F\cos\theta$, in the direction of displacement, $dU = (F\cos\theta\, ds)$. Note that if $0° \le \theta < 90°$, then the force component and the displacement have the *same sense* so that the work is *positive*; whereas if $90° < \theta \le 180°$, these vectors will have *opposite sense*, and therefore the work is *negative*. Also, $dU = 0$ if the force is *perpendicular* to displacement, since $\cos 90° = 0$, or if the force is applied at a *fixed point*, in which case the displacement is zero.

By definition of the dot product (see Eq. B–14) either one of the above equations can also be written as

$$dU = \mathbf{F}\cdot d\mathbf{r}$$

The unit of work in SI units is the ***joule*** (J), which is the amount of work done by a one-newton force when it moves through a distance of one meter in the direction of the force ($1\text{ J} = 1\text{ N}\cdot\text{m}$).

Work of a Variable Force. If the particle acted upon by the force **F** undergoes a finite displacement along its path from $\mathbf{r}_1$ to $\mathbf{r}_2$ or s_1 to s_2, Fig. 14–2*a*, the work of force **F** is determined by integration. Provided **F** and θ can be expressed as a function of position, then

$$U_{1-2} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r} = \int_{s_1}^{s_2} F\cos\theta\, ds \qquad (14\text{–}1)$$

Sometimes, this relation may be obtained by using experimental data to plot a graph of $F\cos\theta$ vs. s. Then the area under this graph bounded by s_1 and s_2 represents the total work, Fig. 14–2*b*.

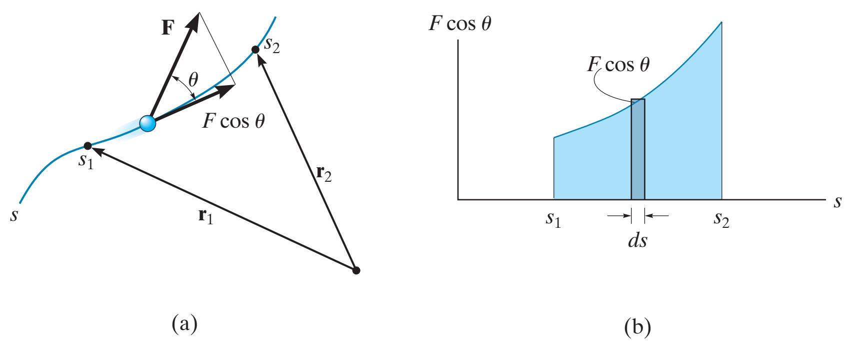


Fig. 14–2

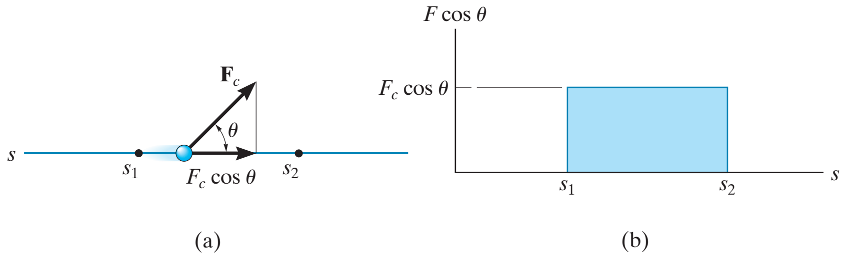


Fig. 14–3

Work of a Constant Force Moving Along a Straight Line. If the force $\mathbf{F}_c$ has a constant magnitude and acts at a constant angle θ from its straight-line path, Fig. 14–3*a*, then the component of $\mathbf{F}_c$ in the direction of displacement is always $F_c \cos\theta$. The work done by $\mathbf{F}_c$ when the particle is displaced from s_1 to s_2 is determined from Eq. 14–1, in which case

$$U_{1-2} = F_c \cos\theta \int_{s_1}^{s_2} ds$$

or

$$U_{1-2} = F_c \cos\theta (s_2 - s_1) \qquad (14\text{–}2)$$

Here the work of $\mathbf{F}_c$ represents the *area of the rectangle* in Fig. 14–3*b*.

The crane must do work in order to hoist the weight of the pipe.

Work of a Weight. Consider a particle of weight $\mathbf{W}$, which moves up along the path s shown in Fig. 14–4 from position s_1 to position s_2. At an intermediate point, the displacement $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$. Since $\mathbf{W} = -W\mathbf{j}$, applying Eq. 14–1 we have

$$U_{1-2} = \int \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} (-W\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$$

$$= \int_{y_1}^{y_2} -W\,dy = -W(y_2 - y_1)$$

or

$$U_{1-2} = -W\,\Delta y \qquad (14\text{–}3)$$

Here the work is independent of the path and is equal to the magnitude of the particle's weight times its vertical displacement. In the case shown, the work is *negative*, since W is downward and Δy is upward. If the particle is displaced *downward* $(-\Delta y)$, then work of the weight is *positive*.

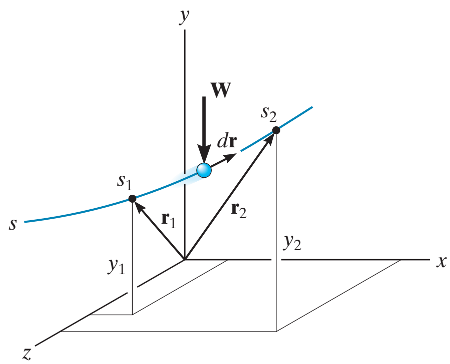


Fig. 14–4

Work of a Spring Force. If an elastic spring is elongated a distance ds, Fig. 14–5a, then the work done by the force that acts on the attached particle is $dU = -F_s\,ds = -ks\,ds$. The work is *negative* since F_s acts in the opposite sense to ds. If the particle displaces from s_1 to s_2, the work of $\mathbf{F}_s$ is then

$$U_{1-2} = \int_{s_1}^{s_2} F_s\,ds = \int_{s_1}^{s_2} -ks\,ds$$

$$\boxed{U_{1-2} = -\left(\tfrac{1}{2}ks_2^2 - \tfrac{1}{2}ks_1^2\right)} \qquad (14\text{–}4)$$

This work represents the trapezoidal area under the line $F_s = ks$, Fig. 14–5*b*.

A mistake in sign can be avoided when applying this equation if one simply notes the direction of the spring force acting on the particle and compares it with the sense of direction of displacement of the particle—if both are in the *same sense, positive work* results; if they are *opposite* to one another, the *work is negative.*

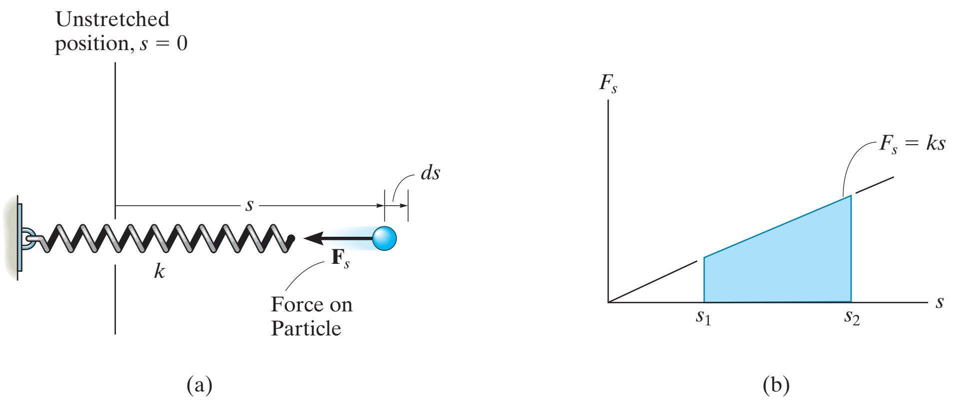
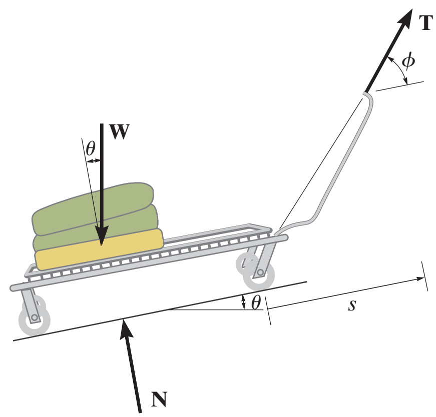


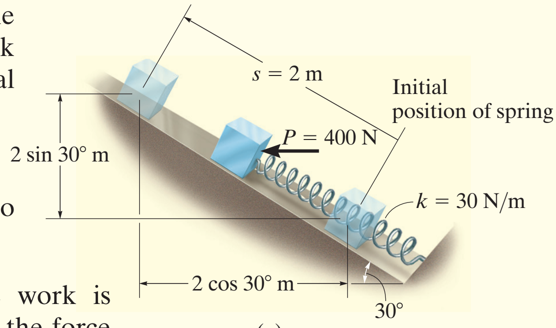
Fig. 14–5



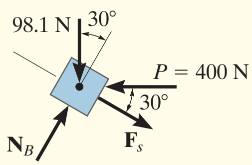
The forces acting on the cart, as it is pulled a distance s up the incline, are shown on its free-body diagram. The constant towing force $\mathbf{T}$ does positive work of $U_T = (T\cos\phi)s$, the weight does negative work of $U_W = -(W\sin\theta)s$, and the normal force $\mathbf{N}$ does no work since there is no displacement of this force along its line of action.

EXAMPLE 14.1

The 10-kg block in Fig. 14–6*a* rests on the smooth incline. If the spring is originally stretched 0.5 m, determine the total work done by all the forces acting on the block when a horizontal force $P = 400$ N pushes the block up the plane $s = 2$ m.



(a)



(b)

Fig 14–6

SOLUTION

First the free-body diagram of the block is drawn in order to account for all the forces that act on the block, Fig. 14–6*b*.

Horizontal Force *P*. Since this force is *constant*, the work is determined using Eq. 14–2. The result can be calculated as the force times the component of displacement in the direction of the force; i.e.,

$$U_P = 400 \text{ N } (2 \text{ m} \cos 30^\circ) = 692.8 \text{ J}$$

or the displacement times the component of force in the direction of displacement, i.e.,

$$U_P = 400 \text{ N} \cos 30^\circ (2 \text{ m}) = 692.8 \text{ J}$$

Spring Force F_s. In the initial position the spring is stretched $s_1 = 0.5$ m and in the final position it is stretched $s_2 = 0.5 \text{ m} + 2 \text{ m} = 2.5$ m. We require the work to be negative since the force and displacement are opposite to each other. The work of the spring force is thus

$$U_s = -\left[\tfrac{1}{2}(30 \text{ N/m})(2.5 \text{ m})^2 - \tfrac{1}{2}(30 \text{ N/m})(0.5 \text{ m})^2\right] = -90 \text{ J}$$

Weight *W*. Since the weight acts in the opposite sense to its vertical displacement, the work is negative; i.e.,

$$U_W = -(98.1 \text{ N})\ (2 \text{ m} \sin 30^\circ) = -98.1 \text{ J}$$

Note that it is also possible to consider the component of weight in the direction of displacement; i.e.,

$$U_W = -(98.1 \sin 30^\circ \text{ N})\ (2 \text{ m}) = -98.1 \text{ J}$$

Normal Force N_B. This force does *no work* since it is *always* perpendicular to the displacement.

Total Work. The work of all the forces when the block is displaced 2 m is therefore

$$U_T = 692.8 \text{ J} - 90 \text{ J} - 98.1 \text{ J} = 505 \text{ J} \qquad \textit{Ans.}$$

14.2 PRINCIPLE OF WORK AND ENERGY

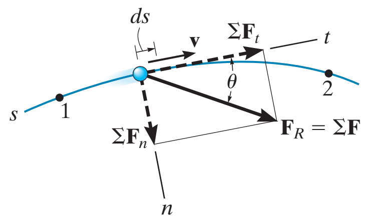


Fig. 14–7

If the particle in Fig. 14–7 has a mass m and is subjected to a system of external forces represented by the resultant $\mathbf{F}_R = \Sigma\mathbf{F}$, then the equation of motion for the particle in the tangential direction is $\Sigma F_t = ma_t$. Substituting the kinematic equation $a_t = v\,dv/ds$ and integrating both sides, assuming initially that the particle has a position $s = s_1$ and a speed $v = v_1$, and later at $s = s_2$, $v = v_2$, we have

$$\Sigma\int_{s_1}^{s_2} F_t\,ds = \int_{v_1}^{v_2} mv\,dv$$

$$\Sigma\int_{s_1}^{s_2} F_t\,ds = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 \tag{14–5}$$

From Fig. 14–7, note that $\Sigma F_t = \Sigma F\cos\theta$, and since all work is defined from Eq. 14–1, the final result can be written as

$$\Sigma U_{1-2} = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 \tag{14–6}$$

This equation is called the ***principle of work and energy*** for the particle. The term on the left is the sum of the work done by *all* the forces acting on the particle as the particle moves from point 1 to point 2. The two terms on the right side, which are of the form $T = \frac{1}{2}mv^2$, define the particle's final and initial ***kinetic energy***. Like work, kinetic energy is a *scalar* and has units of joules (J). However, unlike work, which can be either positive or negative, the kinetic energy is *always positive*, regardless of the direction of motion of the particle. Remember though that it is important that the velocities are measured from an inertial coordinate system.

When Eq. 14–6 is applied, it is often expressed in the form

$$\boxed{T_1 + \Sigma U_{1-2} = T_2} \tag{14–7}$$

which states that the particle's initial kinetic energy plus the work done by all the forces acting on the particle as it moves from its initial to its final position is equal to the particle's final kinetic energy.

PROCEDURE FOR ANALYSIS

The principle of work and energy is used to solve problems involving force, displacement, and velocity, since these terms are involved in the formulation. For application the following procedure is suggested.

Work (Free-Body Diagram).

- Establish the inertial coordinate system and draw a free-body diagram of the particle in order to account for all the forces that do work on the particle as it moves along its path.

Principle of Work and Energy.

- Apply the principle of work and energy, $T_1 + \Sigma U_{1-2} = T_2$.
- The kinetic energy at the initial and final points is *always positive*, since it involves the speed squared $\left(T = \frac{1}{2}mv^2\right)$.
- A force does work when it moves through a displacement in the direction of the force.
- Work is *positive* when the force component is in the *same sense of direction* as its displacement, otherwise it is negative.
- Forces that are functions of displacement must be integrated to obtain the work. Graphically, the work is equal to the area under the force-displacement curve.
- The work of a weight is the product of the weight and the vertical displacement, $U_W = \pm Wy$. It is positive when the weight moves downwards.
- The work of a spring is of the form $U_s = \frac{1}{2}ks^2$, where k is the spring stiffness and s is the stretch or compression of the spring.

Numerical application of this procedure is illustrated in the examples following Sec. 14.3.

If an oncoming car strikes these crash barrels, the car's kinetic energy will be transformed into work, which causes the barrels, and to some extent the car, to be deformed. By knowing the amount of energy that can be absorbed by each barrel it is possible to design a crash barrier such as this.

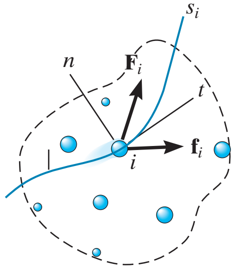
14.3 PRINCIPLE OF WORK AND ENERGY FOR A SYSTEM OF PARTICLES

The principle of work and energy can be extended to include a system of particles isolated within an enclosed region of space as shown in Fig. 14–8. Here the arbitrary ith particle, having a mass m_i, is subjected to a resultant external force $\mathbf{F}_i$ and a resultant internal force $\mathbf{f}_i$ which all the other particles exert on the ith particle. If we apply the principle of work and energy to this and each of the other particles in the system, then since work and energy are scalar quantities, the equations can be summed algebraically, which gives

$$\Sigma T_1 + \Sigma U_{1-2} = \Sigma T_2 \tag{14–8}$$

In this case, the initial kinetic energy of the system plus the work done by all the external and internal forces acting on the system is equal to the final kinetic energy of the system.

If the system represents a *translating rigid body*, or a series of connected translating bodies, then all the particles in each body will undergo the *same displacement*. Therefore, the work of all the internal forces will occur in equal but opposite collinear pairs and so it will cancel out. On the other hand, if the body is assumed to be *nonrigid*, the particles of the body may be displaced along *different paths*, and some of the energy due to force interactions would be given off and lost as heat or stored in the body if permanent deformations occur. We will discuss these effects briefly in what follows, and in Sec. 15.4. Throughout this book, however, the principle of work and energy will be applied to problems where direct accountability of such energy losses does not have to be considered.



Inertial coordinate system

Fig. 14–8

Work of Friction Caused by Sliding. A special class of problems will now be investigated which requires a careful application of Eq. 14–8. These problems involve cases where friction occurs as a body slides over the surface of another body. Consider, for example, a block which is translating a distance s over the rough surface shown in Fig. 14–9*a*. If the applied force $\mathbf{P}$ just balances the *resultant* frictional force $\mu_k N$, Fig. 14–9*b*, then due to equilibrium a constant velocity $\mathbf{v}$ is maintained, and one would expect Eq. 14–8 to be applied as follows:

$$\tfrac{1}{2}mv^2 + Ps - \mu_k N s = \tfrac{1}{2}mv^2$$

Indeed this equation is satisfied if $P = \mu_k N$; however, as one realizes from experience, the sliding motion will *generate heat*, a form of energy which seems not to be accounted for in the work-energy equation. In order to explain this paradox and thereby more closely represent the nature of friction, we should actually model the block so that the surfaces of contact are *deformable* (nonrigid).* Recall that the rough portions at the bottom of the block act as "teeth," and when the block slides these teeth *deform slightly* and either break off or vibrate as they pull away from "teeth" at the contacting surface, Fig. 14–9*c*. As a result, frictional forces that act on the block at these points are displaced slightly, due to the localized deformations, and later they are replaced by other frictional forces as other points of contact are made. At any instant, the *resultant* $\mathbf{F}$ of all these frictional forces remains essentially constant, i.e., $\mu_k N$; however, due to the many *localized deformations*, the actual displacement s' of $\mu_k N$ is *not* the same as the displacement s of the applied force $\mathbf{P}$. Instead, s' will be *less* than s ($s' < s$), and therefore the *external work* done by the resultant frictional force will be $\mu_k N s'$ and not $\mu_k N s$. The remaining amount of work, $\mu_k N(s - s')$, manifests itself as an increase in *internal energy*, which in fact causes the block's temperature to rise.

In summary then, Eq. 14–8 can be applied to problems involving sliding friction; however, it should be fully realized that the work of the resultant frictional force is not represented by $\mu_k N s$; instead, this term represents *both* the external work of friction ($\mu_k N s'$) *and* internal work $[\mu_k N(s - s')]$ which is converted into various forms of internal energy, such as heat.†

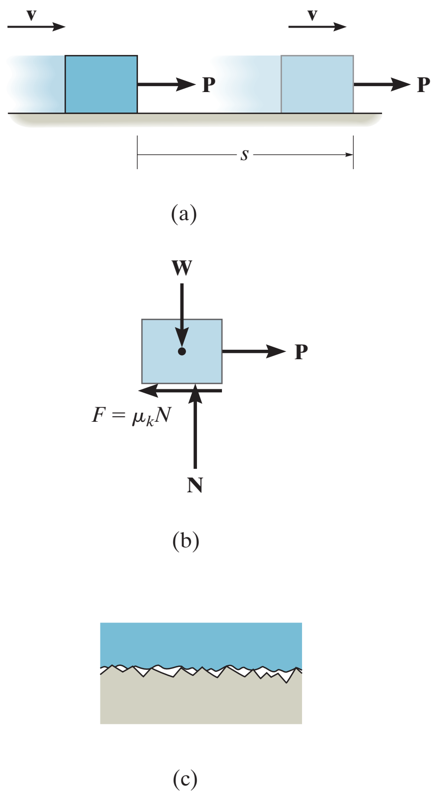


Fig. 14–9

Refer to the companion website for Lecture Summary and Quiz videos.

* See Chapter 8 of *Engineering Mechanics: Statics.*

† See B. A. Sherwood and W. H. Bernard, "Work and Heat Transfer in the Presence of Sliding Friction," *Am. J. Phys.* 52, 1001 (1984).

EXAMPLE 14.2

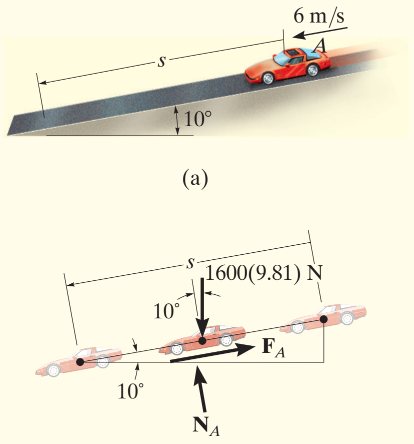


Fig. 14–10

The 1600-kg automobile shown in Fig. 14–10*a* travels down the 10° inclined road at a speed of 6 m/s. If the driver jams on the brakes, causing the wheels to lock, determine how far *s* the tires skid on the road. The coefficient of kinetic friction between the wheels and the road is $\mu_k = 0.5$.

SOLUTION

This problem can be solved using the principle of work and energy, since it involves force, velocity, and displacement.

Work (Free-Body Diagram). As shown in Fig. 14–10*b*, the normal force $\mathbf{N}_A$ does no work since it never undergoes displacement along its line of action. The weight, 1600(9.81) N, is displaced *s* sin 10° and does positive work. Why? The frictional force $\mathbf{F}_A$ does both external and internal work when it undergoes a displacement *s*. This work is negative since it is in the opposite sense of direction to the displacement. Applying the equation of equilibrium normal to the road, we have

$$+\nwarrow \Sigma F_n = 0; \qquad N_A - 1600(9.81)\cos 10^\circ \text{ N} = 0 \qquad N_A = 15\,457.5 \text{ N}$$

Thus,

$$F_A = \mu_k N_A = 0.5\,(15\,457.5 \text{ N}) = 7728.77 \text{ N}$$

Principle of Work and Energy.

$$T_1 + \Sigma U_{1-2} = T_2$$

$$\frac{1}{2}(1600 \text{ kg})(6 \text{ m/s})^2 + 1600(9.81)\text{N}\,(s \sin 10^\circ) - (7728.77 \text{ N})s = 0$$

Solving for *s* yields

$$s = 5.76 \text{ m} \qquad \textit{Ans.}$$

NOTE: If this problem is solved by using the equation of motion, *two steps* are involved. First, from the free-body diagram, Fig. 14–10*b*, the equation of motion is applied along the incline to obtain the deceleration. This yields

$$+\swarrow \Sigma F_s = ma_s; \qquad 1600(9.81)\sin 10^\circ \text{ N} - 7728.77 \text{ N} = (1600 \text{ kg})\,a$$

$$a = -3.127 \text{ m/s}^2$$

Then, since *a* is constant, we have

$$(+\swarrow) \qquad v^2 = v_0^2 + 2a_c(s - s_0);$$

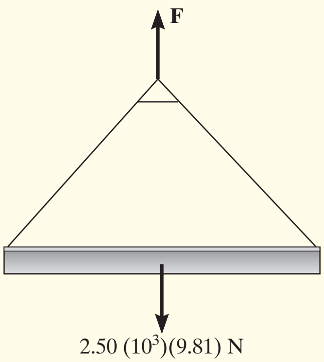
$$(0)^2 = (6 \text{ m/s})^2 + 2(-3.127 \text{ m/s}^2)(s - 0)$$

$$s = 5.76 \text{ m} \qquad \textit{Ans.}$$

EXAMPLE 14.3

For a short time the crane in Fig. 14–11*a* lifts a 2.50-Mg beam with a force of $F = (28 + 3s^2)$ kN. Determine the speed of the beam when it has risen $s = 3$ m. Also, how much time does it take to attain this height starting from rest?

(a)



(b)

Fig. 14–11

SOLUTION

We can solve part of this problem using the principle of work and energy since it involves force, velocity, and displacement. Kinematics must be used to determine the time. Note that at $s = 0$, $F = 28(10^3)\text{ N} > W = 2.50(10^3)(9.81)\text{ N}$, so motion will occur.

Work (Free-Body Diagram). As shown on the free-body diagram, Fig. 14–11*b*, the lifting force **F** does positive work, which must be determined by integration since this force is a variable. Also, the weight is constant and will do negative work since the displacement is upward.

Principle of Work and Energy.

$$T_1 + \Sigma U_{1-2} = T_2$$

$$0 + \int_0^s (28 + 3s^2)(10^3)\,ds - (2.50)(10^3)(9.81)s = \tfrac{1}{2}(2.50)(10^3)v^2$$

$$28(10^3)s + (10^3)s^3 - 24.525(10^3)s = 1.25(10^3)v^2$$

$$v = (2.78s + 0.8s^3)^{\frac{1}{2}} \qquad (1)$$

When $s = 3$ m,

$$v = 5.47\text{ m/s} \qquad \textit{Ans.}$$

Kinematics. Since we were able to express the velocity as a function of displacement, Eq. 1, the time can be determined using $v = ds/dt$. In this case,

$$(2.78s + 0.8s^3)^{\frac{1}{2}} = \frac{ds}{dt}$$

$$t = \int_0^3 \frac{ds}{(2.78s + 0.8s^3)^{\frac{1}{2}}}$$

The integration can be performed numerically using a pocket calculator. The result is

$$t = 1.79\text{ s} \qquad \textit{Ans.}$$

NOTE: The acceleration of the beam can be determined by integrating Eq. 1 using $v\,dv = a\,ds$, or more directly, by applying the equation of motion, $\Sigma F_y = ma_y$.

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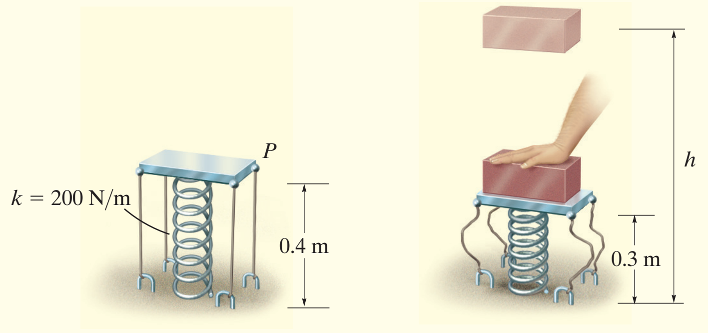
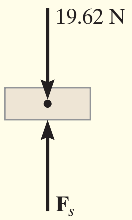
EXAMPLE 14.4

The platform P, shown in Fig. 14–12*a*, has negligible mass and is tied down so that the 0.4-m-long cords keep a 1-m-long spring compressed 0.6 m when *nothing* is on the platform. If a 2-kg block is placed on the platform and released from rest after the platform is pushed down 0.1 m, Fig. 14–12*b*, determine the maximum height h the block rises in the air, measured from the ground.



SOLUTION

Work (Free-Body Diagram). Since the block is released from rest and later reaches its maximum height, the initial and final velocities are zero. The free-body diagram of the block when it is still in contact with the platform is shown in Fig. 14–12*c*. Note that the weight does negative work and the spring force does positive work. Why? In particular, the *initial compression* in the spring is $s_1 = 0.6\text{ m} + 0.1\text{ m} = 0.7\text{ m}$. Due to the cords, the spring's *final compression* is $s_2 = 0.6\text{ m}$ (after the block leaves the platform). The bottom of the block rises from a height of $(0.4\text{ m} - 0.1\text{ m}) = 0.3\text{ m}$ to a final height h.



(c)

Fig. 14–12

Principle of Work and Energy.

$$T_1 + \Sigma U_{1-2} = T_2$$

$$\tfrac{1}{2}mv_1^2 + \left[-\left(\tfrac{1}{2}ks_2^2 - \tfrac{1}{2}ks_1^2\right) - W\,\Delta y\right] = \tfrac{1}{2}mv_2^2$$

Note that here $s_1 = 0.7\text{ m} > s_2 = 0.6\text{ m}$ and so the work of the spring as determined from Eq. 14–4 will indeed be positive once the calculation is made. Thus,

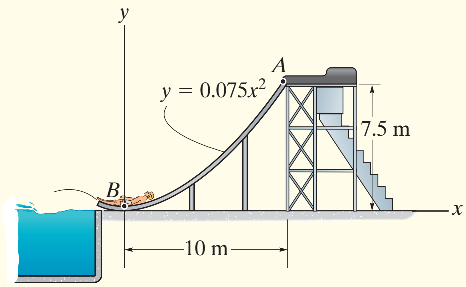
$$0 + \left\{-\left[\tfrac{1}{2}(200\text{ N/m})(0.6\text{ m})^2 - \tfrac{1}{2}(200\text{ N/m})(0.7\text{ m})^2\right] - (19.62\text{ N})[h - (0.3\text{ m})]\right\} = 0$$

Solving yields

$$h = 0.963\text{ m} \qquad \textit{Ans.}$$

EXAMPLE 14.5

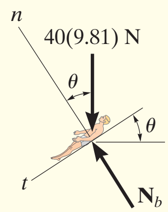
The 40-kg boy in Fig. 14–13*a* slides down the smooth water slide. If he starts from rest at *A*, determine his speed when he reaches *B* and the normal reaction the slide exerts on the boy at this position.



(a)

SOLUTION

Work (Free-Body Diagram). As shown on the free-body diagram, Fig. 14–13*b*, there are two forces acting on the boy as he goes down the slide. Note that the normal force does no work.



(b)

Principle of Work and Energy.

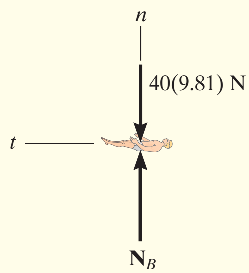
$$T_A + \Sigma U_{A-B} = T_B$$

$$0 + (40(9.81)\text{ N})\,(7.5\text{ m}) = \tfrac{1}{2}(40\text{ kg})v_B^2$$

$$v_B = 12.13\text{ m/s} = 12.1\text{ m/s} \qquad \textit{Ans.}$$

Equation of Motion. Referring to the free-body diagram of the boy when he is at *B*, Fig. 14–13*c*, the normal reaction $\mathbf{N}_B$ can now be obtained by applying the equation of motion along the *n* axis. Here the radius of curvature of the path is

$$\rho_B = \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{|d^2y/dx^2|} = \left.\frac{\left[1 + (0.15x)^2\right]^{3/2}}{|0.15|}\right|_{x=0} = 6.667\text{ m}$$



(c)

Fig. 14–13

Thus,

$$+\uparrow\Sigma F_n = ma_n; \qquad N_B - 40(9.81)\text{ N} = 40\text{ kg}\left(\frac{(12.13\text{ m/s})^2}{6.667\text{ m}}\right)$$

$$N_B = 1275.3\text{ N} = 1.28\text{ kN} \qquad \textit{Ans.}$$

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EXAMPLE 14.6

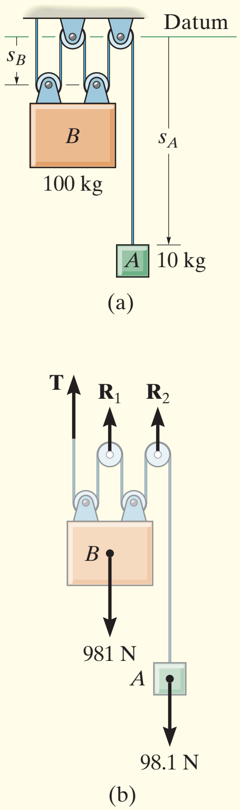


Fig. 14–14

Blocks A and B shown in Fig. 14–14a have a mass of 10 kg and 100 kg, respectively. Determine the distance B travels so that its speed becomes 2 m/s after being released from rest.

SOLUTION

This problem may be solved by considering the blocks separately and applying the principle of work and energy to each block. However, it is easier to eliminate the work of the cable tension from the analysis by considering blocks A and B together as a *single system*.

Work (Free-Body Diagram). As shown on the free-body diagram of the system, Fig. 14–14b, the cable force $\mathbf{T}$ and reactions $\mathbf{R}_1$ and $\mathbf{R}_2$ do *no work*, since these forces represent the reactions at the supports and consequently they do not move while the blocks are displaced. The weights both do positive work if we *assume* both move downward, in the positive sense of direction of s_A and s_B, Fig. 14–14a.

Principle of Work and Energy. Realizing the blocks are released from rest, we have

$$\Sigma T_1 + \Sigma U_{1-2} = \Sigma T_2$$

$$\left[\tfrac{1}{2}m_A(v_A)_1^2 + \tfrac{1}{2}m_B(v_B)_1^2\right] + \left[W_A\,\Delta s_A + W_B\,\Delta s_B\right] = \left[\tfrac{1}{2}m_A(v_A)_2^2 + \tfrac{1}{2}m_B(v_B)_2^2\right]$$

$$[0 + 0] + [98.1\text{ N}\,(\Delta s_A) + 981\text{ N}\,(\Delta s_B)] = \left[\tfrac{1}{2}(10\text{ kg})(v_A)_2^2 + \tfrac{1}{2}(100\text{ kg})(2\text{ m/s})^2\right] \quad (1)$$

Kinematics. Using methods of kinematics, discussed in Sec. 12.9, the total length l of all the vertical segments of cable in Fig. 14–14a can be expressed in terms of the position coordinates s_A and s_B as

$$s_A + 4s_B = l$$

Hence, a change in position yields the displacement equation

$$\Delta s_A + 4\,\Delta s_B = 0$$

$$\Delta s_A = -4\,\Delta s_B \quad (2)$$

Here a downward displacement of one block produces an upward displacement of the other block. Taking the time derivative of Eq. 2 yields

$$v_A = -4v_B = -4(2\text{ m/s}) = -8\text{ m/s} \quad (3)$$

Retaining the negative sign in Eq. 2 and substituting into Eq. 1 yields

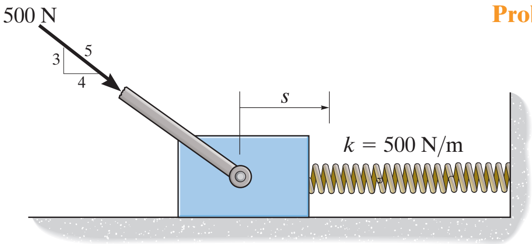
$$\Delta s_B = 0.883\text{ m}\downarrow \qquad \textit{Ans.}$$

NOTE: Since we had to solve Eqs. 1–3 *simultaneously*, it is important to write these equations using the *same* coordinate directions for s_A and s_B.

Refer to the companion website for a self quiz of these Example problems.

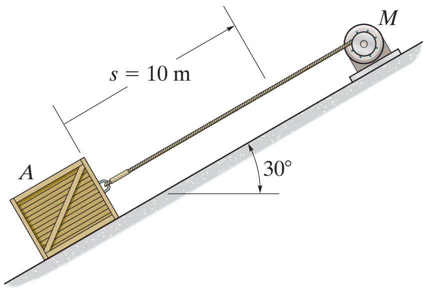
FUNDAMENTAL PROBLEMS

F14–1. The spring is placed between the wall and the 10-kg block. If the block is subjected to a force of $F = 500$ N, determine its velocity when $s = 0.5$ m. When $s = 0$, the block is at rest and the spring is uncompressed. The contact surface is smooth.



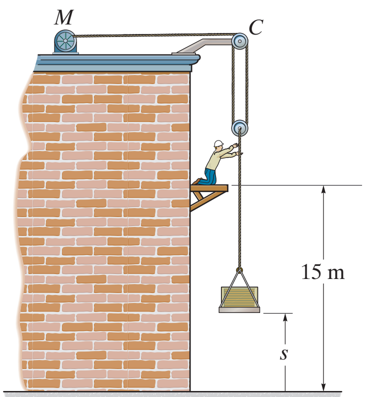
Prob. F14–1

F14–2. If the motor exerts a constant force of 300 N on the cable, determine the speed of the 20-kg crate when it travels $s = 10$ m up the plane, starting from rest. The coefficient of kinetic friction between the crate and the plane is $\mu_k = 0.3$.



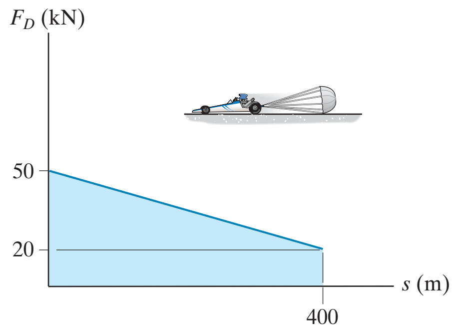
Prob. F14–2

F14–3. If the motor exerts a force of $F = (600 + 2s^2)$ N on the cable, determine the speed of the 100-kg crate when it rises to $s = 15$ m. The crate is initially at rest on the ground.



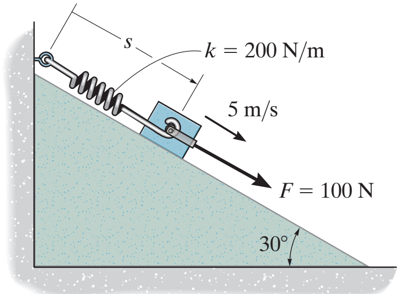
Prob. F14–3

F14–4. The 1.8-Mg dragster is traveling at 125 m/s when the engine is shut off and the parachute is released. If the drag force of the parachute can be approximated by the graph, determine the speed of the dragster when it has traveled 400 m.



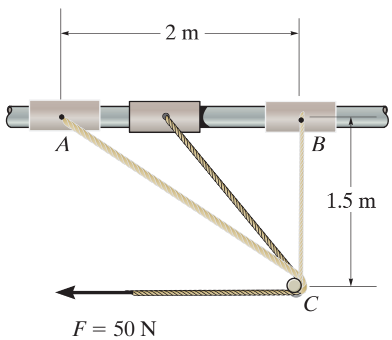
Prob. F14–4

F14–5. When $s = 0.6$ m, the spring is unstretched and the 10-kg block has a speed of 5 m/s down the smooth plane. Determine the distance s when the block stops.



Prob. F14–5

F14–6. The 2.5-kg collar is pulled by a cord that passes around a small peg at C. If the cord is subjected to a constant force of $F = 50$ N, and the collar is at rest when it is at A, determine its speed the instant before it reaches B. Neglect friction.

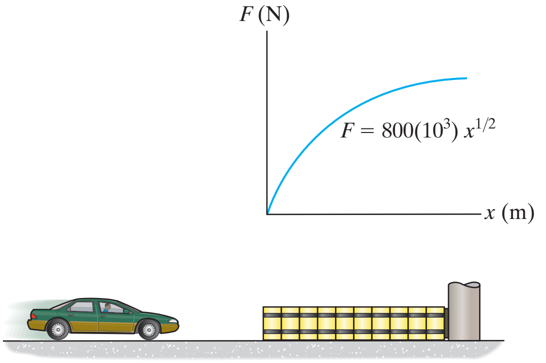


Prob. F14–6

PROBLEMS

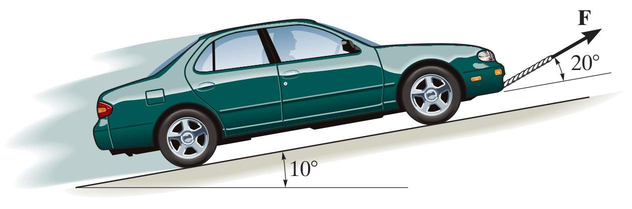
All solutions must include a free-body diagram.

14–1. For protection, the barrel barrier is placed in front of the bridge pier. If the relation between the force and deflection of the barrier is $F = [800(10^3)x^{1/2}]$ N, where x is in m, determine the car's maximum penetration in the barrier. The car has a mass of 2 Mg and it is traveling with a speed of 20 m/s just before it hits the barrier.



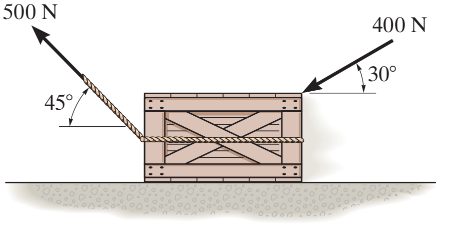
Prob. 14–1

14–2. The car having a mass of 2 Mg is originally traveling at 2 m/s. Determine the distance it must be towed by a force $F = 4$ kN in order to attain a speed of 5 m/s. Neglect friction and the mass of the wheels.



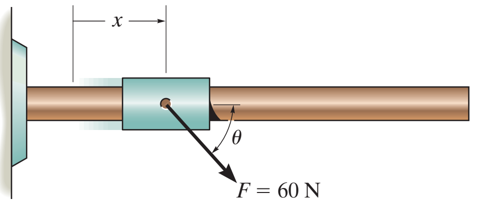
Prob. 14–2

14–3. The 100-kg crate is subjected to the forces shown. If it is originally at rest, determine the distance it slides in order to attain a speed of $v = 8$ m/s. The coefficient of kinetic friction between the crate and the surface is $\mu_k = 0.2$.



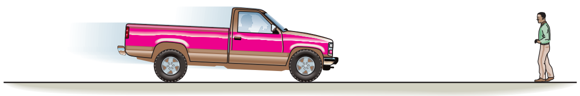
Prob. 14–3

***14–4.** The collar has mass of 5 kg and is moving at 8 m/s when $x = 0$ and a force of $F = 60$ N is applied to it. The direction θ of this force varies such that $\theta = 10x$, where x is in meters and θ is clockwise, measured in degrees. Determine the speed of the collar when $x = 3$ m. The coefficient of kinetic friction between the collar and the rod is $\mu_k = 0.3$.



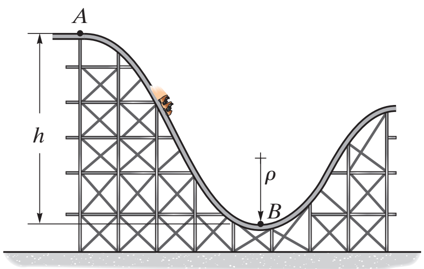
Prob. 14–4

14–5. When the driver applies the brakes of a light truck traveling 40 km/h, it skids 30 m before stopping. How far will the truck skid if it is traveling 80 km/h when the brakes are applied?



Prob. 14–5

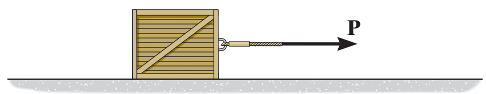
14–6. Determine the required height h of the roller coaster so that when it is essentially at rest at the crest of the hill A it will reach a speed of 100 km/h when it comes to the bottom B. Also, what should be the minimum radius of curvature ρ for the track at B so that the passengers do not experience a normal force greater than $4mg = (39.24m)$ N? Neglect the size of the car.



Prob. 14–6

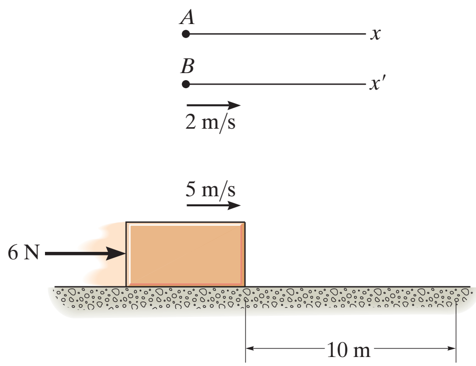
14–7. If the 50-kg crate is subjected to a force of $P = 200$ N, determine its speed when it has traveled 15 m starting from rest. The coefficient of kinetic friction between the crate and the ground is $\mu_k = 0.3$.

***14–8.** If the 50-kg crate starts from rest and attains a speed of 6 m/s when it has traveled a distance of 15 m, determine the force P acting on the crate. The coefficient of kinetic friction between the crate and the ground is $\mu_k = 0.3$.



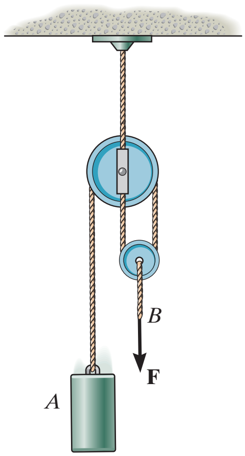
Probs. 14–7/8

14–9. As indicated by the derivation, the principle of work and energy is valid for observers in *any* inertial reference frame. Show that this is so, by considering the 10-kg block which rests on the smooth surface and is subjected to a horizontal force of 6 N. If observer A is in a *fixed* frame x, determine the final speed of the block if it has an initial speed of 5 m/s and travels 10 m, both directed to the right and measured from the fixed frame. Compare the result with that obtained by an observer B, attached to the x' axis and moving at a constant velocity of 2 m/s relative to A. *Hint*: The distance the block travels will first have to be calculated for observer B before applying the principle of work and energy.



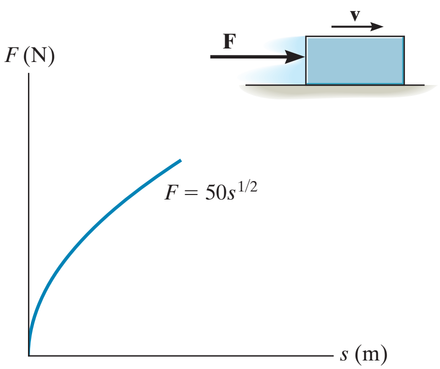
Prob. 14–9

■**14–10.** A force of $F = 250$ N is applied to the end at B. Determine the speed of the 10-kg block when it has moved 1.5 m, starting from rest.



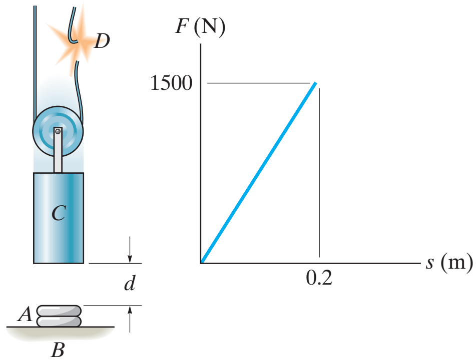
Prob. 14–10

14–11. The force **F**, acting in a constant direction on the 20-kg block, has a magnitude which varies with the positions s of the block. Determine how far the block must slide before its velocity becomes $15\,\text{m/s}$. When $s = 0$ the block is moving to the right at $v = 6\,\text{m/s}$. The coefficient of kinetic friction between the block and surface is $\mu_k = 0.3$.



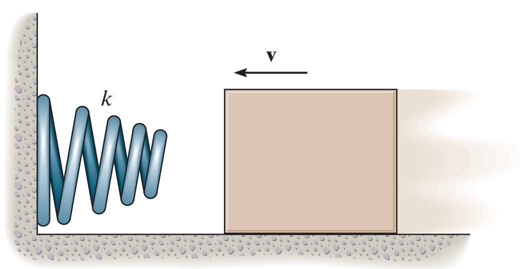
Prob. 14–11

*__14–12.__ The "air spring" A is used to protect the support B and prevent damage to the conveyor-belt tensioning weight C in the event of a belt failure D. The force developed by the air spring as a function of its deflection is shown by the graph. If the block has a mass of 20 kg and is suspended a height $d = 0.4$ m above the top of the spring, determine the maximum deformation of the spring in the event the conveyor belt fails. Neglect the mass of the pulley and belt.



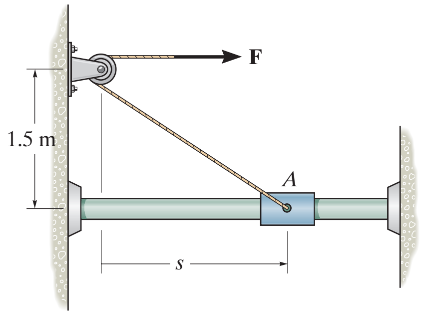
Prob. 14–12

14–13. The 1.5-kg block slides along a smooth plane and strikes a *nonlinear spring* with a speed of $v = 4\,\text{m/s}$. The spring is termed "nonlinear" because it has a resistance of $F_s = ks^2$, where $k = 900\,\text{N/m}^2$. Determine the speed of the block after it has compressed the spring $s = 0.2$ m.



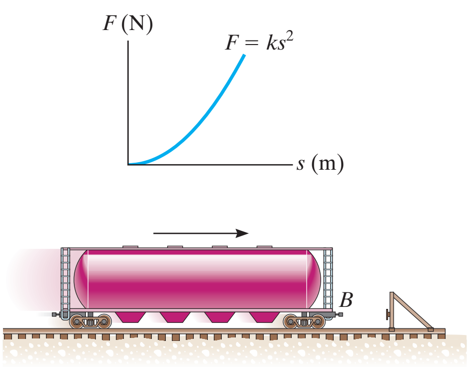
Prob. 14–13

14–14. The force of $F = 50$ N is applied to the cord when $s = 2$ m. If the 6-kg collar is orginally at rest, determine its velocity at $s = 0$. Neglect friction.



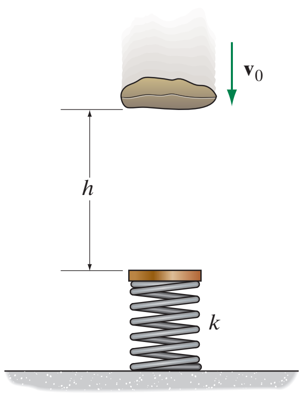
Prob. 14–14

14–15. Design considerations for the bumper B on the 5-Mg train car require use of a nonlinear spring having the load-deflection characteristics shown in the graph. Select the proper value of k so that the maximum deflection of the spring is limited to 0.2 m when the car, traveling at 4 m/s, strikes the rigid stop. Neglect the mass of the car wheels.



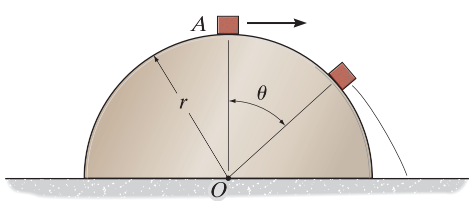
Prob. 14–15

***14–16.** The sack of mass m has a velocity of $\mathbf{v}_0$ when it is at a height h above the unstretched spring. Determine the maximum compression of the spring if it has a stiffness k.



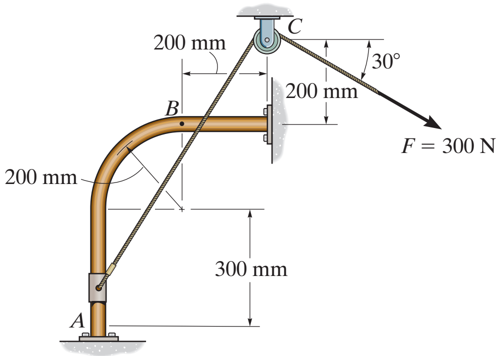
Prob. 14–16

14–17. A small box of mass m is given a speed of $v = \sqrt{\frac{1}{4}gr}$ at the top of the smooth half cylinder. Determine the angle θ at which the box leaves the surface.



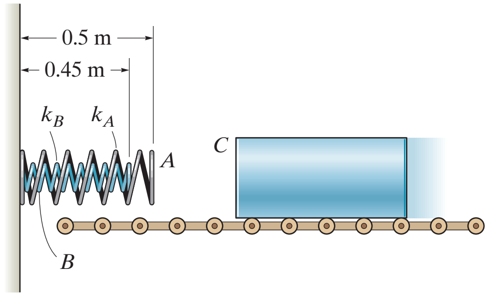
Prob. 14–17

14–18. If the cord is subjected to a constant force of $F = 300$ N and the 15-kg smooth collar starts from rest at A, determine the velocity of the collar when it reaches point B. Neglect the size of the pulley.



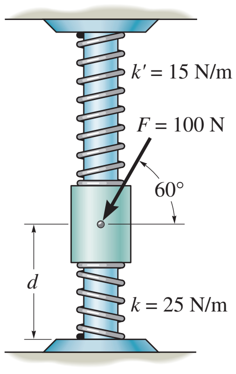
Prob. 14–18

***14–20.** The steel ingot has a mass of 1800 kg. It travels along the conveyor at a speed $v = 0.5$ m/s when it collides with the "nested" spring assembly. Determine the maximum deflection in each spring needed to stop the motion of the ingot. Take $k_A = 5$ kN/m, $k_B = 3$ kN/m.



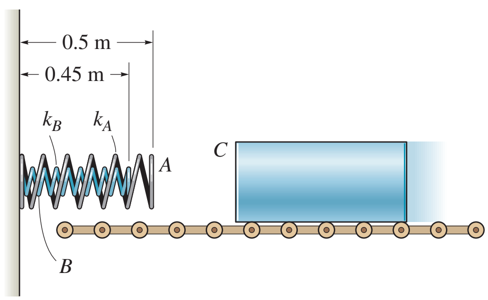
Prob. 14–20

14–19. The collar has a mass of 20 kg and is supported on the smooth rod. The attached springs are undeformed when $d = 0.5$ m. Determine the speed of the collar after the applied force $F = 100$ N causes it to be displaced so that $d = 0.3$ m. When $d = 0.5$ m the collar is at rest.



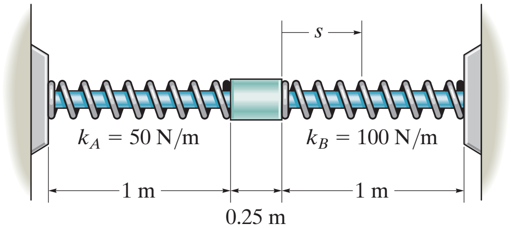
Prob. 14–19

14–21. The steel ingot has a mass of 1800 kg. It travels along the conveyor at a speed $v = 0.5$ m/s when it collides with the "nested" spring assembly. If the stiffness of the outer spring is $k_A = 5$ kN/m determine the required stiffness k_B of the inner spring so that the motion of the ingot is stopped at the moment the front, C, of the ingot is 0.3 m from the wall.



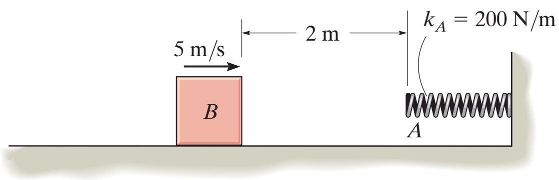
Prob. 14–21

14–22. The collar has a mass of 20 kg and slides along the smooth rod. Two springs are attached to it and the ends of the rod as shown. If each spring has an uncompressed length of 1 m and the collar has a speed of 2 m/s when $s = 0$, determine the maximum compression of each spring due to the back-and-forth (oscillating) motion of the collar.



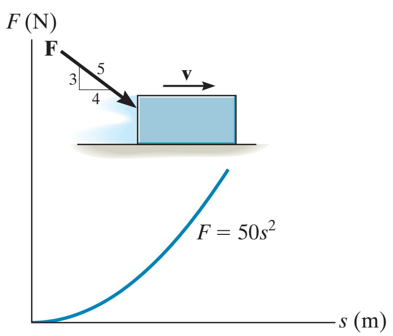
Prob. 14–22

14–23. The 8-kg block is moving with an initial speed of 5 m/s. If the coefficient of kinetic friction between the block and plane is $\mu_k = 0.25$, determine the compression in the spring when the block momentarily stops.



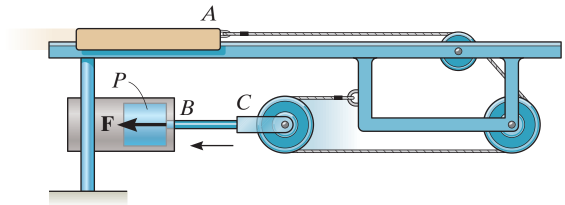
Prob. 14–23

***14–24.** The force **F**, acting in a constant direction on the 20-kg block, has a magnitude which varies with position s of the block. Determine the speed of the block after it slides 3 m. When $s = 0$ the block is moving to the right at 2 m/s. The coefficient of kinetic friction between the block and surface is $\mu_k = 0.3$.



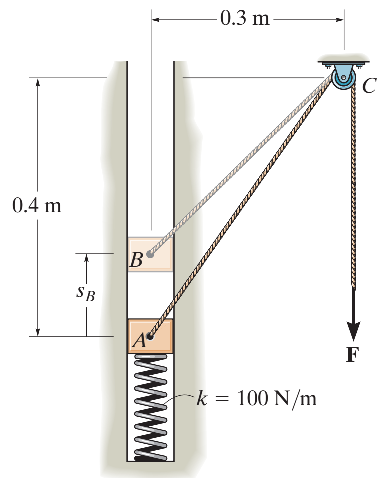
Prob. 14–24

14–25. The catapulting mechanism is used to propel the 10-kg slider A to the right along the smooth track. The propelling action is obtained by drawing the pulley attached to rod BC rapidly to the left by means of a piston P. If the piston applies a constant force $F = 20$ kN to rod BC such that it moves it 0.2 m, determine the speed attained by the slider if it was originally at rest. Neglect the mass of the pulleys, cable, piston, and rod BC.



Prob. 14–25

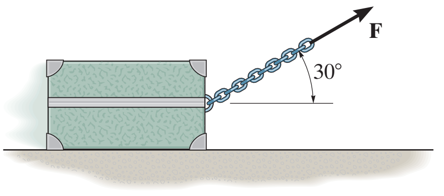
14–26. The block has a mass of 0.8 kg and moves within the smooth vertical slot. If it starts from rest when the *attached* spring is in the unstretched position at A, determine the *constant* vertical force F which must be applied to the cord so that the block attains a speed $v_B = 2.5$ m/s when it reaches B; $s_B = 0.15$ m. Neglect the size and mass of the pulley. *Hint*: The work of **F** can be determined by finding the difference Δl in cord lengths AC and BC and using $U_F = F\,\Delta l$.



Prob. 14–26

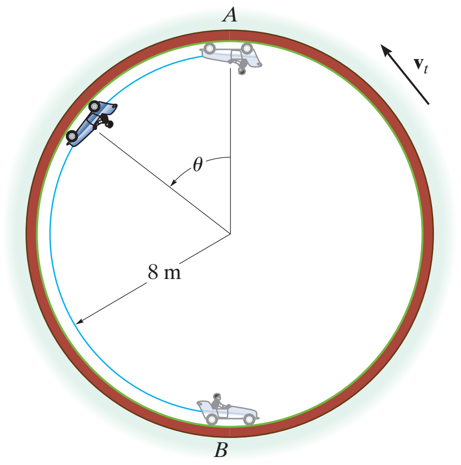
14

14–27. The 20-kg crate is subjected to a force having a constant direction and a magnitude $F = 100$ N. When $s = 15$ m, the crate is moving to the right with a speed of 8 m/s. Determine its speed when $s = 25$ m. The coefficient of kinetic friction between the crate and the ground is $\mu_k = 0.25$.



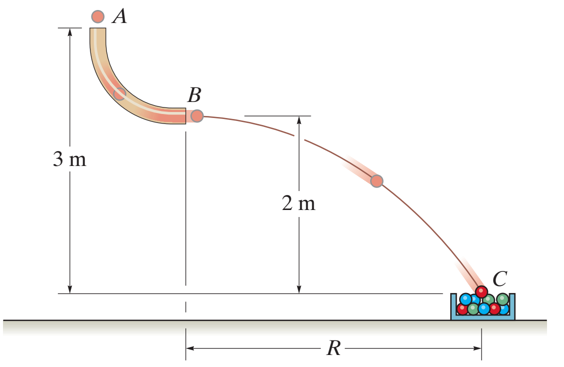
Prob. 14–27

***14–28.** The "flying car" is a ride at an amusement park which consists of a car having wheels that roll along a track mounted inside a rotating drum. By design the car cannot fall off the track, however motion of the car is developed by applying the car's brake, thereby gripping the car to the track and allowing it to move with a constant speed of the track, $v_t = 3$ m/s. If the rider applies the brake when going from B to A and then releases it at the top of the drum, A, so that the car coasts freely down along the track to B ($\theta = \pi$ rad), determine the speed of the car at B and the normal reaction which the drum exerts on the car at B. Neglect friction during the motion from A to B. The rider and car have a total mass of 250 kg, and the center of mass of the car and rider moves along a circular path having a radius of 8 m.



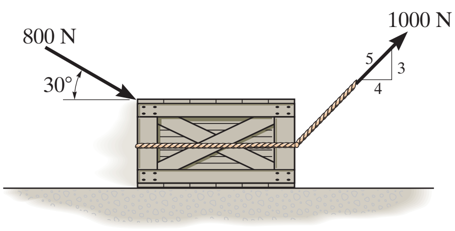
Prob. 14–28

14–29. Marbles having a mass of 5 g are dropped from rest at A through the smooth glass tube and accumulate in the can at C. Determine the placement R of the can from the end of the tube and the speed at which the marbles fall into the can. Neglect the size of the can.



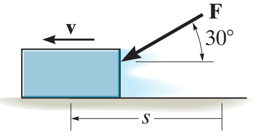
Prob. 14–29

14–30. The crate, which has a mass of 100 kg, is subjected to the action of the two forces. If it is originally at rest, determine the distance it slides in order to attain a speed of 6 m/s. The coefficient of kinetic friction between the crate and the surface is $\mu_k = 0.2$.



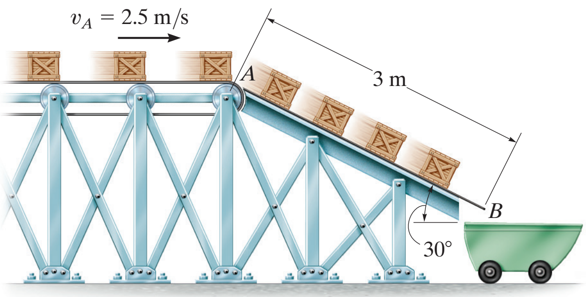
Prob. 14–30

14–31. The 2-kg block is subjected to a force having a constant direction and a magnitude $F = (300/(1 + s))$ N, where s is in meters. When $s = 4$ m, the block is moving to the left with a speed of 8 m/s. Determine its speed when $s = 12$ m. The coefficient of kinetic friction between the block and the ground is $\mu_k = 0.25$.



Prob. 14–31

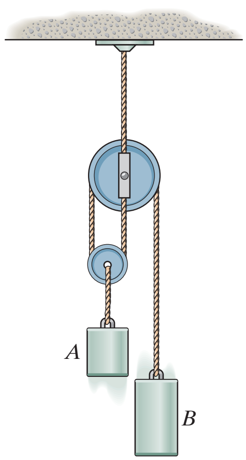
*14–32. The conveyor belt delivers each 12-kg crate to the ramp at A such that the crate's velocity is $v_A = 2.5\text{ m/s}$, directed down *along* the ramp. If the coefficient of kinetic friction between each crate and the ramp is $\mu_k = 0.3$, determine the speed at which each crate slides off the ramp at B. Assume that no tipping occurs.



Prob. 14–32

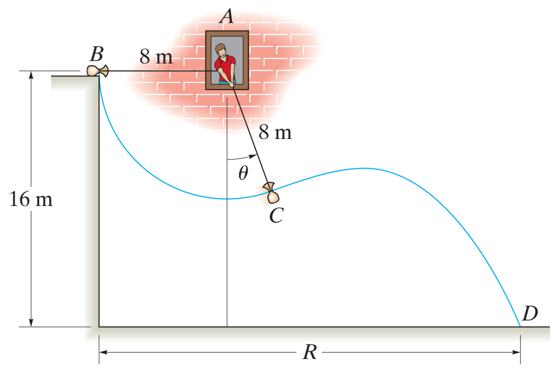
14–33. The 8-kg cylinder A and 3-kg cylinder B are released from rest. Determine the speed of A after it has moved 2 m starting from rest. Neglect the mass of the cord and pulleys.

14-34 Cylinder A has a mass of 3 kg and cylinder B has a mass of 8 kg. Determine the speed of A after it has moved 2 m starting from rest. Neglect the mass of the cord and pulleys.



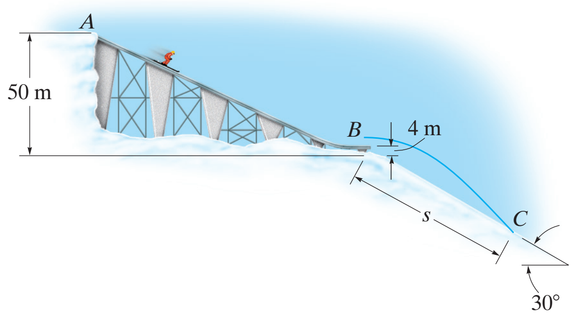
Probs. 14–33/34

14–35. The man at the window A wishes to throw the 30-kg sack on the ground. To do this he allows it to swing from rest at B to point C, when he releases the cord at $\theta = 30°$. Determine the speed at which it strikes the ground and the distance R.



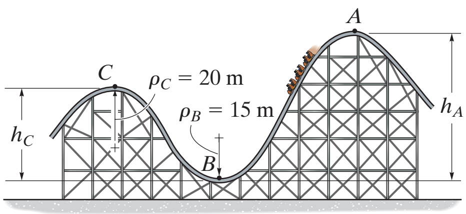
Prob. 14–35

*14–36. The skier starts from rest at A and travels down the ramp. If friction and air resistance can be neglected, determine his speed v_B when he reaches B. Also, find the distance s to where he strikes the ground at C, if he makes the jump traveling horizontally at B. Neglect the skier's size. He has a mass 75 kg.



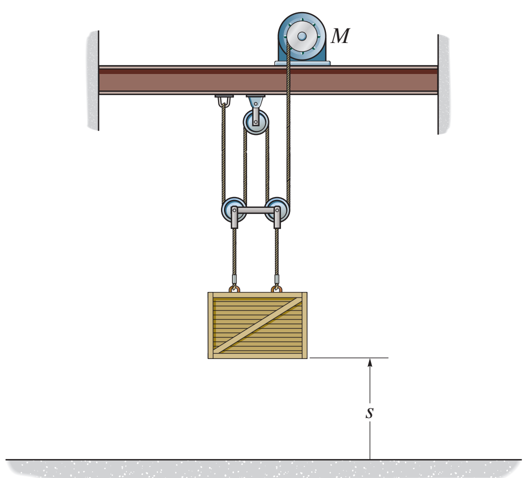
Prob. 14–36

14–37. If the track is to be designed so that the passengers of the roller coaster do not experience a normal force equal to zero or more than 4 times their weight, determine the limiting heights h_A and h_C so that this does not occur. The roller coaster starts from rest at position A. Neglect friction.



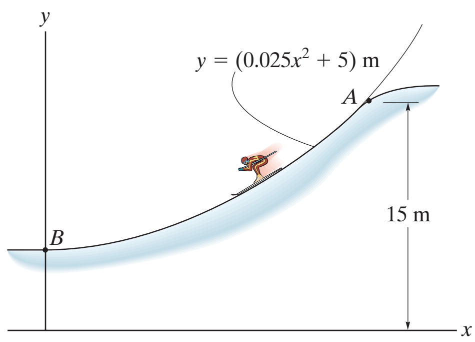
Prob. 14–37

14–38. If the force exerted by the motor M on the cable is 250 N, determine the speed of the 100-kg crate when it is hoisted to $s = 3$ m. The crate is at rest when $s = 0$.



Prob. 14–38

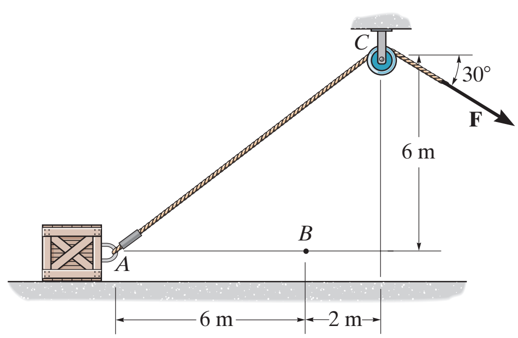
14–39. If the 60-kg skier passes point A with a speed of 5 m/s, determine his speed when he reaches point B. Also find the normal force exerted on him by the slope at this point. Neglect friction.



Prob. 14–39

***14–40.** If the 75-kg crate starts from rest at A, determine its speed when it reaches point B. The cable is subjected to a constant force of $F = 300$ N. Neglect friction and the size of the pulley.

14–41. If the 75-kg crate starts from rest at A, and its speed is 6 m/s when it passes point B, determine the constant force **F** exerted on the cable. Neglect friction and the size of the pulley.



Probs. 14–40/41

14.4 POWER AND EFFICIENCY

Power. The term "power" provides a useful basis for choosing the type of motor or machine which is required to do a certain amount of work in a given time. For example, two pumps may each be able to empty a reservoir if given enough time; however, the pump having the larger power will complete the job sooner.

The ***instantaneous power*** generated by a force that performs an amount of work dU within the time interval dt is therefore

$$P = \frac{dU}{dt}$$

If the work ΔU occurs over a finite time Δt, then the ***average power*** is determined from

$$P_{\text{avg}} = \frac{\Delta U}{\Delta t} \qquad (14\text{–}9)$$

We can also obtain the instantaneous power of a force $\mathbf{F}$ by expressing the work as $dU = \mathbf{F} \cdot d\mathbf{r}$. Then

$$P = \frac{dU}{dt} = \frac{\mathbf{F} \cdot d\mathbf{r}}{dt} = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$$

or

$$\boxed{P = \mathbf{F} \cdot \mathbf{v}} \qquad (14\text{–}10)$$

Hence, power is a *scalar*, where in this formulation $\mathbf{v}$ represents the velocity of the force $\mathbf{F}$ which acts on the particle.

The basic unit of power in the SI system is the watt (W). This unit is defined as

$$1\text{ W} = 1\text{ J/s} = 1\text{ N} \cdot \text{m/s}$$

The power output of this locomotive comes from the driving frictional force developed at its wheels. It is this force that overcomes the frictional resistance of the cars in tow and is able to lift the weight of the train up a grade.

Efficiency. The ***mechanical efficiency*** of a machine is defined as the ratio of the output of useful power produced by the machine to the input of power supplied to the machine. Hence,

$$\boxed{\varepsilon = \frac{\text{power output}}{\text{power input}}} \qquad (14\text{–}11)$$

If energy supplied to the machine occurs during the *same time interval* at which it is drawn, then the efficiency may also be expressed in terms of the ratio

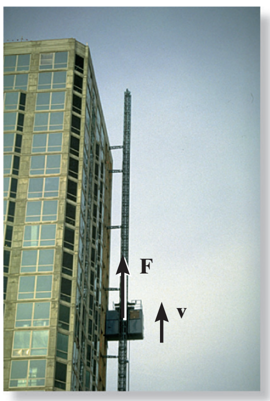
$$\varepsilon = \frac{\text{energy output}}{\text{energy input}} \tag{14–12}$$

Since machines consist of a series of moving parts, frictional forces will always be developed within the machine, and as a result, extra energy or power is needed to overcome these forces. Consequently, power output will be less than power input, and so *the efficiency of a machine is always less than 1*.

PROCEDURE FOR ANALYSIS

To obtain the power developed by a force consider using the following procedure.

- Determine the magnitude of the force **F** acting on the body (or particle) which causes the motion. This force is usually developed by a machine or engine placed either within or external to the body.
- If the body (or particle) is accelerating, it may be necessary to draw its free-body diagram and apply the equation of motion ($\Sigma \mathbf{F} = m\mathbf{a}$) to determine **F**.
- Once **F** and its velocity **v** have been determined, then the power is obtained by multiplying the force magnitude with the component of velocity acting in the direction of **F**, i.e., $P = \mathbf{F} \cdot \mathbf{v} = Fv \cos \theta$.
- In some problems the average power may be found by calculating the work done by **F** during a time period Δt. It is $P_{avg} = \Delta U/\Delta t$.

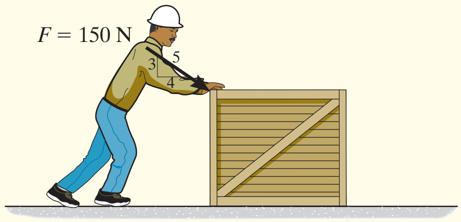


The power requirement of this hoist depends upon the vertical force **F** that acts on the elevator and causes it to move upward. If the velocity of the elevator is **v**, then the instantaneous power output is $P = \mathbf{F} \cdot \mathbf{v}$.

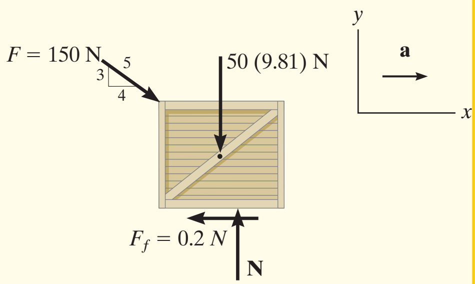
Refer to the companion website for Lecture Summary and Quiz videos.

EXAMPLE 14.7

The man in Fig. 14–15*a* pushes on the 50-kg crate with a force of $F = 150$ N. Determine the instantaneous power supplied by the man when $t = 4$ s and the average power during the four seconds. The coefficient of kinetic friction between the floor and the crate is $\mu_k = 0.2$. Initially the crate is at rest.



(a)



(b)

Fig. 14–15

SOLUTION

Instantaneous Power. To determine the power developed by the man, the velocity of the 150-N force must be obtained when $t = 4$ s. The free-body diagram of the crate is shown in Fig. 14–15*b*. Applying the equations of motion,

$$+\uparrow \Sigma F_y = ma_y; \qquad N - \left(\tfrac{3}{5}\right)150\text{ N} - 50(9.81)\text{ N} = 0$$

$$N = 580.5\text{ N}$$

$$\overset{+}{\rightarrow} \Sigma F_x = ma_x; \qquad \left(\tfrac{4}{5}\right)150\text{ N} - 0.2(580.5\text{ N}) = (50\text{ kg})a$$

$$a = 0.078\text{ m/s}^2$$

The velocity of the crate when $t = 4$ s is therefore

$(\overset{+}{\rightarrow})$

$$v = v_0 + a_c t$$

$$v = 0 + (0.078\text{ m/s}^2)(4\text{ s}) = 0.312\text{ m/s}$$

The power supplied to the crate by the man when $t = 4$ s is therefore

$$P = \mathbf{F}\cdot\mathbf{v} = F_x v = \left(\tfrac{4}{5}\right)(150\text{ N})(0.312\text{ m/s})$$

$$= 37.4\text{ W} \qquad \textit{Ans.}$$

Average Power. The displacement of the crate can be determined since the acceleration is known.

$$s = s_0 + v_0 t + \tfrac{1}{2}a_c t^2$$

$$s = 0 + 0 + \tfrac{1}{2}(0.078\text{ m/s}^2)(4\text{ s})^2 = 0.624\text{ m}$$

The work done by the 150-N force is therefore

$$U = 150\text{ N}\left(\tfrac{4}{5}\right)(0.624\text{ m}) = 74.88\text{ J}$$

Therefore, the average power is

$$P_{avg} = \frac{U}{\Delta t} = \frac{74.88\text{ J}}{4\text{ s}} = 18.7\text{ W} \qquad \textit{Ans.}$$

14

EXAMPLE 14.8

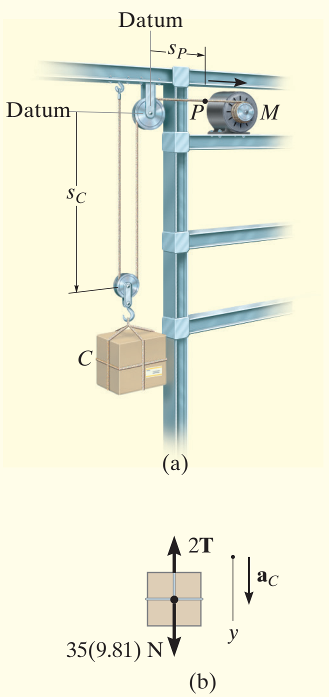


Fig. 14–16

The motor M of the hoist in Fig. 14–16*a* lifts the 35-kg crate C so that the acceleration of point P is $1.2\ \text{m/s}^2$. Determine the power that must be supplied to the motor at the instant P has a velocity of 0.6 m/s. Neglect the mass of the pulley and cable and take $\varepsilon = 0.85$.

SOLUTION

In order to find the power output of the motor, it is first necessary to determine the tension in the cable since this force is developed by the motor.

From the free-body diagram, Fig. 14–16*b*, we have

$$+\downarrow \Sigma F_y = ma_y; \qquad -2T + 35(9.81)\ \text{N} = (35\ \text{kg})\, a_C \qquad (1)$$

The acceleration of the crate can be obtained by using kinematics to relate it to the known acceleration of point P, Fig. 14–16*a*. Using the methods of absolute dependent motion, the coordinates s_C and s_P can be related to a constant portion of cable length l which is changing in the vertical and horizontal directions. We have $2s_C + s_P = l$. Taking the second time derivative of this equation yields

$$2a_C = -a_P \qquad (2)$$

Since $a_P = +1.2\ \text{m/s}^2$, then $a_C = -(1.2\ \text{m/s}^2)/2 = -0.6\ \text{m/s}^2$. What does the negative sign indicate? Substituting this result into Eq. 1 and *retaining* the negative sign since the acceleration in *both* Eq. 1 and Eq. 2 was considered positive downward, we have

$$-2T + 35(9.81)\ \text{N} = (35\ \text{kg})(-0.6\ \text{m/s}^2)$$

$$T = 182.2\ \text{N}$$

The power output required to draw the cable in at a rate of 0.6 m/s is therefore

$$P = \mathbf{T}\cdot\mathbf{v} = (182.2\ \text{N})(0.6\ \text{m/s})$$
$$= 109.3\ \text{W}$$

This *power output* requires that the motor provide a *power input* of

$$\text{power input} = \frac{1}{\varepsilon}(\text{power output})$$

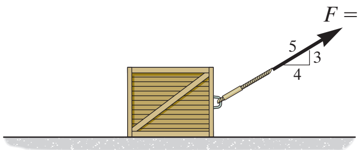
$$= \frac{1}{0.85}(109.3\ \text{W}) = 129\ \text{W} \qquad \textit{Ans.}$$

NOTE: Since the velocity of the crate is constantly changing, the power requirement is *instantaneous*.

Refer to the companion website for a self quiz of these Example problems.

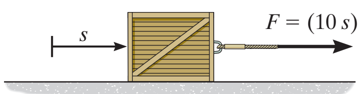
FUNDAMENTAL PROBLEMS

F14–7. If the contact surface between the 20-kg crate and the ground is smooth, determine the power of force **F** when $t = 4$ s. Initially, the crate is at rest.



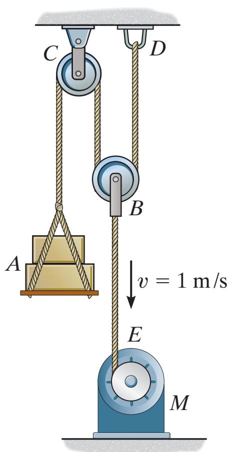
Prob. F14–7

F14–8. If $F = (10s)$ N, where s is in meters, and the contact surface between the 20-kg crate and the ground is smooth, determine the power of force **F** when $s = 5$ m. When $s = 0$, the crate is moving at $v = 1$ m/s.



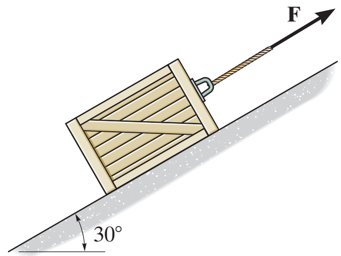
Prob. F14–8

F14–9. If the motor winds in the cable with a constant speed of $v = 1$ m/s, determine the power supplied to the motor at this instant. The load weighs 450 N and the efficiency of the motor is $\varepsilon = 0.8$. Neglect the mass of the pulleys.



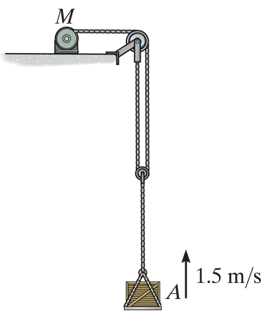
Prob. F14–9

F14–10. The coefficient of kinetic friction between the 20-kg crate and the inclined plane is $\mu_k = 0.2$. If the crate is traveling up the inclined plane with a constant velocity $v = 5$ m/s, determine the power of force **F**.



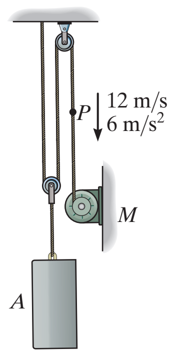
Prob. F14–10

F14–11. If the 50-kg load A is hoisted by motor M so that the load has a constant velocity of 1.5 m/s, determine the power input to the motor, which operates at an efficiency $\varepsilon = 0.8$.



Prob. F14–11

F14–12. At the instant shown, point P on the cable has a velocity $v_P = 12$ m/s, which is increasing at a rate of $a_P = 6$ m/s^2. Determine the power input to the motor M at this instant if the motor operates with an efficiency $\varepsilon = 0.8$. The mass of block A is 50 kg.

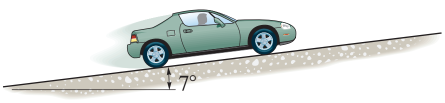


Prob. F14–12

PROBLEMS

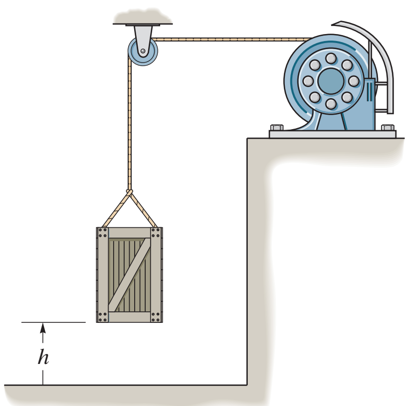
All solutions must include a free-body diagram.

14–42. An automobile having a mass of 2 Mg travels up a 7° slope at a constant speed of $v = 100$ km/h. If mechanical friction and wind resistance are neglected, determine the power developed by the engine if the automobile has an efficiency $\varepsilon = 0.65$.



Prob. 14–42

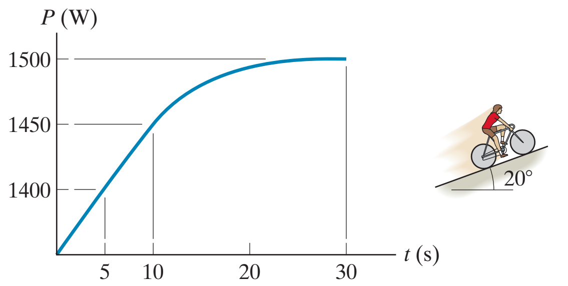
14–43. A motor hoists a 60-kg crate at a constant velocity to a height of $h = 5$ m in 2 s. If the indicated power of the motor is 3.2 kW, determine the motor's efficiency.



Prob. 14–43

***14–44.** A car has a mass m and accelerates along a horizontal straight road from rest such that the power is always a constant amount P. Determine how far it must travel to reach a speed of v.

14–45. Using the biomechanical power curve shown, determine the maximum speed attained by the rider and his bicycle, which have a total mass of 92 kg, as the rider ascends the 20° slope starting from rest.



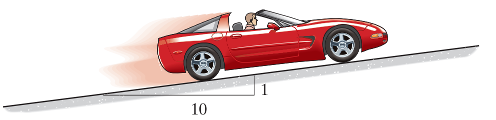
Prob. 14–45

14–46. A spring having a stiffness of 5 kN/m is compressed 400 mm. The stored energy in the spring is used to drive a machine which requires 90 W of power. Determine how long the spring can supply energy at the required rate.

14–47. To dramatize the loss of energy in an automobile, consider a car having a weight of 25 000 N that is traveling at 56 km/h. If the car is brought to a stop, determine how long a 100-W light bulb must burn to expend the same amount of energy.

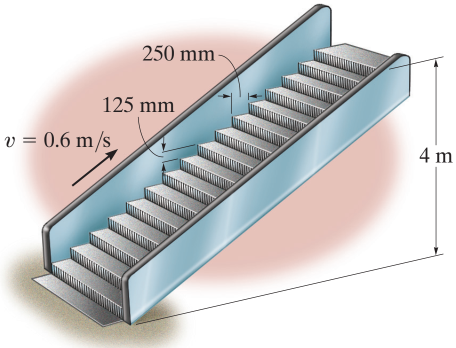
***14–48.** If the engine of a 1.5-Mg car generates a constant power of 15 kW, determine the speed of the car after it has traveled a distance of 200 m on a level road starting from rest. Neglect friction.

14–49. The 2-Mg car increases its speed uniformly from rest to 25 m/s in 30 s up the inclined road. Determine the maximum power that must be supplied by the engine, which operates with an efficiency of $\varepsilon = 0.8$. Also, find the average power supplied by the engine.



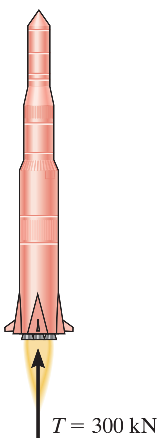
Prob. 14–49

14–50. The escalator steps move with a constant speed of 0.6 m/s. If the steps are 125 mm high and 250 mm in length, determine the power of a motor needed to lift an average mass of 150 kg per step. There are 32 steps.



Prob. 14–50

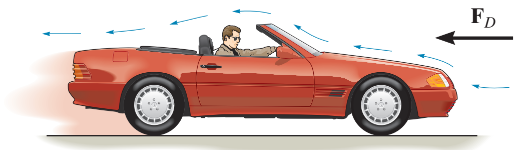
14–51. A rocket having a total mass of 8 Mg is fired vertically from rest. If the engines provide a constant thrust of $T = 300$ kN, determine the power output of the engines as a function of time. Neglect the effect of drag resistance and the loss of fuel mass and weight.



Prob. 14–51

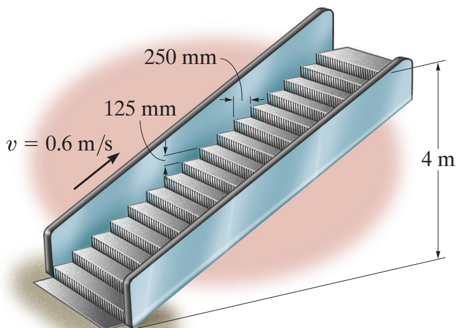
***14–52.** The sports car has a mass of 2.3 Mg, and while it is traveling at 28 m/s the driver causes it to accelerate at 5 m/s^2. If the drag resistance on the car due to the wind is $F_D = (0.3v^2)$ N, where v is the velocity in m/s, determine the power supplied to the engine at this instant. The engine has a running efficiency of $\varepsilon = 0.68$.

14–53. The sports car has a mass of 2.3 Mg and accelerates at 6 m/s^2, starting from rest. If the drag resistance on the car due to the wind is $F_D = (10v)$ N, where v is the velocity in m/s, determine the power supplied to the engine when $t = 5$ s. The engine has a running efficiency of $\varepsilon = 0.68$.



Probs. 14–52/53

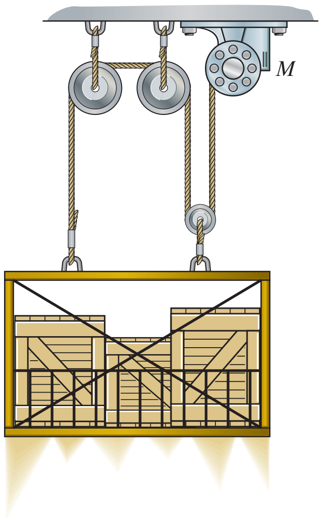
14–54. If the escalator in Prob. 14–50 is not moving, determine the constant speed at which a man having a mass of 80 kg must walk up the steps to generate 100 W of power—the same amount that is needed to power a standard light bulb.



Prob. 14–54

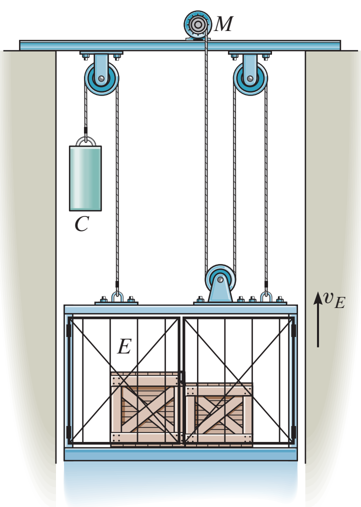
14–55. The motor is used to lift the loaded 500-kg elevator with a constant velocity $v_E = 8$ m/s. If the motor draws 60 kW of electrical power, determine the motor's efficiency. Neglect the mass of the pulleys and cable.

***14–56.** The 500-kg elevator starts from rest and moves upward with a constant acceleration $a_c = 2\,\text{m/s}^2$. Determine the power output of the motor M when $t = 3$ s. Neglect the mass of the pulleys and cable.



Probs. 14–55/56

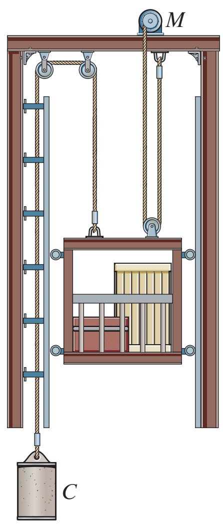
14–57. The elevator E and its freight have a total mass of 400 kg. Hoisting is provided by the motor M and the 60-kg block C. If the motor has an efficiency of $\varepsilon = 0.6$, determine the power that must be supplied to the motor when the elevator is hoisted upward at a constant speed of $v_E = 4$ m/s.



Prob. 14–57

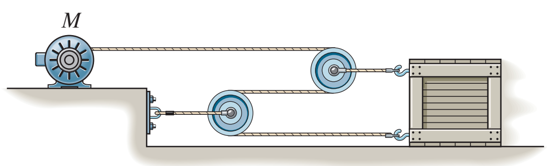
14–58. The material hoist and the load have a total mass of 800 kg and the counterweight C has a mass of 150 kg. At a given instant, the hoist has an upward velocity of 2 m/s and an acceleration of 1.5 m/s^2. Determine the power generated by the motor M at this instant if it operates with an efficiency of $\varepsilon = 0.8$.

14–59. The material hoist and the load have a total mass of 800 kg and the counterweight C has a mass of 150 kg. If the upward speed of the hoist increases uniformly from 0.5 m/s to 1.5 m/s in 1.5 s, determine the average power generated by the motor M during this time. The motor operates with an efficiency of $\varepsilon = 0.8$.



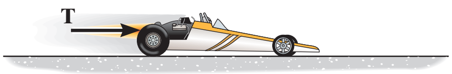
Probs. 14–58/59

***14–60.** The crate has a mass of 150 kg and rests on a surface for which the coefficients of static and kinetic friction are $\mu_s = 0.3$ and $\mu_k = 0.2$, respectively. If the motor M supplies a cable force of $F = (8t^2 + 20)$ N, where t is in seconds, determine the power output developed by the motor when $t = 5$ s.



Prob. 14–60

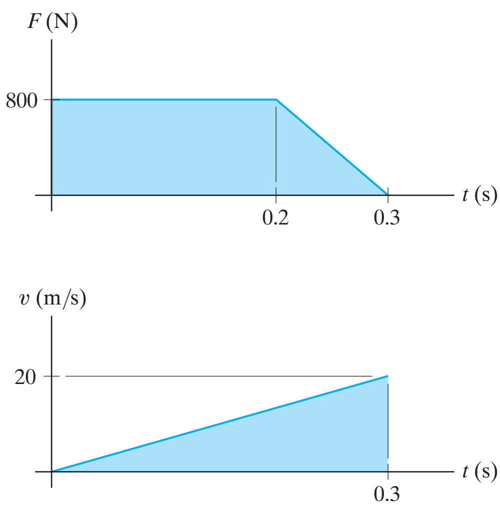
14–61. If the jet on the dragster supplies a constant thrust of $T = 20$ kN, determine the power generated by the jet as a function of time. Neglect drag and rolling resistance, and the loss of fuel. The dragster has a mass of 1 Mg and starts from rest.



Prob. 14–61

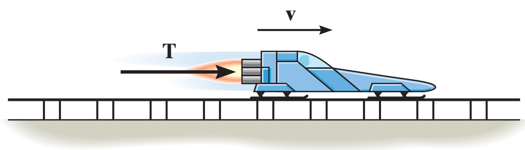
14–62. An athlete pushes against an exercise machine with a force that varies with time as shown in the first graph. Also, the velocity of the athlete's arm acting in the same direction as the force varies with time as shown in the second graph. Determine the power applied as a function of time and the work done in $t = 0.3$ s.

14–63. An athlete pushes against an exercise machine with a force that varies with time as shown in the first graph. Also, the velocity of the athlete's arm acting in the same direction as the force varies with time as shown in the second graph. Determine the maximum power developed during the 0.3-second time period.



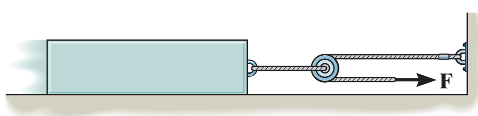
Probs. 14–62/63

***14–64.** The rocket sled has a mass of 4 Mg and travels from rest along the horizontal track for which the coefficient of kinetic friction is $\mu_k = 0.2$. If the engine provides a constant thrust $T = 150$ kN, determine the power output of the engine as a function of time. Neglect the loss of fuel mass and air resistance.



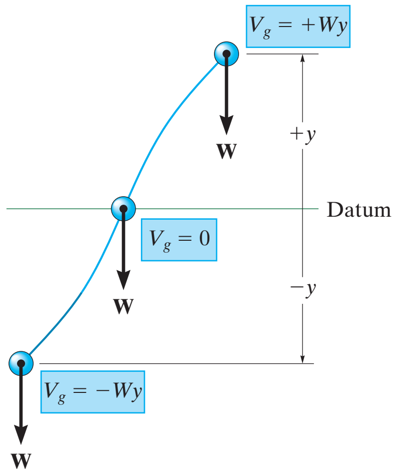
Prob. 14–64

14–65. The block has a mass of 150 kg and rests on a surface for which the coefficients of static and kinetic friction are $\mu_s = 0.5$ and $\mu_k = 0.4$, respectively. If a force $F = (60t^2)$ N, where t is in seconds, is applied to the cable, determine the power developed by the force when $t = 5$ s. *Hint*: First determine the time needed for the force to cause motion.



Prob. 14–65

14.5 CONSERVATIVE FORCES AND POTENTIAL ENERGY



Gravitational potential energy

Fig. 14–17

Conservative Force. If the work of a force is independent of its path and depends only on the force's initial and final positions on the path, then the force is called a ***conservative force***. Examples of conservative forces are the weight of a particle and the force developed by a spring. The work done by the weight depends only on the *vertical* displacement of the weight, and the work done by a spring force depends only on the spring's elongation or compression.

In contrast to a conservative force, consider the force of friction exerted on a sliding object by a fixed surface. The work done depends on the path—the longer the path, the greater the work. Consequently, frictional forces are ***nonconservative***. The work is dissipated from the body in the form of heat.

Energy. ***Energy*** is defined as the capacity for doing work. There are essentially two types of energy, kinetic and potential. Kinetic energy is associated with the motion of the particle. For example, if a particle is originally at rest, then the principle of work and energy requires $\Sigma U_{1\to 2} = T_2$. In other words, the work done on the particle is transferred into kinetic energy, which gives the particle a speed v. When the energy depends upon the position of the particle, measured from a fixed datum or reference plane, it is called ***potential energy***. Thus, potential energy is a measure of the amount of work a conservative force must do to move a particle from a given position to the datum. In mechanics, the potential energy created by gravity (weight) and an elastic spring is important.

Gravitational Potential Energy. If a particle is located a distance y above an arbitrarily selected datum, as shown in Fig. 14–17, the particle's weight **W** has positive ***gravitational potential energy***, V_g, since **W** has the capacity of doing positive work when the particle is moved back down to the datum. Likewise, if the particle is located a distance y below the datum, V_g is negative since the weight does negative work when the particle is moved back up to the datum. At the datum $V_g = 0$.*

In general then, if y is positive upward, the gravitational potential energy of the particle of weight W is

$$V_g = Wy \tag{14–13}$$

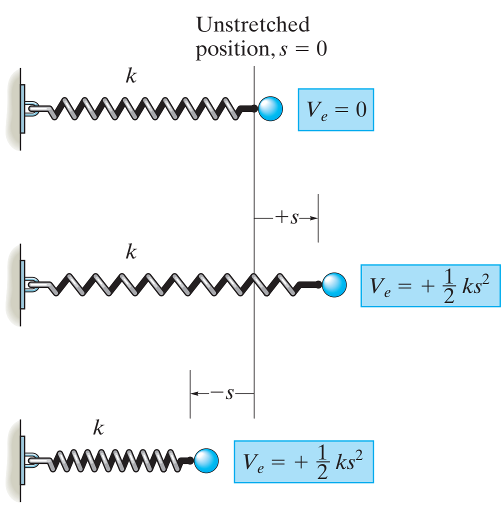
Gravitational potential energy of this weight is increased as it is hoisted upward.

*Here the weight is assumed to be *constant*. This assumption is suitable for small differences in elevation Δy. If the elevation change is significant, however, a variation of weight with elevation must be taken into account (see Prob. 14–83).

Elastic Potential Energy. When an elastic spring is elongated or compressed a distance s from its unstretched position, the force on the spring does work and thereby stores elastic potential energy V_e in the spring. This energy is

$$V_e = +\tfrac{1}{2}ks^2 \tag{14–14}$$

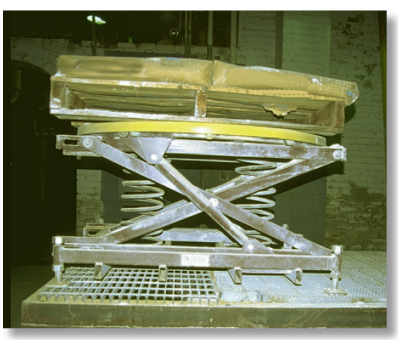
Here V_e is always positive since, in the deformed position, the force of the spring has the *capacity* or "potential" for always doing positive work on the particle when the spring is returned to its unstretched position, Fig. 14–18.



Elastic potential energy

Fig. 14–18

14



The weight of the sacks resting on this platform causes potential energy to be stored in the supporting springs. As each sack is removed, the platform will *rise* slightly since some of the potential energy within the springs will be transformed into an increase in gravitational potential energy of the remaining sacks. Such a device is useful for removing the sacks without having to bend over to pick them up as they are unloaded.

Potential Function. In the general case, if a particle is subjected to both gravitational and elastic forces, the particle's potential energy can be expressed as a ***potential function***, which is the algebraic sum

$$V = V_g + V_e \tag{14–15}$$

Measurement of V depends on the location of the particle with respect to a selected datum in accordance with Eqs. 14–13 and 14–14.

The work done by the gravitational and elastic forces in moving the particle from one point to another point is measured by the *difference* of this function, i.e.,

$$U_{1-2} = V_1 - V_2 \tag{14–16}$$

For example, the potential function for a particle of weight W suspended from a spring can be expressed in terms of its position, s, measured from a datum located at the unstretched length of the spring, Fig. 14–19. We have

$$V = V_g + V_e$$
$$= -Ws + \tfrac{1}{2}ks^2$$

If the particle moves from s_1 to a lower position s_2, then applying Eq. 14–16 the work of $\mathbf{W}$ and $\mathbf{F}_s$ is

$$U_{1-2} = V_1 - V_2 = \left(-Ws_1 + \tfrac{1}{2}ks_1^2\right) - \left(-Ws_2 + \tfrac{1}{2}ks_2^2\right)$$
$$= W(s_2 - s_1) - \left(\tfrac{1}{2}ks_2^2 - \tfrac{1}{2}ks_1^2\right)$$

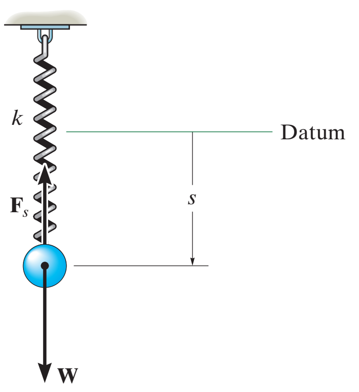


Fig. 14–19

When the displacement along the path is infinitesimal, i.e., from point (x, y, z) to $(x + dx, y + dy, z + dz)$, Eq. 14–16 becomes

$$\begin{aligned} dU &= V(x, y, z) - V(x + dx, y + dy, z + dz) \\ &= -dV(x, y, z) \end{aligned} \qquad (14\text{–}17)$$

If we represent both the force and its displacement as Cartesian vectors, then the work can also be expressed as

$$\begin{aligned} dU &= \mathbf{F} \cdot d\mathbf{r} = (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\ &= F_x\,dx + F_y\,dy + F_z\,dz \end{aligned}$$

Substituting this result into Eq. 14–17 and expressing the differential $dV(x, y, z)$ in terms of its partial derivatives yields

$$F_x\,dx + F_y\,dy + F_z\,dz = -\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right)$$

Since changes in x, y, and z are all independent of one another, this equation is satisfied provided

$$F_x = -\frac{\partial V}{\partial x}, \qquad F_y = -\frac{\partial V}{\partial y}, \qquad F_z = -\frac{\partial V}{\partial z} \qquad (14\text{–}18)$$

Thus,

$$\mathbf{F} = -\frac{\partial V}{\partial x}\mathbf{i} - \frac{\partial V}{\partial y}\mathbf{j} - \frac{\partial V}{\partial z}\mathbf{k} \qquad (14\text{–}19)$$

This equation relates a force $\mathbf{F}$ to its potential function V and thereby provides a mathematical criterion for proving that $\mathbf{F}$ is conservative. For example, the gravitational potential function for a weight located a distance y above a datum is $V_g = Wy$. To prove that $\mathbf{W}$ is conservative, it is necessary to show that it satisfies Eq. 14–18 (or Eq. 14–19), in which case

$$F_y = -\frac{\partial V}{\partial y}; \qquad F_y = -\frac{\partial}{\partial y}(Wy) = -W$$

The negative sign indicates that $\mathbf{W}$ acts downward, opposite to positive y, which is upward. Use Eq. 14–19 to show the spring force $F = kx$ is a conservative force.

14

14.6 CONSERVATION OF ENERGY

When a particle is acted upon by a system of *both* conservative and nonconservative forces, the portion of the work done by the *conservative forces* can be written in terms of the difference in their potential energies using Eq. 14–16, i.e., $(\Sigma U_{1-2})_{\text{cons.}} = V_1 - V_2$. As a result, the principle of work and energy can be written as

$$T_1 + V_1 + (\Sigma U_{1-2})_{\text{noncons.}} = T_2 + V_2 \qquad (14\text{–}20)$$

Here $(\Sigma U_{1-2})_{\text{noncons.}}$ represents the work of the nonconservative forces acting on the particle. If *only conservative forces* do work then we have

$$\boxed{T_1 + V_1 = T_2 + V_2} \qquad (14\text{–}21)$$

This equation is referred to as the ***conservation of mechanical energy*** or simply the ***conservation of energy***. It states that during the motion the sum of the particle's kinetic and potential energies remains *constant*. For this to occur, kinetic energy must be transformed into potential energy, and vice versa.

For example, if a ball of weight **W** is dropped from a height h above the ground (datum), Fig. 14–20, the potential energy of the ball is maximum before it is dropped, at which time its kinetic energy is zero. The total mechanical energy of the ball in its initial position is thus

$$E = T_1 + V_1 = 0 + Wh = Wh$$

When the ball has fallen a distance $h/2$, its speed can be determined by using $v^2 = v_0^2 + 2a_c(y - y_0)$, which yields $v = \sqrt{2g(h/2)} = \sqrt{gh}$. The energy of the ball at the mid-height position is therefore

$$E = T_2 + V_2 = \frac{1}{2}\frac{W}{g}\left(\sqrt{gh}\right)^2 + W\left(\frac{h}{2}\right) = Wh$$

Just before the ball strikes the ground, its potential energy is zero and its speed is $v = \sqrt{2gh}$. Here, again, the total energy of the ball is

$$E = T_3 + V_3 = \frac{1}{2}\frac{W}{g}\left(\sqrt{2gh}\right)^2 + 0 = Wh$$

Note that when the ball comes in contact with the ground, it deforms somewhat, and provided the ground is hard enough, the ball will rebound off the surface, reaching a new height h', which will be *less* than the height h from which it was first released. Neglecting air friction, the difference in height accounts for an energy loss, $E_l = W(h - h')$, which occurs during the collision. Portions of this loss produce noise, localized deformation of the ball and ground, and heat.

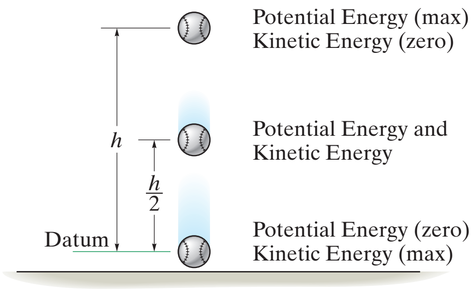


Fig. 14–20

System of Particles. If a system of particles is *subjected only to conservative forces*, then an equation similar to Eq. 14–21 can be written for the particles. Applying the ideas of the preceding discussion, Eq. 14–8 ($\Sigma T_1 + \Sigma U_{1-2} = \Sigma T_2$) becomes

$$\Sigma T_1 + \Sigma V_1 = \Sigma T_2 + \Sigma V_2 \qquad (14\text{–}22)$$

Here, the sum of the system's initial kinetic and potential energies is equal to the sum of the system's final kinetic and potential energies. In other words, for the system $\Sigma T + \Sigma V = \text{const}$.

PROCEDURE FOR ANALYSIS

The conservation of energy equation can be used to solve problems involving *velocity*, *displacement*, and *conservative force systems*. It is generally easier to apply than the principle of work and energy because this equation requires specifying the particle's or system's kinetic and potential energies at only *two points* along the path, rather than finding the work when the particle moves through a displacement. For application it is suggested that the following procedure be used.

Potential Energy.

- Draw two diagrams showing the particle or system in its initial and final position along the path.
- If the particle or system is subjected to a vertical displacement, establish the fixed horizontal datum from which to measure the particle's gravitational potential energy V_g.
- Determine the elevation y of the particle from the datum and the stretch or compression s of any connecting springs.
- Recall $V_g = Wy$, where y is positive upward from the datum and negative downward from the datum; also for a spring, $V_e = \frac{1}{2}ks^2$, which is *always positive*.

Conservation of Energy.

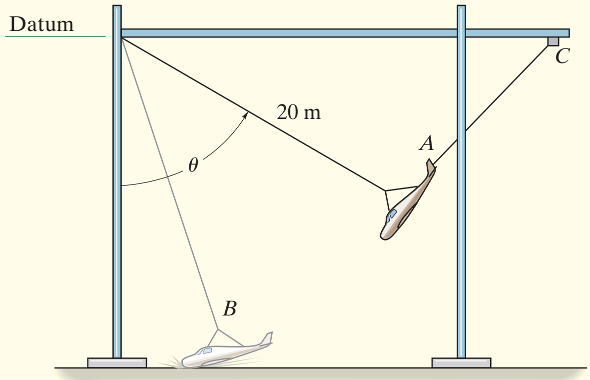
- Apply the equation $T_1 + V_1 = T_2 + V_2$.
- When determining the kinetic energy, $T = \frac{1}{2}mv^2$, remember that the particle's speed v must be measured from an inertial reference frame.

14

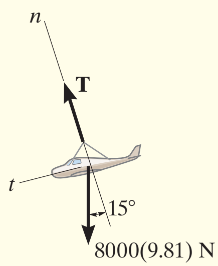
Refer to the companion website for Lecture Summary and Quiz videos.

EXAMPLE 14.9

The gantry structure in the photo is used to test the response of an airplane during a crash. As shown in Fig. 14–21*a*, the plane, having a mass of 8 Mg, is hoisted back until $\theta = 60^\circ$, and then the pull-back cable AC is released when the plane is at rest. Determine the speed of the plane just before it crashes into the ground, $\theta = 15^\circ$. Also, what is the maximum tension developed in the supporting cable at this instant? Neglect the size of the airplane and the effect of lift caused by the wings during the motion.



(a)



(b)

Fig. 14–21

SOLUTION

Since the force of the cable does *no work* on the plane, it must be obtained using the equation of motion. First, however, we must determine the plane's speed at B.

Potential Energy. For convenience, the datum has been established at the top of the gantry, Fig. 14–21*a*.

Conservation of Energy.

$$T_A + V_A = T_B + V_B$$

$$0 - 8000\text{ kg }(9.81\text{ m/s}^2)(20 \cos 60^\circ\text{ m}) =$$

$$\tfrac{1}{2}(8000\text{ kg})v_B^2 - 8000\text{ kg }(9.81\text{ m/s}^2)(20 \cos 15^\circ\text{ m})$$

$$v_B = 13.52\text{ m/s} = 13.5\text{ m/s} \qquad \textit{Ans.}$$

Equation of Motion. From the free-body diagram when the plane is at B, Fig. 14–21*b*, we have

$$+\nwarrow \quad \Sigma F_n = ma_n;$$

$$T - (8000(9.81)\text{ N}) \cos 15^\circ = (8000\text{ kg})\frac{(13.52\text{ m/s})^2}{20\text{ m}}$$

$$T = 149\text{ kN} \qquad \textit{Ans.}$$

EXAMPLE 14.10

The ram R shown in Fig. 14–22*a* has a mass of 100 kg and is released from rest 0.75 m from the top of a spring, A, that has a stiffness $k_A = 12$ kN/m. If a second spring B, having a stiffness $k_B = 15$ kN/m, is "nested" in A, determine the maximum displacement of A needed to stop the downward motion of the ram. The unstretched length of each spring is indicated in the figure. Neglect the mass of the springs.

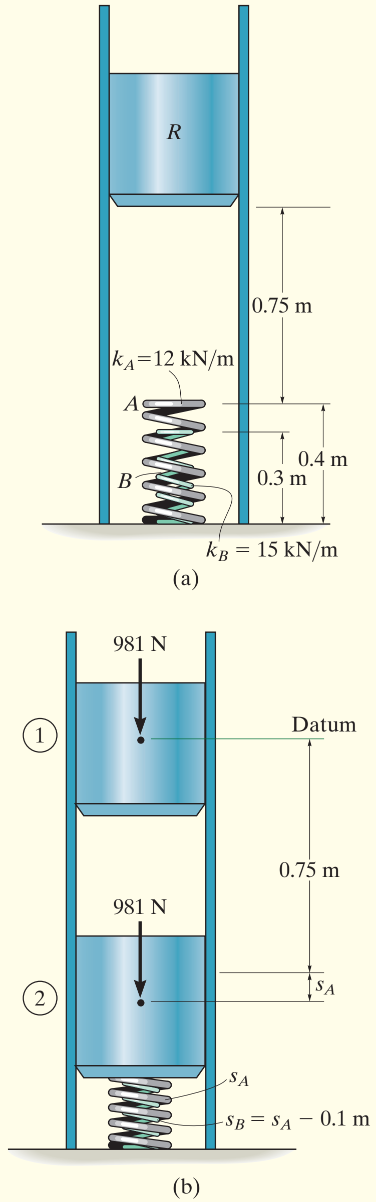


Fig. 14–22

SOLUTION

Here we will consider the system to be the ram and the two springs.

Potential Energy. We will *assume* that the ram compresses *both* springs at the instant it comes to rest. The datum is located through the center of gravity of the ram at its initial position, Fig. 14–22*b*. When the kinetic energy is reduced to zero ($v_2 = 0$), A is compressed a distance s_A and B compresses $s_B = s_A - 0.1$ m.

Conservation of Energy.

$$T_1 + V_1 = T_2 + V_2$$

$$0 + 0 = 0 + \left[\tfrac{1}{2}k_A s_A^2 + \tfrac{1}{2}k_B(s_A - 0.1)^2 - Wh\right]$$

$$0 + 0 = 0 + \left[\tfrac{1}{2}(12\,000 \text{ N/m})s_A^2 + \tfrac{1}{2}(15\,000 \text{ N/m})(s_A - 0.1 \text{ m})^2 - 981 \text{ N}\,(0.75 \text{ m} + s_A)\right]$$

Rearranging the terms,

$$13\,500 s_A^2 - 2481 s_A - 660.75 = 0$$

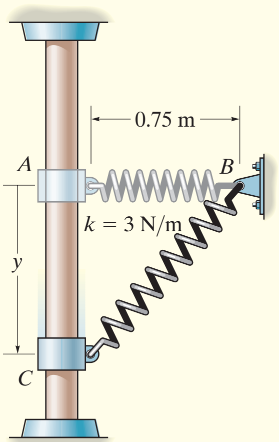
Using the quadratic formula and solving for the positive root, we have

$$s_A = 0.331 \text{ m} \qquad \textit{Ans.}$$

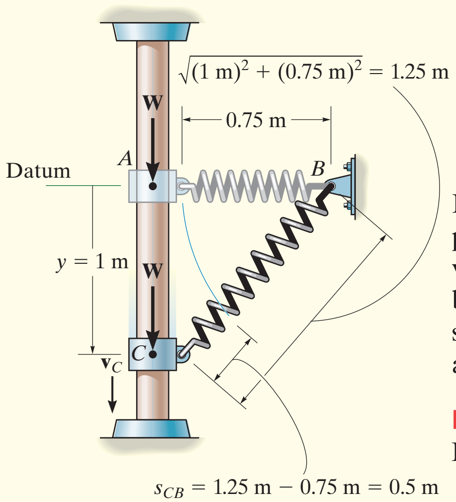
Since $s_B = 0.331 \text{ m} - 0.1 \text{ m} = 0.231$ m, which is positive, the assumption that *both* springs are compressed by the ram is correct.

NOTE: The second root, $s_A = -0.148$ m, does not represent the physical situation. Since positive s is measured downward, the negative sign indicates that spring A would have to be "extended" by an amount of 0.148 m to stop the ram.

EXAMPLE 14.11



(a)



(b)

Fig. 14–23

A smooth 2-kg collar, shown in Fig. 14–23*a*, fits loosely on the vertical shaft. If the spring is unstretched when the collar is in the position *A*, determine the speed at which the collar is moving when $y = 1$ m, if (a) it is released from rest at *A*, and (b) it is released at *A* with an *upward* velocity $v_A = 2$ m/s.

SOLUTION

The system here consists of the collar and spring.

Part (a) Potential Energy. For convenience, the datum is established through *AB*, Fig. 14–23*b*. When the collar is at *C*, the gravitational potential energy is $-(mg)y$, since the collar is *below* the datum. The elastic potential energy is $\frac{1}{2}ks_{CB}^2$, where $s_{CB} = 0.5$ m, which represents the *stretch* in the spring as shown in the figure.

Conservation of Energy.

$$T_A + V_A = T_C + V_C$$

$$0 + 0 = \tfrac{1}{2}mv_C^2 + \left[\tfrac{1}{2}ks_{CB}^2 - mgy\right]$$

$$0 + 0 = \left[\tfrac{1}{2}(2\text{ kg})v_C^2\right] + \left[\tfrac{1}{2}(3\text{ N/m})(0.5\text{ m})^2 - 2(9.81)\text{ N}\,(1\text{ m})\right]$$

$$v_C = 4.39\text{ m/s}\downarrow \qquad \textit{Ans.}$$

If this problem was solved using the equation of motion or the principle of work and energy, then for *both* of these methods, the variation of the magnitude and direction of the spring force must be taken into account (see Example 13.4). Here, however, the above solution is clearly advantageous since *only* the energy at the initial and final points of the path must be determined.

Part (b) Conservation of Energy. If $v_A = 2$ m/s, using the data in Fig. 14–23*b*, we have

$$T_A + V_A = T_C + V_C$$

$$\tfrac{1}{2}mv_A^2 + 0 = \tfrac{1}{2}mv_C^2 + \left[\tfrac{1}{2}ks_{CB}^2 - mgy\right]$$

$$\tfrac{1}{2}(2\text{ kg})(2\text{ m/s})^2 + 0 = \tfrac{1}{2}(2\text{ kg})v_C^2 + \left[\tfrac{1}{2}(3\text{ N/m})(0.5\text{ m})^2 - 2(9.81)\text{ N}\,(1\text{ m})\right]$$

$$v_C = 4.82\text{ m/s}\downarrow \qquad \textit{Ans.}$$

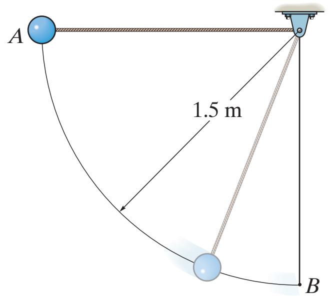
NOTE: The kinetic energy of the collar depends only on the *magnitude* of velocity, and therefore it is immaterial if the collar is moving up or down at 2 m/s when released at *A*.

Refer to the companion website for a self quiz of these Example problems.

14

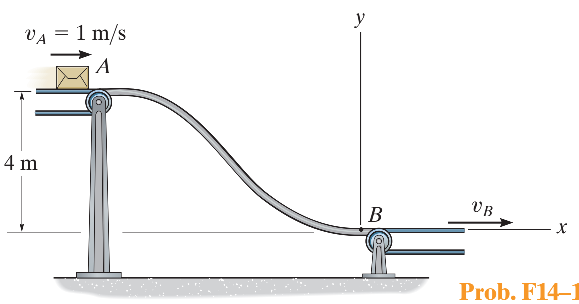
FUNDAMENTAL PROBLEMS

F14–13. The 2-kg pendulum bob is released from rest when it is at A. Determine the speed of the bob and the tension in the cord when the bob passes through its lowest position, B.



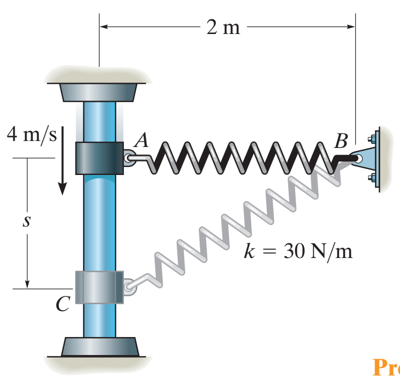
Prob. F14–13

F14–14. The 2-kg package leaves the conveyor belt at A with a speed of $v_A = 1$ m/s and slides down the smooth ramp. Determine the required speed of the conveyor belt at B so that the package can be delivered without slipping on the belt. Also, find the normal reaction the curved portion of the ramp exerts on the package at B if $\rho_B = 2$ m.



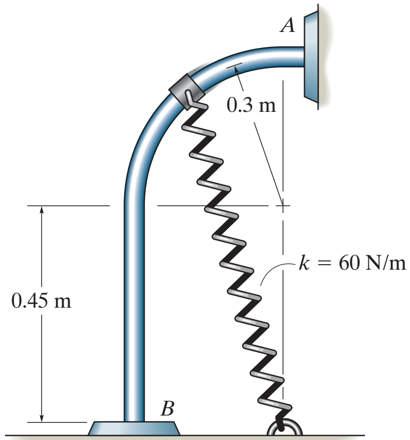
Prob. F14–14

F14–15. The 2-kg collar is given a downward velocity of 4 m/s when it is at A. If the spring has an unstretched length of 1 m and a stiffness of $k = 30$ N/m, determine the velocity of the collar at $s = 1$ m.



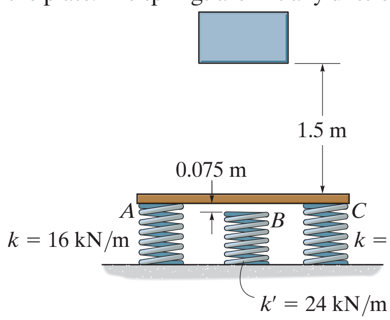
Prob. F14–15

F14–16. The 2.5-kg smooth collar is released from rest at A. Determine its speed when it strikes the stop B. The spring has an unstretched length of 0.15 m.



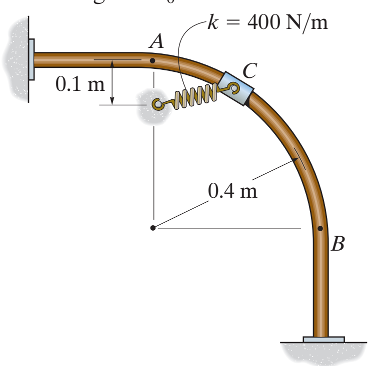
Prob. F14–16

F14–17. The 35-kg block is released from rest 1.5 m above the plate. Determine the compression of each spring when the block momentarily comes to rest after striking the plate. Neglect the mass of the plate. The springs are initially unstretched.



Prob. F14–17

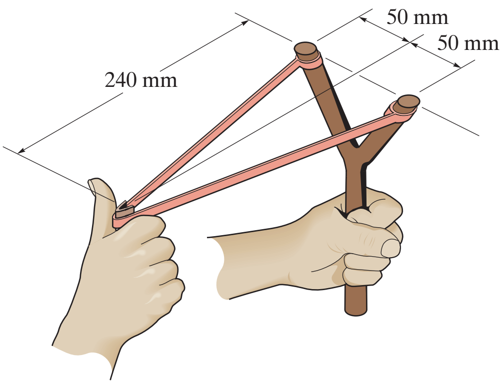
F14–18. The 4-kg collar C has a velocity of $v_A = 2$ m/s when it is at A. If the guide rod is smooth, determine the speed of the collar when it is at B. The spring has an unstretched length of $l_0 = 0.2$ m.



Prob. F14–18

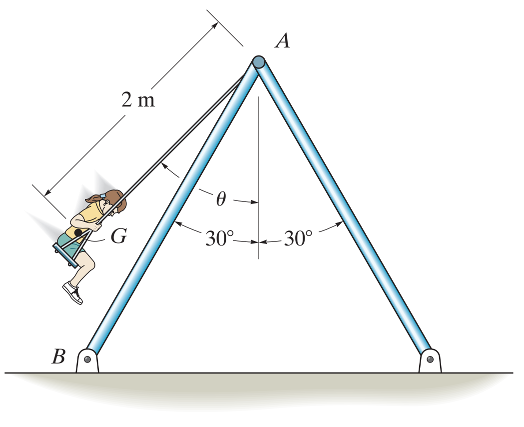
PROBLEMS

14–66. Each of the two elastic rubber bands of the slingshot has an unstretched length of 180 mm. If they are pulled back to the position shown and released from rest, determine the maximum height the 30-g pellet will reach if it is fired vertically upward. Neglect the mass of the rubber bands and the change in elevation of the pellet while it is constrained by the rubber bands. Each rubber band has a stiffness $k = 80$ N/m.



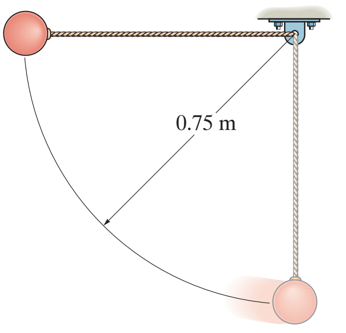
Prob. 14–66

14–67. The girl has a mass of 40 kg and center of mass at G. If she is swinging to a maximum height defined by $\theta = 60°$, determine the force developed along each of the four supporting posts such as AB at the instant $\theta = 0°$. The swing is centrally located between the posts.



Prob. 14–67

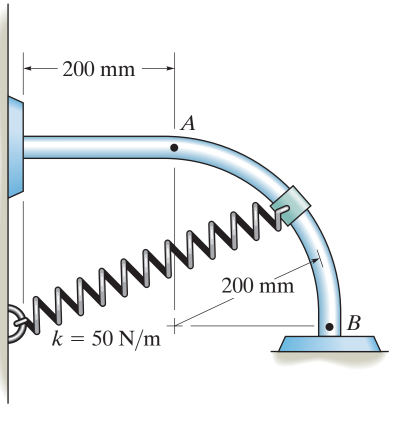
***14–68.** The bob of the pendulum has a mass of 0.2 kg and is released from rest when it is in the horizontal position shown. Determine its speed and the tension in the cord at the instant the bob passes through its lowest position.



Prob. 14–68

14–69. The 5-kg collar has a velocity of 5 m/s to the right when it is at A. It then travels down along the smooth guide. Determine the speed of the collar when it reaches point B, which is located just before the end of the curved portion of the rod. The spring has an unstretched length of 100 mm. Also, what is the normal force on the collar at this instant?

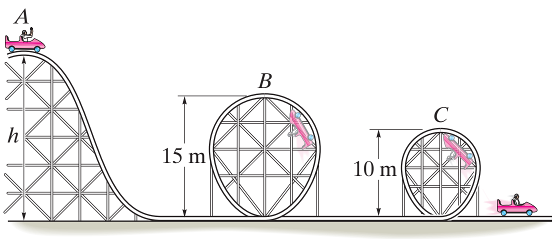
14–70. The 10-kg collar has a velocity of 3 m/s to the right when it is at A. It then travels along the smooth guide. Determine its speed when it reaches point B, which is located just before the end of the curved portion of the rod, and the normal force it exerts on the rod at this point. The spring has an unstretched length of 150 mm.



Probs. 14–69/70

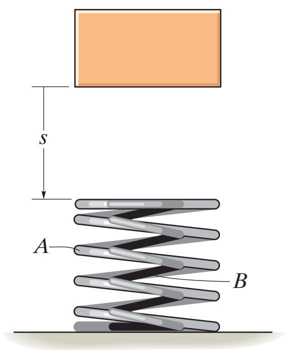
14–71. The roller coaster car has a mass of 700 kg, including its passenger. If it starts from the top of the hill A with a speed $v_A = 3$ m/s, determine the minimum height h of the hill crest so that the car travels around the inside loops without leaving the track. Neglect friction, the mass of the wheels, and the size of the car. What is the normal reaction on the car when the car is at B and when it is at C? Take $\rho_B = 7.5$ m and $\rho_C = 5$ m.

***14–72.** The roller coaster car has a mass of 700 kg, including its passenger. If it is released from rest at the top of the hill A, determine the minimum height h of the hill crest so that the car travels around both inside loops without leaving the track. Neglect friction, the mass of the wheels, and the size of the car. What is the normal reaction on the car when the car is at B and when it is at C? Take $\rho_B = 7.5$ m and $\rho_C = 5$ m.



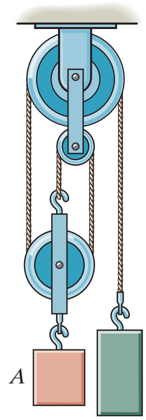
Probs. 14–71/72

14–73. Two equal-length springs are "nested" together in order to form a shock absorber. If it is designed to arrest the motion of a 2-kg mass that is dropped $s = 0.5$ m above the top of the springs from an at-rest position, and the maximum compression of the springs is to be 0.2 m, determine the required stiffness of the inner spring, k_B, if the outer spring has a stiffness $k_A = 400$ N/m.



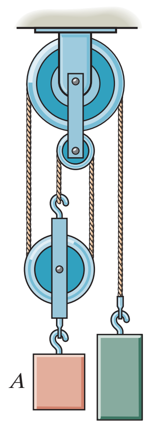
Prob. 14–73

14–74. The assembly consists of two blocks A and B which have a mass of 20 kg and 30 kg, respectively. Determine the speed of each block when B descends 1.5 m. The blocks are released from rest. Neglect the mass of the pulleys and cords.



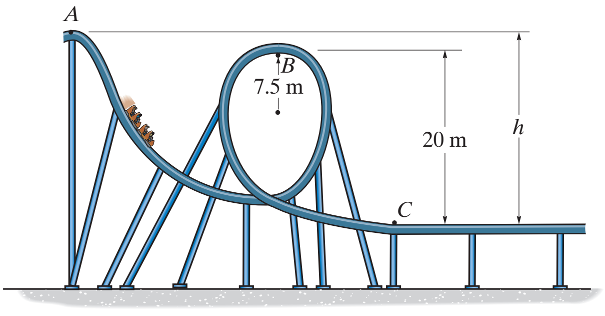
Prob. 14–74

14–75. The assembly consists of two blocks A and B, which have a mass of 20 kg and 30 kg, respectively. Determine the distance B must descend in order for A to achieve a speed of 3 m/s starting from rest.



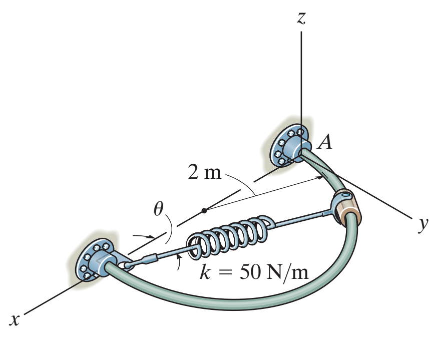
Prob. 14–75

***14–76.** The roller coaster car having a mass m is released from rest at point A. If the track is to be designed so that the car does not leave it at B, determine the required height h. Also, find the speed of the car when it reaches point C. Neglect friction.



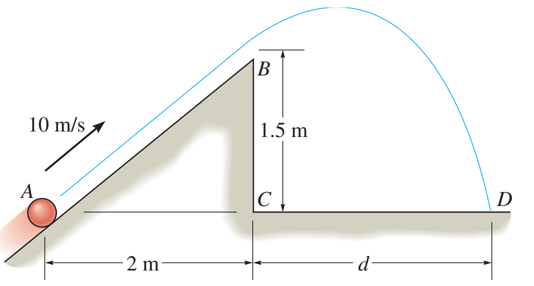
Prob. 14–76

14–77. The spring has a stiffness $k = 50$ N/m and an unstretched length of 0.3 m. If it is attached to the 2-kg smooth collar and the collar is released from rest at A ($\theta = 0°$), determine the speed of the collar when $\theta = 60°$. The motion occurs in the horizontal plane. Neglect the size of the collar.



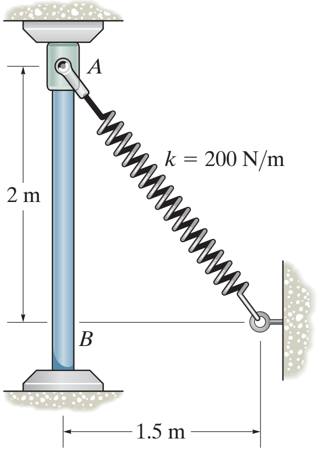
Prob. 14–77

14–78. The 2-kg ball of negligible size is fired from point A with an initial velocity of 10 m/s up the smooth inclined plane. Determine the distance from point C to where it hits the horizontal surface at D. Also, what is its velocity when it strikes the surface?



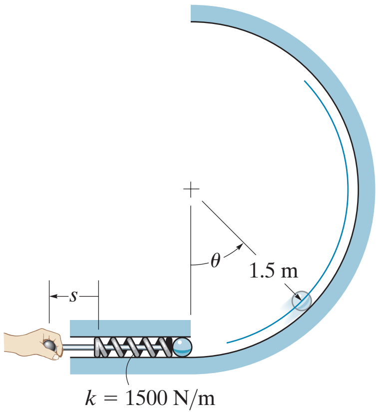
Prob. 14–78

14–79. The spring has a stiffness $k = 200$ N/m and an unstretched length of 0.5 m. If it is attached to the 3-kg smooth collar and the collar is released from rest at A, determine the speed of the collar when it reaches B. Neglect the size of the collar.



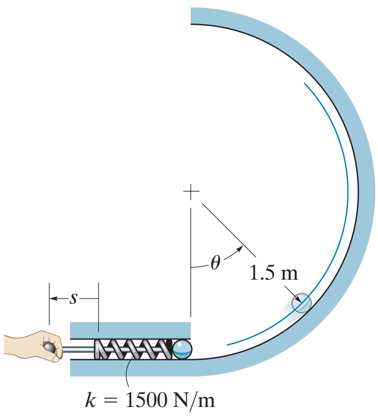
Prob. 14–79

***14-80.** When $s = 0$, the spring on the firing mechanism is unstretched. If the arm is pulled back such that $s = 100$ mm and released, determine the maximum angle θ the 0.3-kg ball will travel without leaving the circular track. Assume all surfaces of contact to be smooth. Neglect the mass of the spring and the size of the ball.



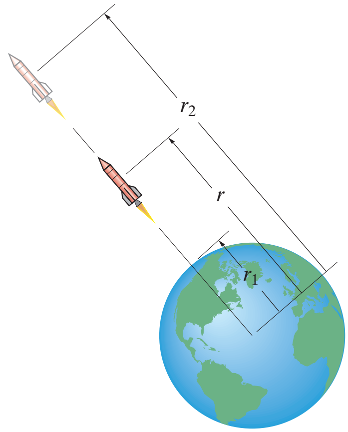
Prob. 14–80

14–81. When $s = 0$, the spring on the firing mechanism is unstretched. If the arm is pulled back such that $s = 100$ mm and released, determine the speed of the 0.3-kg ball and the normal reaction of the circular track on the ball when $\theta = 60°$. Assume all surfaces of contact to be smooth. Neglect the mass of the spring and the size of the ball.



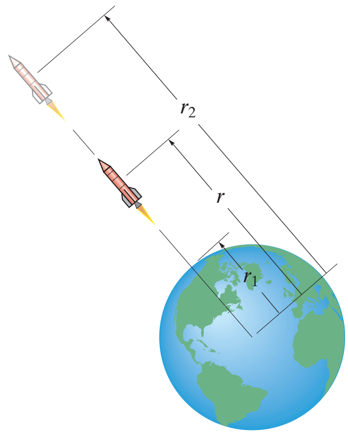
Prob. 14–81

14–82. A rocket of mass m is fired vertically from the surface of the earth, i.e., at $r = r_1$. Assuming that no mass is lost as it travels upward, determine the work it must do against gravity to reach a distance r_2. The force of gravity is $F = GM_em/r^2$ (Eq. 13–1), where M_e is the mass of the earth and r the distance between the rocket and the center of the earth.



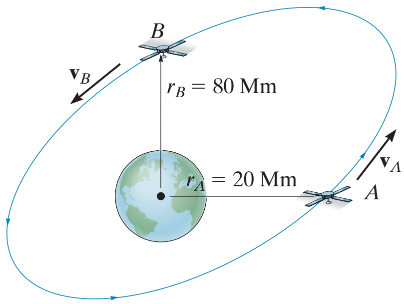
Prob. 14–82

14–83. If the mass of the earth is M_e, show that the gravitational potential energy of a body of mass m located a distance r from the center of the earth is $V_g = -GM_em/r$. Recall that the gravitational force acting between the earth and the body is $F = G(M_em/r^2)$, Eq. 13–1. For the calculation, locate the datum at $r \rightarrow \infty$. Also, prove that F is a conservative force.



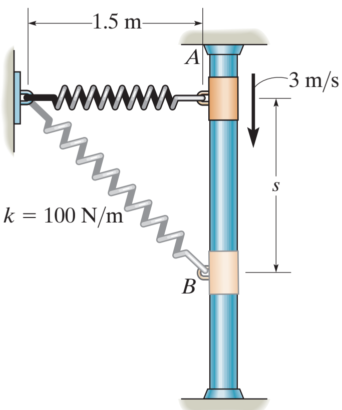
Prob. 14–83

***14–84.** A 60-kg satellite travels in free flight along an elliptical orbit such that at A, where $r_A = 20$ Mm, it has a speed $v_A = 40$ Mm/h. What is the speed of the satellite when it reaches point B, where $r_B = 80$ Mm? *Hint*: See Prob. 14–83, where $M_e = 5.976(10^{24})$ kg and $G = 66.73(10^{-12})\ \text{m}^3/(\text{kg}\cdot\text{s}^2)$.



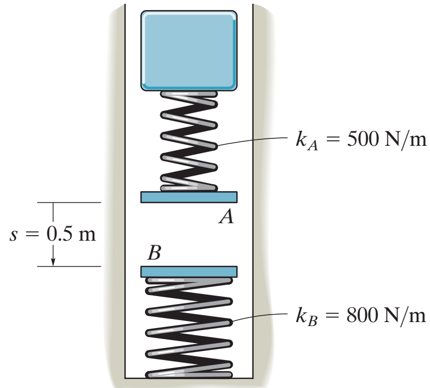
Prob. 14–84

14–85. The 4-kg smooth collar has a speed of 3 m/s when it is at $s = 0$. Determine the maximum distance s it travels before it stops momentarily. The spring has an unstretched length of 1 m.



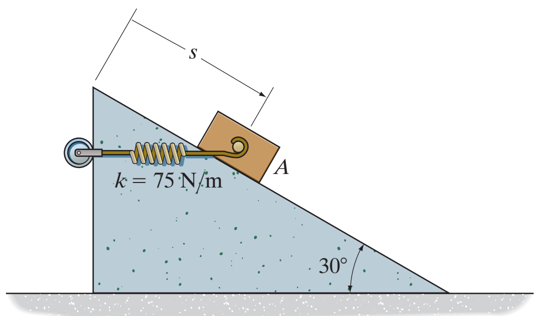
Prob. 14–85

14–86. The block has a mass of 20 kg and is released from rest when $s = 0.5$ m. If the mass of the bumpers A and B can be neglected, determine the maximum deformation of each spring due to the collision.



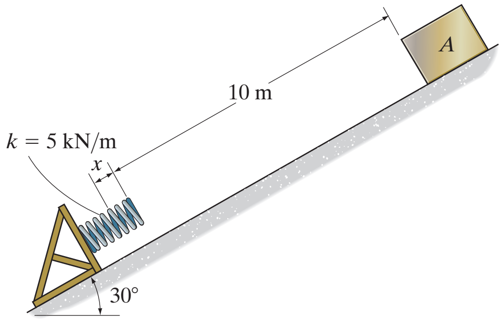
Prob. 14–86

14–87. The spring is unstretched when $s = 1$ m and the 15-kg block is released from rest at this position. Determine the speed of the block when $s = 3$ m. The spring remains horizontal during the motion, and the contact surfaces between the block and the inclined plane are smooth.



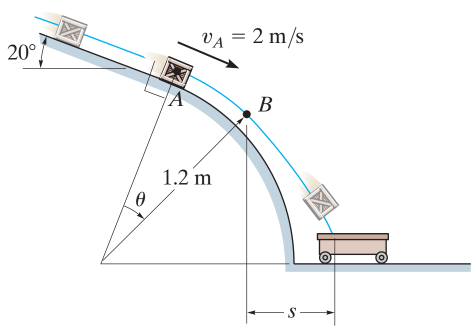
Prob. 14–87

***14–88.** The 10-kg block A is released from rest and slides down the smooth plane. Determine the compression x of the spring when the block momentarily stops.



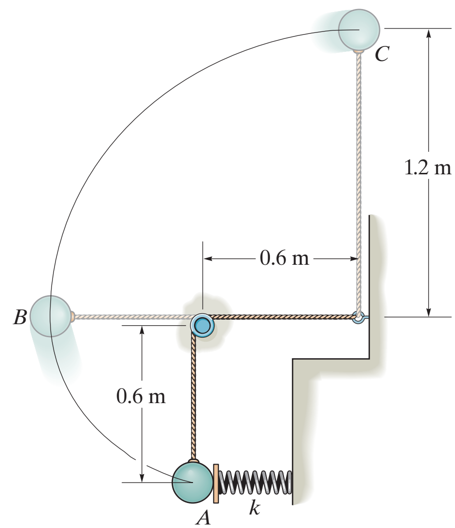
Prob. 14–88

14–89. When the 6-kg box reaches point A it has a speed of $v_A = 2$ m/s. Determine the angle θ at which it leaves the smooth circular ramp and the distance s to where it falls into the cart. Neglect friction.



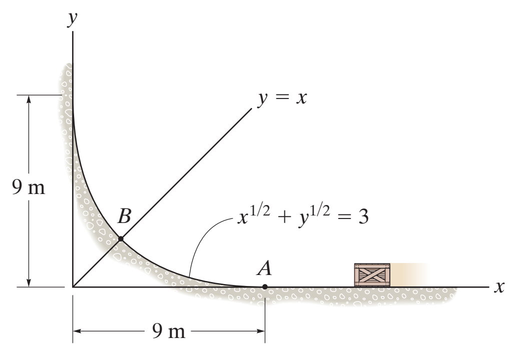
Prob. 14–89

14–90. The 0.75-kg bob of a pendulum is fired from rest at position A, by a spring which has a stiffness of $k = 6$ kN/m and is compressed 125 mm. Determine the speed of the bob and the tension in the cord when the bob is at positions B and C. Point B is located on the path where the radius of curvature is still 0.6 m, i.e., just before the cord becomes horizontal.



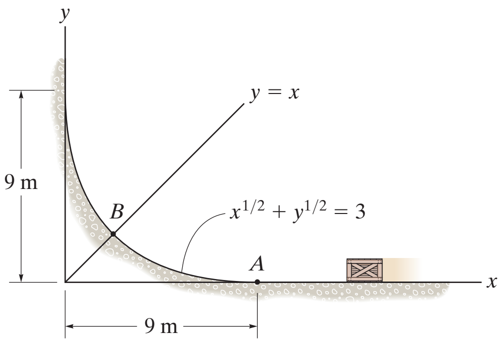
Prob. 14–90

14–91. When the 5-kg box reaches point A it has a speed $v_A = 10$ m/s. Determine the normal force the box exerts on the surface when it reaches point B. Neglect friction and the size of the box.



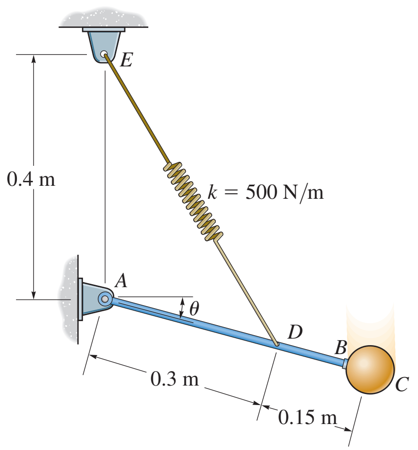
Prob. 14–91

***14–92.** When the 5-kg box reaches point A it has a speed $v_A = 10\ \text{m/s}$. Determine how high the box reaches up the surface before it comes to a stop. Also, what is the resultant normal force on the surface at this point and the acceleration? Neglect friction and the size of the box.



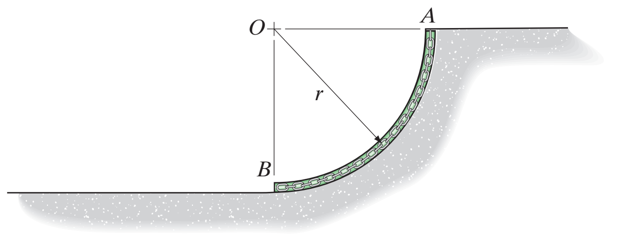
Prob. 14–92

14–93. The 10-kg sphere C is released from rest when $\theta = 0°$ and the tension in the spring is 100 N. Determine the speed of the sphere at the instant $\theta = 90°$. Neglect the mass of rod AB and the size of the sphere.



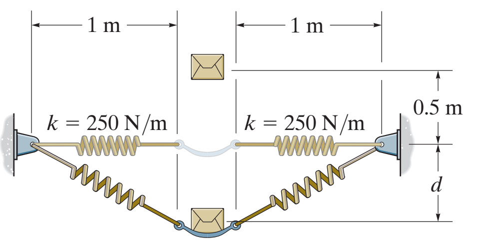
Prob. 14–93

14–94. A quarter-circular tube AB of mean radius r contains a smooth chain that has a mass per unit length of m_0. If the chain is released from rest from the position shown, determine its speed when it emerges completely from the tube.



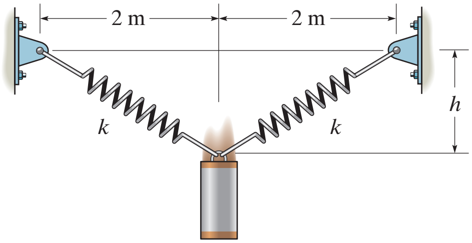
Prob. 14–94

14–95. A pan of negligible mass is attached to two identical springs of stiffness $k = 250\ \text{N/m}$. If a 10-kg box is dropped from a height of 0.5 m above the pan, determine the maximum vertical displacement d. Initially each spring has a tension of 50 N.



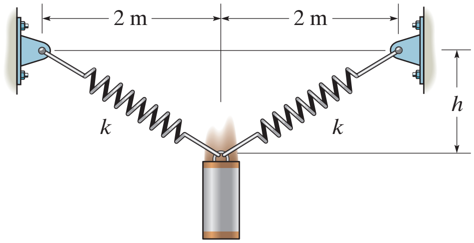
Prob. 14–95

***14–96.** If the 20-kg cylinder is released from rest at $h = 0$, determine the required stiffness k of each spring so that its motion is arrested or stops when $h = 0.5$ m. Each spring has an unstretched length of 1 m.



Prob. 14–96

14–97. The cylinder has a mass of 20 kg and is released from rest when $h = 0$. Determine its speed when $h = 3$ m. Each spring has a stiffness $k = 40$ N/m and an unstretched length of 2 m.



Prob. 14–97

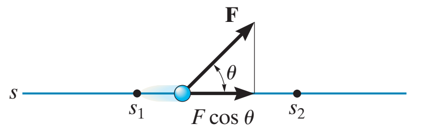
CHAPTER REVIEW

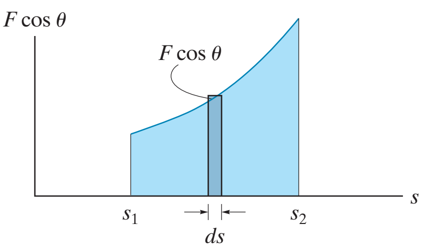
14

Work of a Force

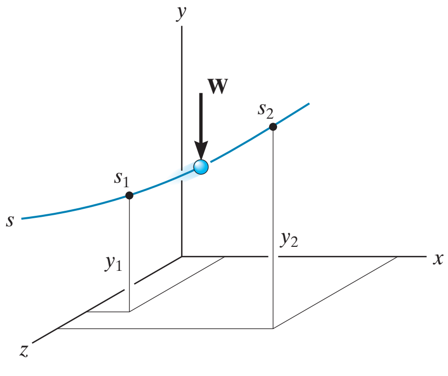
A force does work when it undergoes a displacement along its line of action. If the force varies with the displacement, then the work is $U = \int F \cos\theta \, ds$.

Graphically, this represents the area under the F–s diagram.



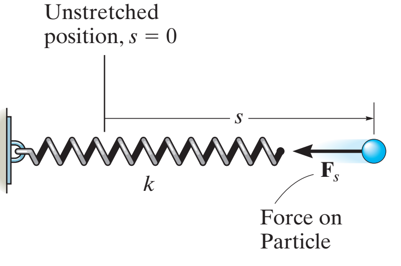


If the force is constant, then for a displacement Δs in the direction of the force, $U = F_c \, \Delta s$. A typical example of this case is the work of a weight, $U = -W(y_2 - y_1)$. Here, Δy is the vertical displacement.



The work done by a spring force, $F = ks$, depends upon the stretch or compression s of the spring.

$$U = -\left(\tfrac{1}{2}ks_2^2 - \tfrac{1}{2}ks_1^2\right)$$



The Principle of Work and Energy

If the equation of motion in the tangential direction, $\Sigma F_t = ma_t$, is combined with the kinematic equation, $a_t \, ds = v \, dv$, we obtain the principle of work and energy. This equation states that the initial kinetic energy of a particle, plus the work done on the particle, is equal to the final kinetic energy of the particle.

$$T_1 + \Sigma U_{1-2} = T_2$$

The principle of work and energy is useful for solving problems that involve force, velocity, and displacement. For application, the free-body diagram of the particle should be drawn in order to identify the forces that do work.

Power and Efficiency

Power is the time rate of doing work. For application, the force **F** creating the power and its velocity **v** must be specified.

$$P = \mathbf{F} \cdot \mathbf{v}$$

Average power is equal to the work done divided by the time.

$$P_{avg} = \frac{\Delta U}{\Delta t}$$

Efficiency represents the ratio of power output to power input. Due to frictional losses, it is always less than one.

$$\varepsilon = \frac{\text{power output}}{\text{power input}}$$

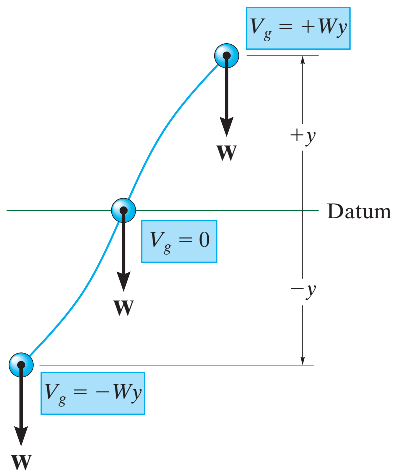
Conservation of Energy

A conservative force does work that is independent of its path. Two examples are the weight of a particle and the spring force.

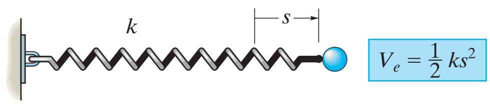
Friction is a nonconservative force since the work depends upon the length of the path. The longer the path, the more work.

The work of a conservative force depends upon its position relative to a datum. When this work is referenced from a datum, it is called potential energy. For a weight, it is $V_g = \pm Wy$, and for a spring it is $V_e = +\frac{1}{2}ks^2$.

Mechanical energy consists of kinetic energy T and gravitational and elastic potential energies V. According to the conservation of energy, this sum is constant for a system of particles and it has the same value at any position on the path.



Gravitational potential energy

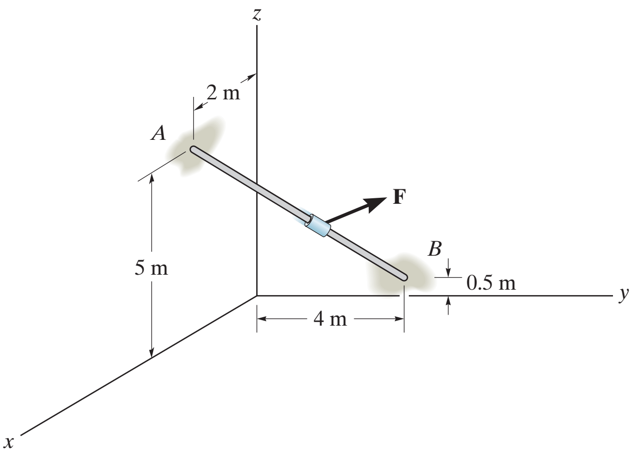


Elastic potential energy

$$T_1 + V_1 = T_2 + V_2$$

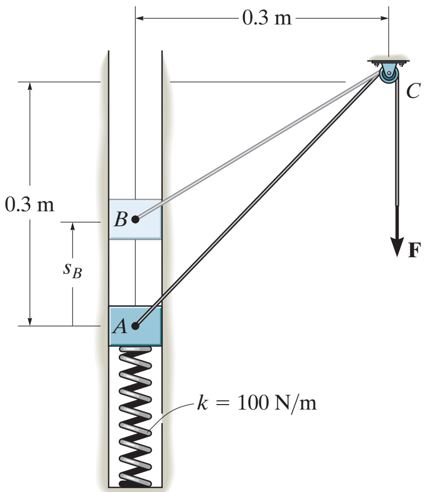
REVIEW PROBLEMS

R14–1. The small 1-kg collar starting from rest at A slides down along the smooth rod. During the motion, the collar is acted upon by a force $\mathbf{F} = \{50\mathbf{i} + 60y\mathbf{j} + 20z\mathbf{k}\}$ N, where y and z are in meters. Determine the collar's speed when it strikes the wall at B.



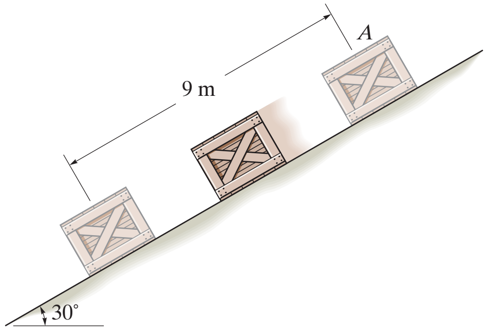
Prob. R14–1

R14–2. The block has a mass of 0.5 kg and moves within the smooth vertical slot. If the block starts from rest when the *attached* spring is in the unstretched position at A, determine the *constant* vertical force F which must be applied to the cord so that the block attains a speed $v_B = 2.5$ m/s when it reaches B; $s_B = 0.15$ m. Neglect the mass of the cord and pulley.



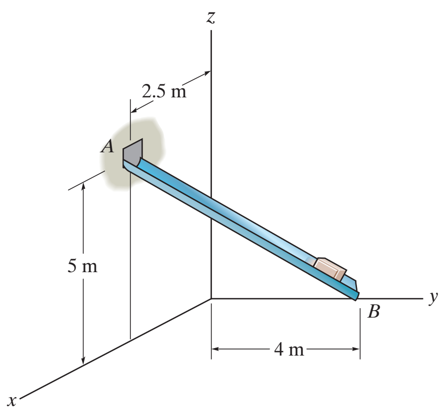
Prob. R14–2

R14–3. If a 70-kg crate is released from rest at A, determine its speed after it slides 9 m down the plane. The coefficient of kinetic friction between the crate and plane is $\mu_k = 0.3$.



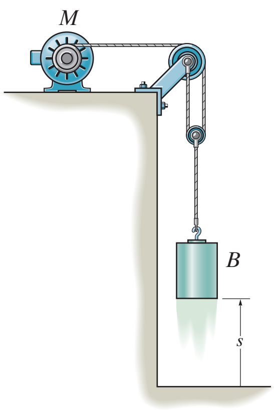
Prob. R14–3

R14–4. The block has a mass of 0.75 kg and slides along the smooth chute AB. It is released from rest at A, which has coordinates of $A(2.5\text{ m}, 0, 5\text{ m})$. Determine the speed at which it slides off at B, which has coordinates of $B(0, 4\text{ m}, 0)$.



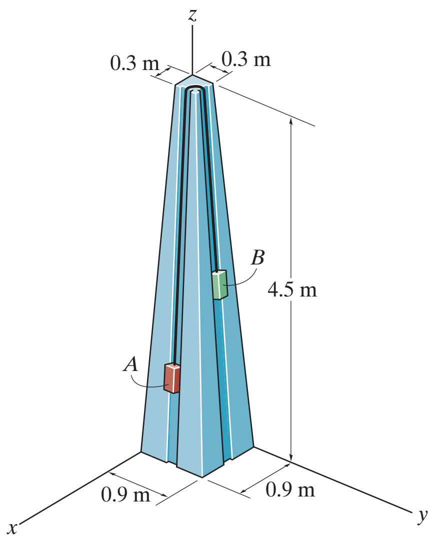
Prob. R14–4

R14–5. The 25-kg load is hoisted by the pulley system and motor M. If the motor exerts a constant force of 150 N on the cable, determine the power that must be supplied to the motor if the load has been hoisted $s = 3$ m starting from rest. The motor has an efficiency of $\varepsilon = 0.76$.



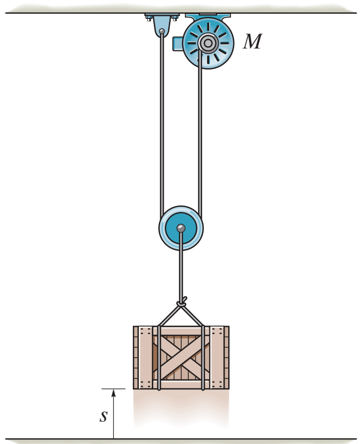
Prob. R14–5

R14–6. The blocks A and B have a mass of 5 kg and 15 kg, respectively. They are connected together by a light cord and ride in the frictionless grooves. Determine the speed of each block after block A moves 1.8 m up along the plane. The blocks are released from rest.



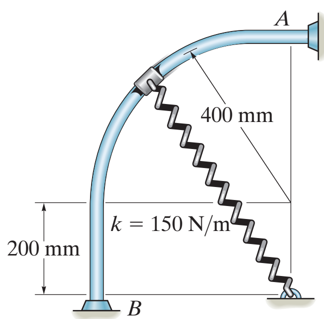
Prob. R14–6

R14–7. The crate, having a mass of 25 kg, is hoisted by the pulley system and motor M. If the crate starts from rest and, by constant acceleration, attains a speed of 6 m/s after rising 5 m, determine the power that must be supplied to the motor at the instant $s = 5$ m. The motor has an efficiency $\varepsilon = 0.74$.



Prob. R14–7

R14–8. The collar of negligible size has a mass of 0.25 kg and is attached to a spring having an unstretched length of 100 mm. If the collar is released from rest at A and travels along the smooth guide, determine its speed just before it strikes B.



Prob. R14–8

CHAPTER 15

Riders on this amusement park ride obey the conservation of angular momentum. As their motion changes, it can be determined as discussed in this chapter.

KINETICS OF A PARTICLE: IMPULSE AND MOMENTUM

Lecture Summary and Quiz, Example, and Problem-solving videos are available where this icon appears.

CHAPTER OBJECTIVES

- To develop the principle of linear impulse and momentum for a particle and apply it to solve problems that involve force, velocity, and time.
- To discuss the conservation of linear momentum for particles.
- To analyze the mechanics of impact.
- To introduce the concept of angular impulse and momentum.
- To solve problems involving steady fluid streams and propulsion with variable mass.

15.1 PRINCIPLE OF LINEAR IMPULSE AND MOMENTUM

In this section we will integrate the equation of motion with respect to time and thereby obtain the principle of impulse and momentum. Since $a = dv/dt$, we have

$$\Sigma \mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} \qquad (15\text{–}1)$$

Rearranging the terms and integrating between the limits $\mathbf{v} = \mathbf{v}_1$ when $t = t_1$ and $\mathbf{v} = \mathbf{v}_2$ when $t = t_2$, we have

$$\Sigma \int_{t_1}^{t_2} \mathbf{F}\,dt = m\int_{\mathbf{v}_1}^{\mathbf{v}_2} d\mathbf{v}$$

The impulse tool is used to remove the dent in the trailer fender. To do so its end is first screwed into a hole drilled in the fender, then the weight is gripped and jerked upwards, striking the stop ring. The impulse developed is transferred along the shaft of the tool and pulls suddenly on the dent.

15

or

$$\Sigma \int_{t_1}^{t_2} \mathbf{F} dt = m\mathbf{v}_2 - m\mathbf{v}_1 \qquad (15\text{–}2)$$

This equation is referred to as the ***principle of linear impulse and momentum***. It provides a *direct means* of obtaining the particle's final velocity $\mathbf{v}_2$ after a specified time period when the particle's initial velocity is known and the forces acting on the particle are either constant or can be expressed as functions of time.

Linear Momentum. Each of the two vectors of the form $\mathbf{L} = m\mathbf{v}$ in Eq. 15–2 is referred to as the particle's ***linear momentum***. Since the particle's mass m is a positive scalar, the linear-momentum vector has the same direction as $\mathbf{v}$, and its magnitude has units of mass times velocity, e.g., $\text{kg} \cdot \text{m/s}$.

Linear Impulse. The integral $\mathbf{I} = \int \mathbf{F}\, dt$ in Eq. 15–2 is referred to as the ***linear impulse***. This term is a vector quantity which measures the effect of a force during the time the force acts. Since time is a positive scalar, the impulse acts in the same direction as the force, and its magnitude has units of force times time, e.g., $\text{N} \cdot \text{s}$.*

If the force is expressed as a function of time, the impulse can be determined by direct evaluation of the integral. As a special case, if the force is constant in both magnitude and direction, then the impulse becomes

$$\mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}_c dt = \mathbf{F}_c(t_2 - t_1)$$

Graphically the magnitude of the impulse can be represented by the shaded area under the curve of force versus time, Fig. 15–1. A constant force creates the shaded rectangular area shown in Fig. 15–2.

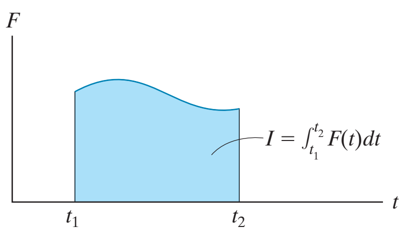


Fig. 15–1

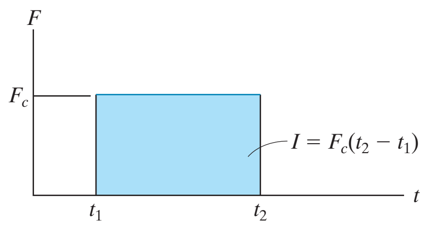


Fig. 15–2

* Although the units for impulse and momentum are defined differently, it can be shown that Eq. 15–2 is dimensionally homogeneous.

Principle of Linear Impulse and Momentum. For problem solving, Eq. 15–2 will be rewritten in the form

$$m\mathbf{v}_1 + \Sigma \int_{t_1}^{t_2} \mathbf{F}\, dt = m\mathbf{v}_2 \qquad (15\text{–}3)$$

which states that the initial momentum of the particle at time t_1 plus the sum of all the impulses applied to the particle from t_1 to t_2 is equivalent to the final momentum of the particle at time t_2. These three terms are illustrated graphically on the impulse and momentum diagrams shown in Fig. 15–3.

If each of the vectors in Eq. 15–3 is resolved into its x, y, z components, we can write the following three scalar equations of linear impulse and momentum.

$$\begin{aligned} m(v_x)_1 + \Sigma \int_{t_1}^{t_2} F_x\, dt &= m(v_x)_2 \\ m(v_y)_1 + \Sigma \int_{t_1}^{t_2} F_y\, dt &= m(v_y)_2 \\ m(v_z)_1 + \Sigma \int_{t_1}^{t_2} F_z\, dt &= m(v_z)_2 \end{aligned} \qquad (15\text{–}4)$$

The dynamics of many types of sports, such as golf, requires application of the principle of linear impulse and momentum.

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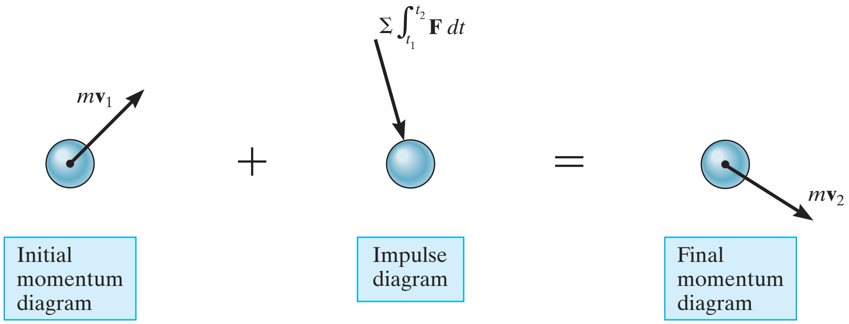

Fig. 15–3

15.2 PRINCIPLE OF LINEAR IMPULSE AND MOMENTUM FOR A SYSTEM OF PARTICLES

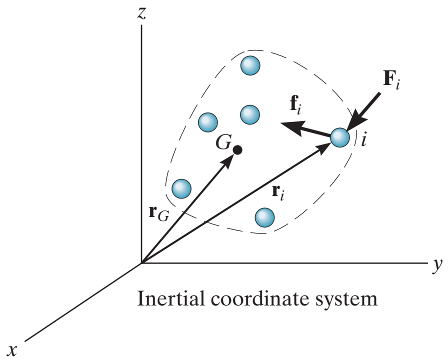

Fig. 15–4

The principle of linear impulse and momentum for a system of particles moving relative to an inertial reference, Fig. 15–4, is obtained from the equation of motion applied to all the particles in the system, i.e.,

$$\Sigma \mathbf{F}_i = \Sigma m_i \frac{d\mathbf{v}_i}{dt}$$

The term on the left side represents only the sum of the external forces $\mathbf{F}_i$ acting on the particles. The internal forces $\mathbf{f}_i$ acting between particles do not appear with this summation, since by Newton's third law they occur in equal but opposite collinear pairs and therefore cancel out. Multiplying both sides of the above equation by dt and integrating between the limits $t = t_1, \mathbf{v}_i = (\mathbf{v}_i)_1$ and $t = t_2, \mathbf{v}_i = (\mathbf{v}_i)_2$, we have

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$$\Sigma m_i(\mathbf{v}_i)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{F}_i \, dt = \Sigma m_i(\mathbf{v}_i)_2 \qquad (15\text{–}5)$$

Here the initial linear momentum of the system plus the impulses of all the external forces acting on the system from t_1 to t_2 is equal to the system's final linear momentum.

This equation can be simplified by noting that the location of the mass center G of the system is determined from $m\mathbf{r}_G = \Sigma m_i \mathbf{r}_i$, where $m = \Sigma m_i$ is the total mass of all the particles, Fig. 15–4. Taking the time derivative, we have

$$m\mathbf{v}_G = \Sigma m_i \mathbf{v}_i \qquad (15\text{–}6)$$

In words, the total linear momentum of the system of particles, $\Sigma m_i \mathbf{v}_i$, is equivalent to the linear momentum of a "fictitious" aggregate particle of mass $m = \Sigma m_i$ moving with the velocity of the mass center of the system, $\mathbf{v}_G$. Substituting into Eq. 15–6 yields

$$m(\mathbf{v}_G)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{F}_i \, dt = m(\mathbf{v}_G)_2 \qquad (15\text{–}7)$$

If the system of particles collectively represents the particles composing a rigid body, then this result justifies application of the principle of linear impulse and momentum to a body of finite size.

PROCEDURE FOR ANALYSIS

The principle of linear impulse and momentum is used to solve problems involving *force*, *time*, and *velocity*, since these terms are involved in the formulation. For application it is suggested that the following procedure be used.*

Free-Body Diagram.

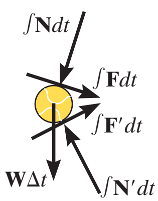
- Establish the x, y, z inertial frame of reference and draw the particle's free-body diagram in order to account for all the forces that produce impulses on the particle.
- The direction and sense of the particle's initial and final velocities should be established.
- If a vector is unknown, assume that the sense of its components is in the direction of the positive inertial coordinate(s).
- As an alternative procedure, draw the impulse and momentum diagrams for the particle as shown in Fig. 15–3.

Principle of Impulse and Momentum.

- Apply the principle of linear impulse and momentum, $m\mathbf{v}_1 + \Sigma \int_{t_1}^{t_2} \mathbf{F}\,dt = m\mathbf{v}_2$. If motion occurs in the x–y plane, the two scalar component equations can be formulated by either resolving the vector components of $\mathbf{F}$ from the free-body diagram, or by using the data on the impulse and momentum diagrams.
- Realize that every force acting on the particle's free-body diagram will create an impulse, even though some of these forces will do no work.
- Forces that are functions of time must be integrated to obtain the impulse. Graphically, the impulse is equal to the area under the force–time curve.

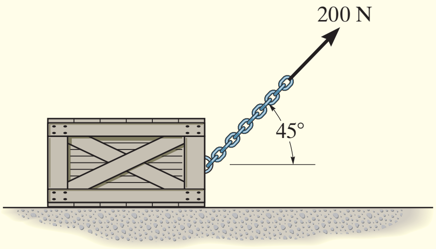
Refer to the companion website for Lecture Summary and Quiz videos.

As the wheels of the pitching machine rotate, they apply frictional impulses to the ball, thereby giving it a linear momentum. These impulses are shown on the impulse diagram. Here both the frictional and normal impulses vary with time. By comparison, the weight impulse is constant and is very small since the time Δt the ball is in contact with the wheels will be very small.



* This procedure will be followed when developing the proofs and theory in the book.

EXAMPLE 15.1



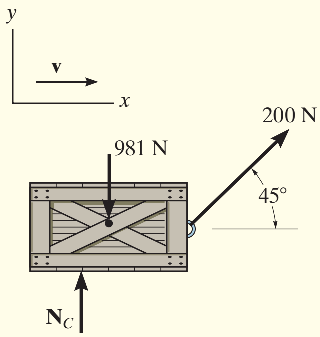
(a)

The 100-kg crate shown in Fig. 15–5*a* is originally at rest on the smooth horizontal surface. If a towing force of 200 N, acting at an angle of 45°, is applied for 10 s, determine the final velocity of the crate and the normal force which the surface exerts on the crate during this time interval.

SOLUTION

This problem can be solved using the principle of impulse and momentum since it involves force, velocity, and time.

Free-Body Diagram. Since all the forces acting on the free-body diagram, Fig. 15–5*b*, are *constant*, the impulses are simply the product of the force magnitude and 10 s $[\mathbf{I} = \mathbf{F}_c(t_2 - t_1)]$. Note the alternative procedure of drawing the crate's impulse and momentum diagrams, Fig. 15–5*c*.



(b)

Principle of Impulse and Momentum. Applying Eqs. 15–4, we have

$$(\xrightarrow{+}) \qquad m(v_x)_1 + \Sigma\int_{t_1}^{t_2} F_x\,dt = m(v_x)_2$$

$$0 + 200\text{ N}\cos 45°(10\text{ s}) = (100\text{ kg})v_2$$

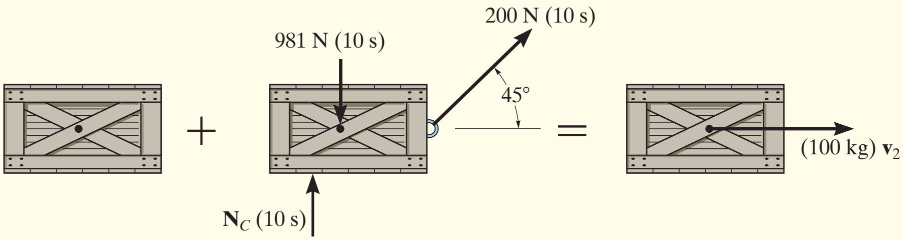
$$v_2 = 14.1\text{ m/s} \qquad \textit{Ans.}$$

$$(+\uparrow) \qquad m(v_y)_1 + \Sigma\int_{t_1}^{t_2} F_y\,dt = m(v_y)_2$$

$$0 + N_C(10\text{ s}) - 981\text{ N}(10\text{ s}) + 200\text{ N}\sin 45°(10\text{ s}) = 0$$

$$N_C = 840\text{ N} \qquad \textit{Ans.}$$

NOTE: Since no motion occurs in the y direction, direct application of the equilibrium equation $\Sigma F_y = 0$ gives the same result for N_C. Try to solve the problem by first applying $\Sigma F_x = ma_x$, then $v = v_0 + a_c t$.

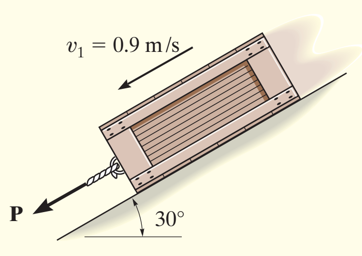


(c)

Fig. 15–5

EXAMPLE 15.2

The 25-kg crate shown in Fig. 15–6*a* is acted upon by a force having a variable magnitude $P = (90t)$ N, where t is in seconds. Determine the crate's velocity when $t = 2$ s if the initial velocity of the crate is $v_1 = 0.9$ m/s down the plane. The coefficient of kinetic friction between the crate and the plane is $\mu_k = 0.3$.



(a)

SOLUTION

Free-Body Diagram. See Fig. 15–6*b*. Since the magnitude of force $P = 90t$ varies with time, the impulse it creates must be determined by integrating over the 2-s time interval.

Principle of Impulse and Momentum. Applying Eqs. 15–4 in the x direction, we have

$$(+\swarrow) \qquad m(v_x)_1 + \Sigma\int_{t_1}^{t_2} F_x\,dt = m(v_x)_2$$

$$(25\text{ kg})(0.9\text{ m/s}) + \int_0^{2\text{ s}} 90t\,dt - 0.3N_C(2\text{ s}) + (25\text{ kg})(9.81\text{ m/s}^2)\sin 30°(2\text{ s}) = (25\text{ kg})\,v_2$$

$$22.5 + 180 - 0.6N_C + 245.25 = 25v_2$$

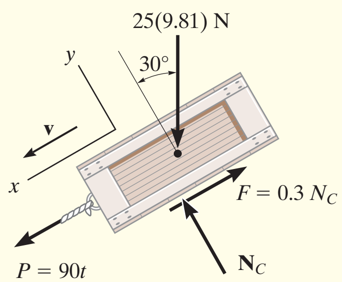
The equation of equilibrium can be applied in the y direction. Why?

$$+\nwarrow\Sigma F_y = 0; \qquad N_C - 25(9.81)\cos 30°\text{ N} = 0$$

Solving,

$$N_C = 212.39\text{ N}$$

$$v_2 = 12.8\text{ m/s}\swarrow \qquad \textit{Ans.}$$



(b)

Fig. 15–6

NOTE: We can also solve this problem using the equation of motion. From Fig. 15–6*b*,

$$+\swarrow\Sigma F_x = ma_x; \quad 90t - 0.3(212.39) + 25(9.81)\sin 30° = 25a$$

$$a = 3.6t + 2.356$$

Using kinematics,

$$+\swarrow dv = a\,dt; \qquad \int_{0.9\text{ m/s}}^{v_2} dv = \int_0^{2\text{ s}} (3.6t + 2.356)dt$$

$$v_2 = 12.8\text{ m/s} \qquad \textit{Ans.}$$

By comparison, application of the principle of impulse and momentum eliminates the need for using kinematics ($a = dv/dt$) and thereby yields an easier method for solution.

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EXAMPLE 15.3

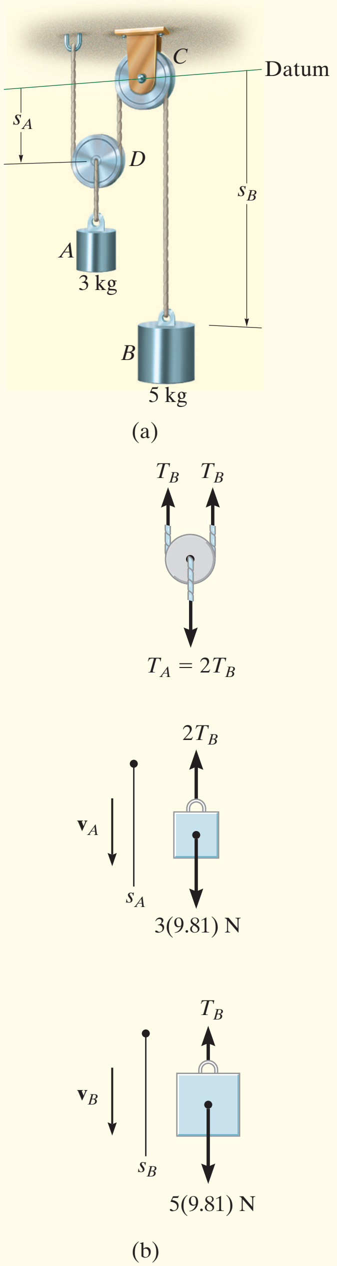


Fig. 15–7

Blocks A and B shown in Fig. 15–7a have a mass of 3 kg and 5 kg, respectively. If the system is released from rest, determine the velocity of block B in 6 s. Neglect the mass of the pulleys and cord.

SOLUTION

Free-Body Diagram. See Fig. 15–7b. Since the weight of each block is constant, the cord tensions will also be constant. Furthermore, since the mass of pulley D is neglected, the cord tension $T_A = 2T_B$. Note that the blocks are both assumed to be moving downward in the positive coordinate directions, s_A and s_B.

Principle of Impulse and Momentum.

Block A:

$(+\downarrow)$
$$m(v_A)_1 + \Sigma \int_{t_1}^{t_2} F_y\,dt = m(v_A)_2$$

$$0 - 2T_B(6\text{ s}) + 3(9.81)\text{ N}(6\text{ s}) = (3\text{ kg})(v_A)_2 \qquad (1)$$

Block B:

$(+\downarrow)$
$$m(v_B)_1 + \Sigma \int_{t_1}^{t_2} F_y\,dt = m(v_B)_2$$

$$0 + 5(9.81)\text{ N}(6\text{ s}) - T_B(6\text{ s}) = (5\text{ kg})(v_B)_2 \qquad (2)$$

Kinematics. Since the blocks are subjected to dependent motion, the velocity of A can be related to that of B by using the kinematic analysis discussed in Sec. 12–9. A horizontal datum is established through the fixed point at C, Fig. 15–7a, and the position coordinates, s_A and s_B, are related to the constant total length l of the vertical segments of the cord by the equation

$$2s_A + s_B = l$$

Taking the time derivative yields

$$2v_A = -v_B \qquad (3)$$

As indicated by the negative sign, when B moves downward A moves upward. Substituting this result into Eq. 1 and solving Eqs. 1 and 2 yields

$$(v_B)_2 = 35.8\text{ m/s}\downarrow$$

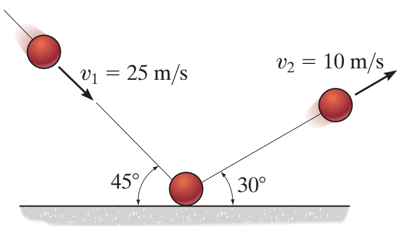
$$T_B = 19.2\text{ N} \qquad \textit{Ans.}$$

NOTE: Realize that the *positive* (downward) direction for $\mathbf{v}_A$ and $\mathbf{v}_B$ is *consistent* in Figs. 15–7a and 15–7b and in Eqs. 1 to 3. This is important since we are seeking a simultaneous solution of equations.

Refer to the companion website for a self quiz of these Example problems.

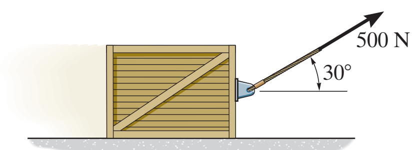
FUNDAMENTAL PROBLEMS

F15–1. The 0.5-kg ball strikes the rough ground and rebounds with the velocities shown. Determine the magnitude of the impulse the ground exerts on the ball. Assume that the ball does not slip when it strikes the ground, and neglect the size of the ball and the impulse produced by the weight of the ball.



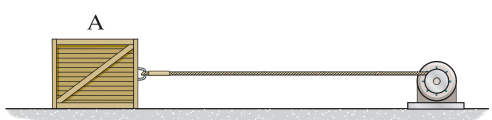
Prob. F15–1

F15–2. If the coefficient of kinetic friction between the 75-kg crate and the ground is $\mu_k = 0.2$, determine the speed of the crate when $t = 4$ s. The crate starts from rest and is towed by the 500-N force.



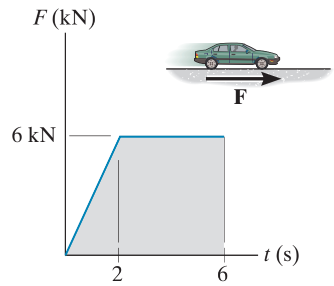
Prob. F15–2

F15–3. The motor exerts a force of $F = (20t^2)$ N on the cable, where t is in seconds. Determine the speed of the 25-kg crate when $t = 4$ s. The coefficients of static and kinetic friction between the crate and the plane are $\mu_s = 0.3$ and $\mu_k = 0.25$, respectively.



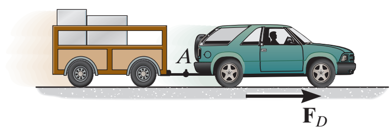
Prob. F15–3

F15–4. The wheels of the 1.5-Mg car generate the traction force **F** described by the graph. If the car starts from rest, determine its speed when $t = 6$ s.



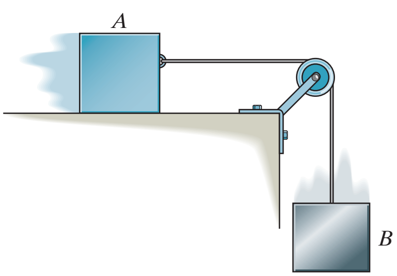
Prob. F15–4

F15–5. The 2.5-Mg four-wheel-drive SUV tows the 1.5-Mg trailer. The traction force developed at the wheels is $F_D = 9$ kN. Determine the speed of the truck in 20 s, starting from rest. Also, determine the tension developed in the coupling, A, between the SUV and the trailer. Neglect the mass of the wheels.



Prob. F15–5

F15–6. The 4.5-kg block A attains a velocity of 0.3 m/s in 5 seconds, starting from rest. Determine the tension in the cord and the coefficient of kinetic friction between block A and the horizontal plane. Neglect the weight of the pulley. Block B has a mass of 3.6 kg.

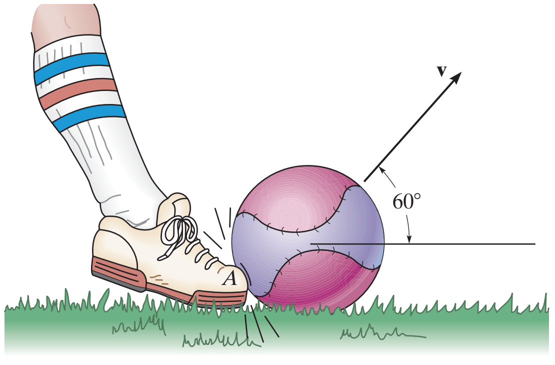


Prob. F15–6

PROBLEMS

All solutions must include a free-body diagram.

15–1. A man kicks the 150-g ball such that it leaves the ground at an angle of 60° and strikes the ground at the same elevation a distance of 12 m away. Determine the impulse of his foot on the ball at A. Neglect the impulse caused by the ball's weight while it's being kicked.

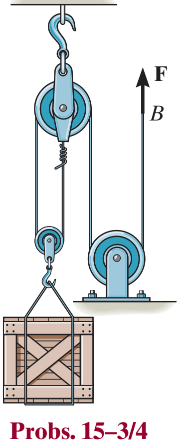


Prob. 15–1

15–2. A 2.5-kg block is given an initial velocity of 3 m/s up a 45° smooth slope. Determine the time for it to travel up the slope before it stops.

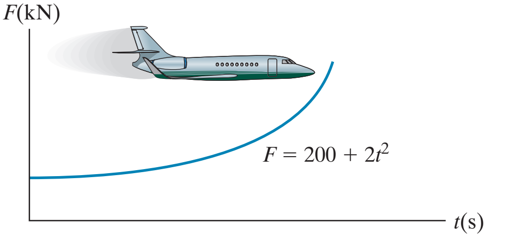
15–3. The 20-kg crate is lifted by a force of $F = (100 + 5t^2)$ N, where t is in seconds. Determine the speed of the crate when $t = 3$ s, starting from rest.

***15–4.** The 20-kg crate is lifted by a force of $F = (100 + 5t^2)$ N, where t is in seconds. Determine how high the crate has moved upward when $t = 3$ s, starting from rest.



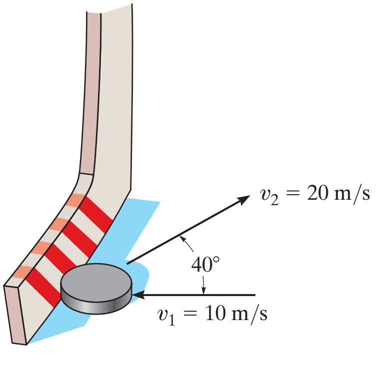
Probs. 15–3/4

15–5. The jet plane has a mass of 250 Mg and a horizontal velocity of 100 m/s when $t = 0$. If *both* engines provide a horizontal thrust which varies as shown in the graph, determine the plane's velocity in $t = 5$ s. Neglect air resistance and the loss of fuel during the motion.



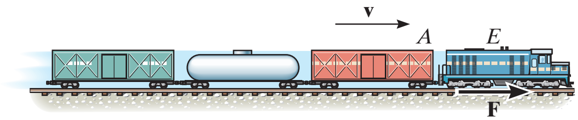
Prob. 15–5

15–6. A hockey puck is traveling to the left with a velocity of $v_1 = 10$ m/s when it is struck by a hockey stick and given a velocity of $v_2 = 20$ m/s as shown. Determine the magnitude of the net impulse exerted by the hockey stick on the puck. The puck has a mass of 0.2 kg.



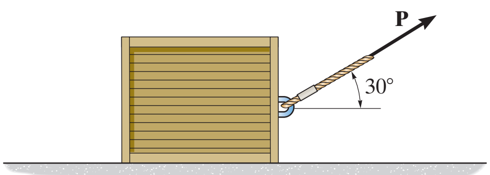
Prob. 15–6

15–7. A train consists of a 50-Mg engine and three cars, each having a mass of 30 Mg. If it takes 80 s for the train to increase its speed uniformly to 40 km/h, starting from rest, determine the force T developed at the coupling between the engine E and the first car A. The wheels of the engine provide a resultant frictional tractive force $\mathbf{F}$ which gives the train forward motion, whereas the car wheels roll freely. Also, determine F acting on the engine wheels.



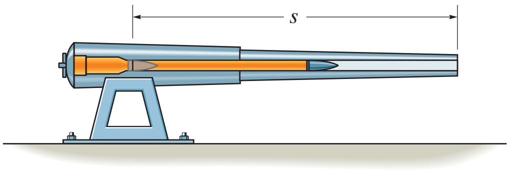
Prob. 15–7

***15–8.** The 50-kg crate is pulled by the constant force $\mathbf{P}$. If the crate starts from rest and achieves a speed of 10 m/s in 5 s, determine the magnitude of $\mathbf{P}$. The coefficient of kinetic friction between the crate and the ground is $\mu_k = 0.2$.



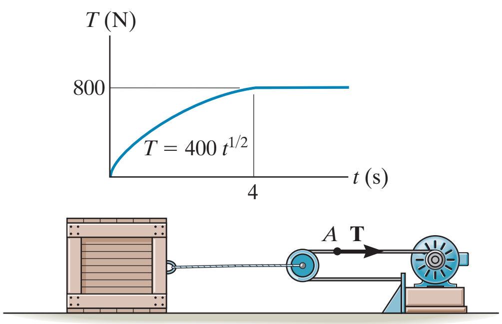
Prob. 15–8

15–9. The force acting on a projectile having a mass m as it passes horizontally through the barrel of the cannon is $F = C \sin (\pi t/t')$. Determine the projectile's velocity when $t = t'$. If the projectile reaches the end of the barrel at this instant, determine the length s.



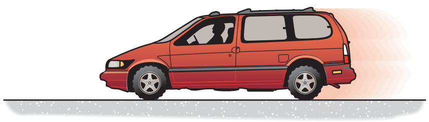
Prob. 15–9

15–10. The 200-kg crate rests on the ground for which the coefficients of static and kinetic friction are $\mu_s = 0.5$ and $\mu_k = 0.4$, respectively. The winch delivers a horizontal towing force T to its cable at A which varies as shown in the graph. Determine the speed of the crate when $t = 4$ s. Originally the tension in the cable is zero. *Hint*: First determine the force needed to begin moving the crate.



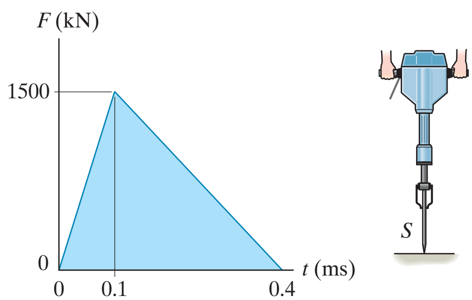
Prob. 15–10

15–11. The 2.5-Mg van is traveling with a speed of 100 km/h when the brakes are applied and all four wheels lock. If the speed decreases to 40 km/h in 5 s, determine the coefficient of kinetic friction between the tires and the road.



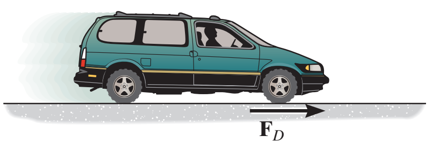
Prob. 15–11

***15–12.** During operation the jack hammer strikes the concrete surface with a force which is indicated in the graph. To achieve this the 2-kg spike *S* is fired into the surface at 90 m/s. Determine the speed of the spike just after rebounding.



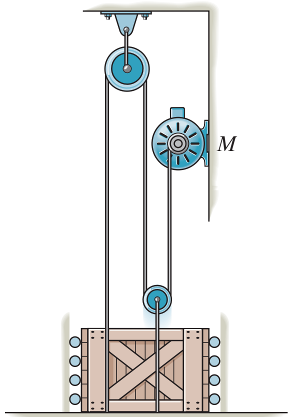
Prob. 15–12

15–13. For a short period of time, the frictional driving force acting on the wheels of the 2.5-Mg van is $F_D = (600t^2)$ N, where t is in seconds. If the van has a speed of 20 km/h when $t = 0$, determine its speed when $t = 5$ s.



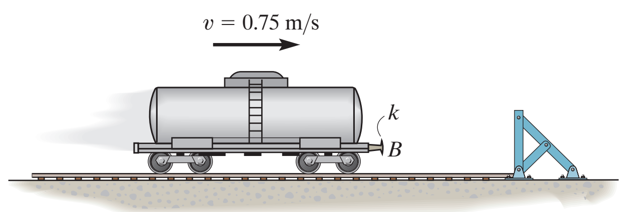
Prob. 15–13

15–14. The motor, *M*, pulls on the cable with a force $F = (10t^2 + 300)$ N, where t is in seconds. If the 100 kg crate is originally at rest at $t = 0$, determine its speed when $t = 4$ s. Neglect the mass of the cable and pulleys. *Hint:* First find the time needed to begin lifting the crate.



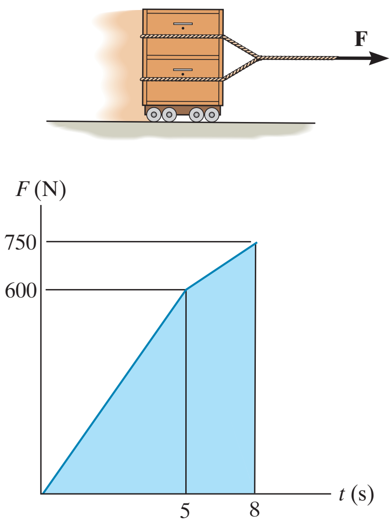
Prob. 15–14

15–15. A tankcar has a mass of 20 Mg and is freely rolling to the right with a speed of 0.75 m/s. If it strikes the barrier, determine the horizontal impulse needed to stop the car if the spring in the bumper *B* has a stiffness (a) $k \rightarrow \infty$ (bumper is rigid), and (b) $k = 15$ kN/m.



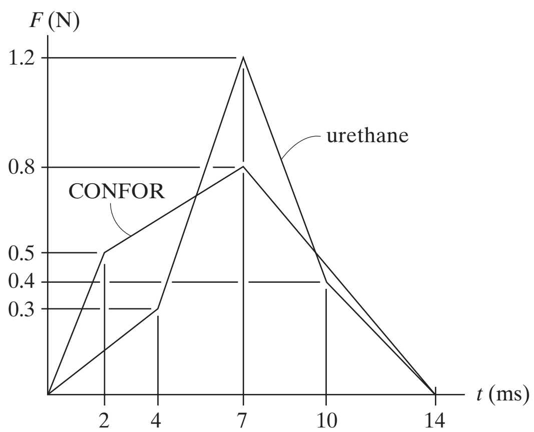
Prob. 15–15

***15–16.** The towing force acting on the 400-kg safe varies as shown in the graph. Determine its speed, starting from rest, when $t = 8$ s. How far has it traveled during this time?



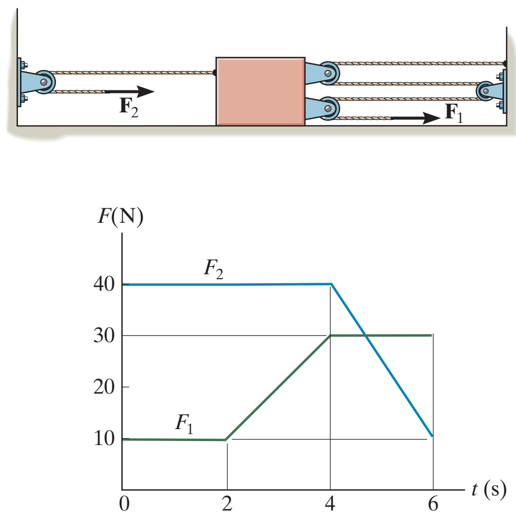
Prob. 15–16

15–17. The choice of a seating material for moving vehicles depends upon its ability to resist shock and vibration. From the data shown in the graphs, determine the impulses created by a falling weight onto a sample of urethane foam and CONFOR foam.



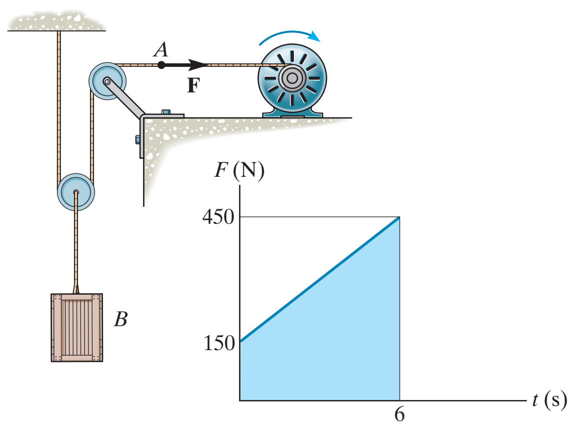
Prob. 15–17

15–18. The 30-kg slider block is moving to the left with a speed of 5 m/s when it is acted upon by the forces $\mathbf{F}_1$ and $\mathbf{F}_2$. If these loadings vary in the manner shown in the graph, determine the speed of the block at $t = 6$ s. Neglect friction and the mass of the pulleys and cords.



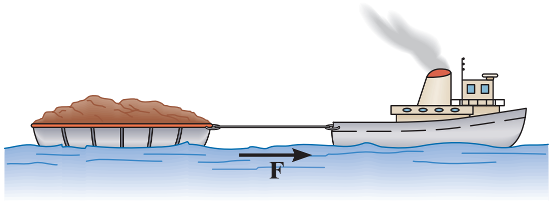
Prob. 15–18

15–19. The motor exerts a force F on the 40-kg crate as shown in the graph. Determine the speed of the crate when $t = 3$ s and when $t = 6$ s. When $t = 0$, the crate is moving downward at 10 m/s.



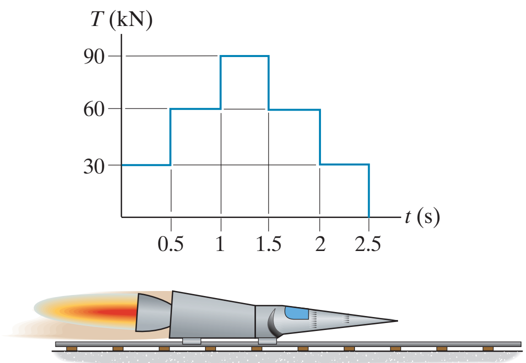
Prob. 15–19

***15–20.** If it takes 35 s for the 50-Mg tugboat to increase its speed uniformly to 25 km/h, starting from rest, determine the force of the rope on the tugboat. The propeller provides the propulsion force **F** on the tugboat which gives it forward motion, whereas the barge moves freely. Also, determine the magnitude of **F**. The barge has a mass of 75 Mg.



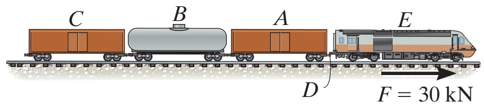
Prob. 15–20

15–21. Determine the maximum speed attained by the 1.5-Mg rocket sled if the rockets provide the thrust shown in the graph. Initailly, the sled is at rest. Neglect friction and the loss of mass due to fuel consumption.



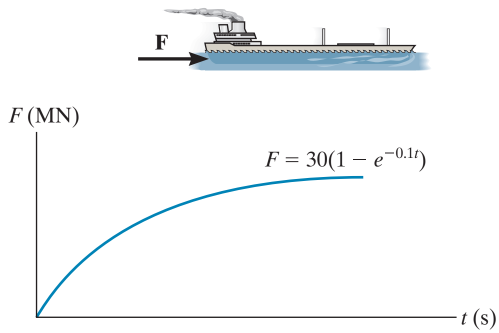
Prob. 15–21

15–22. The train consists of a 30-Mg engine E, and cars A, B, and C, which have a mass of 15 Mg, 10 Mg, and 8 Mg, respectively. If the tracks provide a traction force of $F = 30$ kN on the engine wheels, determine the speed of the train when $t = 30$ s, starting from rest. Also, find the horizontal coupling force at D between the engine E and car A. Neglect rolling resistance.



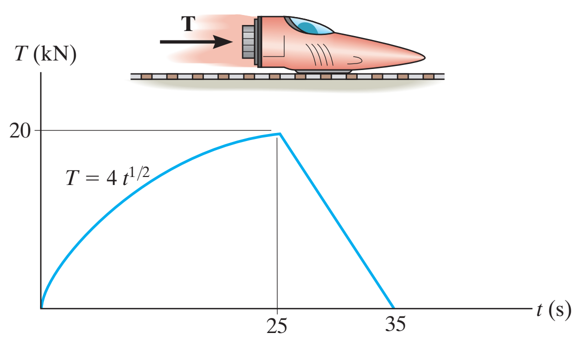
Prob. 15–22

15–23. The tanker has a mass of 130 Gg. If it is originally at rest, determine its speed when $t = 10$ s. The horizontal thrust provided by its propeller varies with time as shown in the graph. Neglect the effect of water resistance.



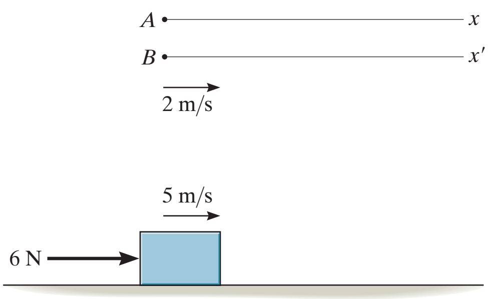
Prob. 15–23

***15–24.** The thrust on the 4-Mg rocket sled is shown in the graph. Determine the sleds maximum velocity and the distance the sled travels when $t = 35$ s. Neglect friction.



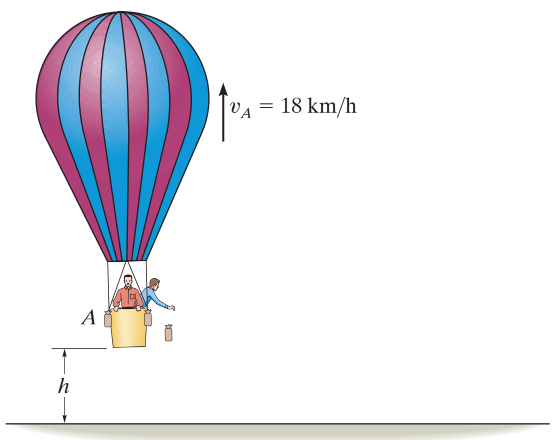
Prob. 15–24

15–25. As indicated by the derivation, the principle of impulse and momentum is valid for observers in *any* inertial reference frame. Show that this is so, by considering the 10-kg block which slides along the smooth surface and is subjected to a horizontal force of 6 N. If observer A is in a *fixed* frame x, determine the final speed of the block in 4 s if it has an initial speed of 5 m/s measured from the fixed frame. Compare the result with that obtained by an observer B, attached to the x' axis that moves at a constant velocity of 2 m/s relative to A.



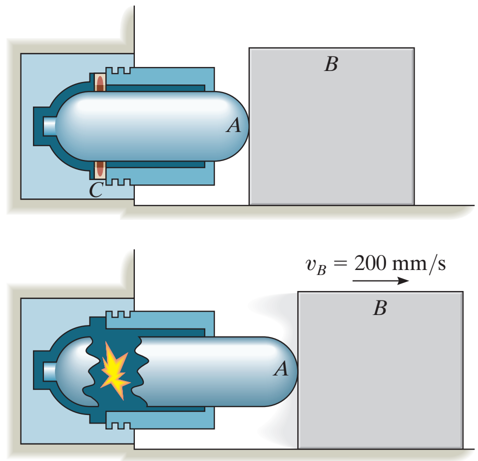
Prob. 15–25

15–26. The balloon has a total mass of 400 kg including the passengers and ballast. The balloon is rising at a constant velocity of 18 km/h when $h = 10$ m. If the man drops the 40-kg sand bag, determine the velocity of the balloon when the bag strikes the ground. Neglect air resistance.



Prob. 15–26

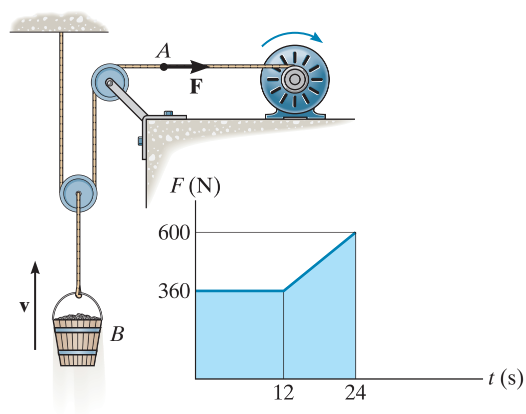
15–27. In case of emergency, the gas actuator is used to move a 75-kg block B by exploding a charge C near a pressurized cylinder of negligible mass. As a result of the explosion, the cylinder fractures and the released gas forces the front part of the cylinder, A, to move B forward, giving it a speed of 200 mm/s in 0.4 s. If the coefficient of kinetic friction between B and the floor is $\mu_k = 0.5$, determine the impulse that the actuator imparts to B.



Prob. 15–27

***15–28.** The winch delivers a horizontal towing force **F** to its cable at A which varies as shown in the graph. Determine the speed of the 70-kg bucket when $t = 18$ s. Originally the bucket is moving upward at $v_1 = 3$ m/s.

15–29. The winch delivers a horizontal towing force **F** to its cable at A which varies as shown in the graph. Determine the speed of the 80-kg bucket when $t = 24$ s. Originally the bucket is released from rest.

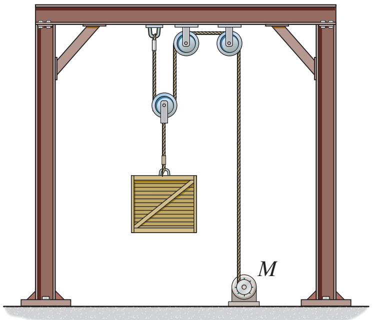


Probs. 15–28/29

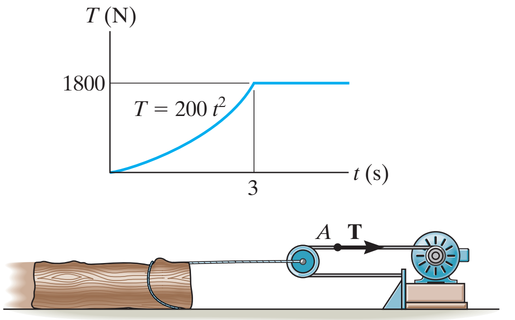
15–30. The 100-kg crate is hoisted by the motor M. If the velocity of the crate increases uniformly from 1.5 m/s to 4.5 m/s in 5 s, determine the tension developed in the cable during the motion.

15–31. The 100-kg crate is hoisted by the motor M. The motor exerts a force on the cable of $T = (200t^{1/2} + 150)$ N, where t is in seconds. If the crate starts from rest at the ground, determine the speed of the crate when $t = 5$ s.

15–33. The log has a mass of 500 kg and rests on the ground for which the coefficients of static and kinetic friction are $\mu_s = 0.5$ and $\mu_k = 0.4$, respectively. The winch delivers a horizontal towing force T to its cable at A which varies as shown in the graph. Determine the speed of the log when $t = 5$ s. Originally the tension in the cable is zero. *Hint:* First determine the force needed to begin moving the log.



Probs. 15–30/31

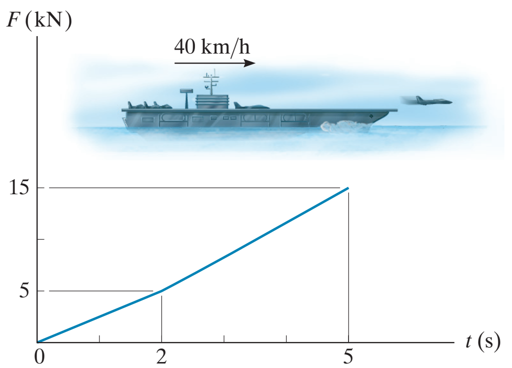


Prob. 15–33

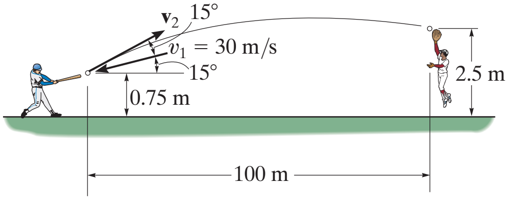
15

***15–32.** A jet plane having a mass of 7 Mg takes off from an aircraft carrier such that the engine thrust varies as shown by the graph. If the carrier is traveling forward with a speed of 40 km/h, determine the plane's airspeed after 5 s.

15–34. The 0.15-kg baseball has a speed of $v_1 = 30$ m/s just before it is struck by the bat. It then travels along the trajectory shown before the outfielder catches it. Determine the magnitude of the average impulsive force imparted to the ball if it is in contact with the bat for 0.75 ms.



Prob. 15–32



Prob. 15–34

15.3 CONSERVATION OF LINEAR MOMENTUM FOR A SYSTEM OF PARTICLES

When the sum of the *external impulses* acting on a system of particles is *zero*, Eq. 15–5 reduces to a simplified form, namely,

$$\Sigma m_i(\mathbf{v}_i)_1 = \Sigma m_i(\mathbf{v}_i)_2 \tag{15–8}$$

This equation is referred to as the ***conservation of linear momentum***. It states that the total linear momentum for a system of particles remains constant during the time period t_1 to t_2. From Eq. 15–7, we can also write

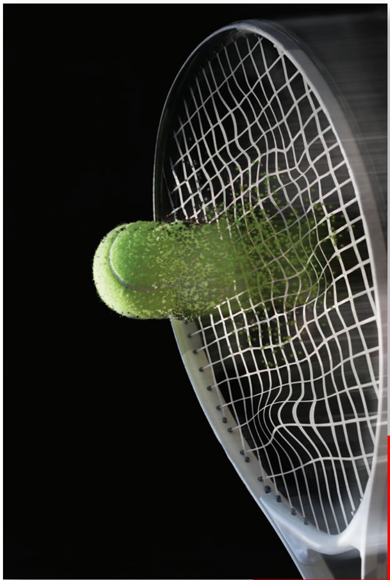
$$(\mathbf{v}_G)_1 = (\mathbf{v}_G)_2 \tag{15–9}$$

which indicates that the velocity $\mathbf{v}_G$ of the mass center for the system of particles does not change if no external impulses are applied to the system.

The conservation of linear momentum is often applied when particles collide or interact. For application, a careful study of the free-body diagram for the *entire* system of particles should be made in order to identify the forces which create external impulses and thereby determine in what direction(s) linear momentum is conserved. If the time over which the motion is studied is *very short*, some of the impulses may also be neglected or considered approximately equal to zero. The forces causing these negligible impulses are called ***nonimpulsive forces***. By comparison, forces which are very large and act for a very short period of time produce a significant change in momentum and are called ***impulsive forces***. They, of course, cannot be neglected in the impulse–momentum analysis.

Impulsive forces normally occur due to an explosion or the striking of one body against another, whereas nonimpulsive forces may include the weight of a body, the force imparted by a slightly deformed spring having a relatively small stiffness, or for that matter, any force that is very small compared to other larger (impulsive) forces. When making this distinction between impulsive and nonimpulsive forces, it is important to realize that this only applies during the time t_1 to t_2. To illustrate, consider the effect of striking a tennis ball with a racket as shown in the photo on the next page. During the *very short* time of interaction, the force of the racket on the ball is impulsive since it changes the ball's momentum drastically. By

The hammer in the top photo applies an impulsive force to the stake. During this extremely short time of contact the weight of the stake can be considered nonimpulsive, and provided the stake is driven into soft ground, the impulse of the ground acting on the stake can also be considered nonimpulsive. By contrast, if the stake is used in a concrete chipper to break concrete, then two impulsive forces act on the stake: one at its top due to the chipper and the other on its bottom due to the rigidity of the concrete.



comparison, the ball's weight will have a negligible effect on the change in momentum, and therefore it is nonimpulsive. Consequently, it can be neglected from an impulse–momentum analysis during this time. If an impulse–momentum analysis is considered during the much longer time of flight after the racket–ball interaction, then the impulse of the ball's weight is important since it, along with air resistance, causes the change in the momentum of the ball.

PROCEDURE FOR ANALYSIS

Generally, the principle of linear impulse and momentum or the conservation of linear momentum are applied to a *system of particles* in order to determine the final velocities of the particles *just after* the time period considered. By applying these principles to the entire system, the internal impulses acting within the system are *eliminated* from the analysis. For application it is suggested that the following procedure be used.

Free-Body Diagram.

- Establish the x, y, z inertial frame of reference and draw the free-body diagram for each particle of the system in order to identify the internal and external forces.
- The conservation of linear momentum applies to the system in a direction which either has no external forces or the forces can be considered nonimpulsive.
- Establish the direction and sense of the particles' initial and final velocities. If the sense is unknown, assume it is in the positive coordinate direction.

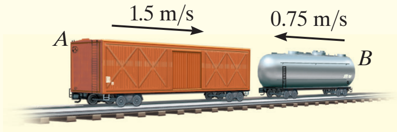
Momentum Equations.

- Apply the principle of linear impulse and momentum or the conservation of linear momentum in the appropriate directions.
- If it is necessary to determine the internal impulse $\int F\,dt$ acting on only one particle of a system, then the particle must be *isolated* (free-body diagram), and the principle of linear impulse and momentum must be applied to this particle.
- After the impulse is calculated, and provided the time Δt for which the impulse acts is known, then the average impulsive force F_{avg} can be determined from $F_{avg} = \int F\,dt/\Delta t$.

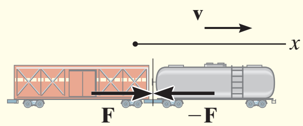
Refer to the companion website for Lecture Summary and Quiz videos.

EXAMPLE 15.4

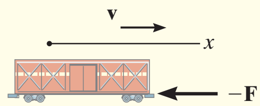
The 15-Mg boxcar A is coasting at 1.5 m/s on the horizontal track when it encounters a 12-Mg tank car B coasting at 0.75 m/s toward it as shown in Fig. 15–8*a*. If the cars collide and couple together, determine (a) the speed of both cars just after the coupling, and (b) the average force between them if the coupling takes place in 0.8 s.



(a)



(b)



(c)

Fig. 15–8

SOLUTION

Part (a) Free-Body Diagram.* As shown in Fig. 15–8*b*, we have considered *both* cars as a single system. By inspection, momentum is conserved in the x direction since the coupling force **F** is *internal* to the system and will therefore cancel out. It is assumed that when coupled, both cars move at $\mathbf{v}_2$ in the positive x direction.

Conservation of Linear Momentum.

$$(\overset{+}{\rightarrow}) \qquad m_A(v_A)_1 + m_B(v_B)_1 = (m_A + m_B)v_2$$

$$(15\,000 \text{ kg})(1.5 \text{ m/s}) - 12\,000 \text{ kg}(0.75 \text{ m/s}) = (27\,000 \text{ kg})v_2$$

$$v_2 = 0.5 \text{ m/s} \rightarrow \qquad \textit{Ans.}$$

Part (b). The average (impulsive) coupling force, $\mathbf{F}_{avg}$, can be determined by applying the principle of linear momentum to *either one* of the cars.

Free-Body Diagram. As shown in Fig. 15–8*c*, by isolating the boxcar the coupling force is *external* to the car.

Principle of Impulse and Momentum. Since $\int F\,dt = F_{avg}\,\Delta t = F_{avg}(0.8 \text{ s})$, we have

$$(\overset{+}{\rightarrow}) \qquad m_A(v_A)_1 + \Sigma\int F\,dt = m_A v_2$$

$$(15\,000 \text{ kg})(1.5 \text{ m/s}) - F_{avg}(0.8 \text{ s}) = (15\,000 \text{ kg})(0.5 \text{ m/s})$$

$$F_{avg} = 18.8 \text{ kN} \qquad \textit{Ans.}$$

NOTE: Try solving for F_{avg} by applying the principle of impulse and momentum to the tank car.

* Only horizontal forces are shown on the free-body diagram.

EXAMPLE 15.5

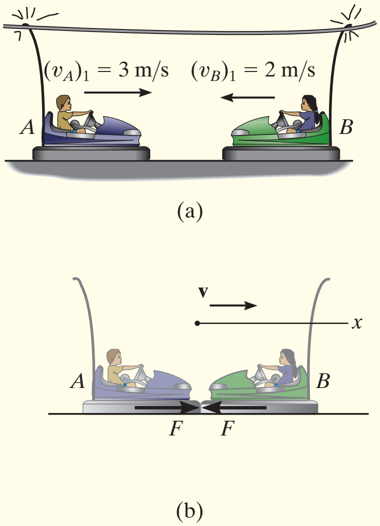


Fig. 15–9

The bumper cars A and B in Fig. 15–9*a* each have a mass of 150 kg and are coasting with the velocities shown before they freely collide head on. If no energy is lost during the collision, determine their velocities after collision.

SOLUTION

Free-Body Diagram. The cars will be considered as a single system. The free-body diagram is shown in Fig. 15–9*b*.

Conservation of Momentum.

$$(\overset{+}{\rightarrow}) \qquad m_A(v_A)_1 + m_B(v_B)_1 = m_A(v_A)_2 + m_B(v_B)_2$$

$$(150\text{ kg})(3\text{ m/s}) + (150\text{ kg})(-2\text{ m/s}) = (150\text{ kg})(v_A)_2 + (150\text{ kg})(v_B)_2$$

$$(v_A)_2 = 1 - (v_B)_2 \qquad (1)$$

Conservation of Energy. Since no energy is lost, the conservation of energy theorem gives

$$T_1 + V_1 = T_2 + V_2$$

$$\frac{1}{2}m_A(v_A)_1^2 + \frac{1}{2}m_B(v_B)_1^2 + 0 = \frac{1}{2}m_A(v_A)_2^2 + \frac{1}{2}m_B(v_B)_2^2 + 0$$

$$\frac{1}{2}(150\text{ kg})(3\text{ m/s})^2 + \frac{1}{2}(150\text{ kg})(2\text{ m/s})^2 + 0 = \frac{1}{2}(150\text{ kg})(v_A)_2^2 + \frac{1}{2}(150\text{ kg})(v_B)_2^2 + 0$$

$$(v_A)_2^2 + (v_B)_2^2 = 13 \qquad (2)$$

Substituting Eq. 1 into Eq. 2 and simplifying, we get

$$(v_B)_2^2 - (v_B)_2 - 6 = 0$$

Solving for the two roots,

$$(v_B)_2 = 3\text{ m/s} \qquad \text{and} \qquad (v_B)_2 = -2\text{ m/s}$$

Since $(v_B)_2 = -2$ m/s refers to the velocity of B just *before* collision, then the velocity of B just after the collision must be

$$(v_B)_2 = 3\text{ m/s} \rightarrow \qquad \textit{Ans.}$$

Substituting this result into Eq. 1, we obtain

$$(v_A)_2 = 1 - 3\text{ m/s} = -2\text{ m/s} = 2\text{ m/s} \leftarrow \qquad \textit{Ans.}$$

EXAMPLE 15.6

An 800-kg rigid pile P shown in Fig. 15–10a is driven into the ground using a 300-kg hammer. The hammer falls from rest at a height $y_0 = 0.5$ m and strikes the top of the pile. Determine the impulse which the pile exerts on the hammer if the pile is surrounded entirely by loose sand, so that after striking, the hammer does *not* rebound off the pile.

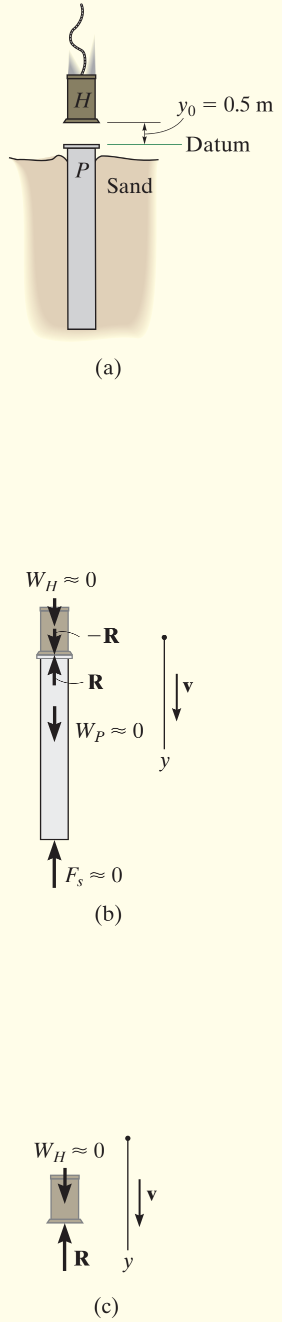


Fig. 15–10

SOLUTION

Conservation of Energy. The velocity at which the hammer strikes the pile can be determined using the conservation of energy equation applied to the hammer. With the datum at the top of the pile, Fig. 15–10a, we have

$$T_0 + V_0 = T_1 + V_1$$

$$\frac{1}{2}m_H(v_H)_0^2 + W_H y_0 = \frac{1}{2}m_H(v_H)_1^2 + W_H y_1$$

$$0 + 300(9.81)\text{ N}(0.5\text{ m}) = \frac{1}{2}(300\text{ kg})(v_H)_1^2 + 0$$

$$(v_H)_1 = 3.132\text{ m/s}$$

Free-Body Diagram. From the physical aspects of the problem, the free-body diagram of the hammer and pile, Fig. 15–10b, indicates that during the *short time* from *just before* to *just after* the *collision*, the weights of the hammer and pile and the resistance force $\mathbf{F}_s$ of the sand are all *nonimpulsive*. The impulsive force $\mathbf{R}$ is internal to the system and therefore cancels. Consequently, momentum is conserved in the vertical direction during this short time.

Conservation of Momentum. Since the hammer does not rebound off the pile just after collision, then $(v_H)_2 = (v_P)_2 = v_2$.

$$(+\downarrow) \qquad m_H(v_H)_1 + m_P(v_P)_1 = m_H v_2 + m_P v_2$$

$$(300\text{ kg})(3.132\text{ m/s}) + 0 = (300\text{ kg})v_2 + (800\text{ kg})v_2$$

$$v_2 = 0.8542\text{ m/s}$$

Principle of Impulse and Momentum. The impulse which the pile imparts to the hammer can now be determined since v_2 is known. From the free-body diagram for the hammer, Fig. 15–10c, we have

$$(+\downarrow) \qquad m_H(v_H)_1 + \Sigma\int_{t_1}^{t_2} F_y\,dt = m_H v_2$$

$$(300\text{ kg})(3.132\text{ m/s}) - \int R\,dt = (300\text{ kg})(0.8542\text{ m/s})$$

$$\int R\,dt = 683\text{ N}\cdot\text{s} \qquad \textit{Ans.}$$

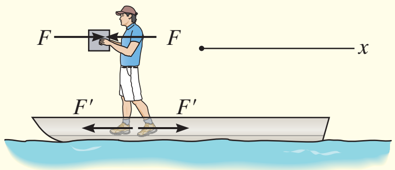
NOTE: The equal but opposite impulse acts on the pile. Try finding this impulse by applying the principle of impulse and momentum to the pile.

EXAMPLE 15.7

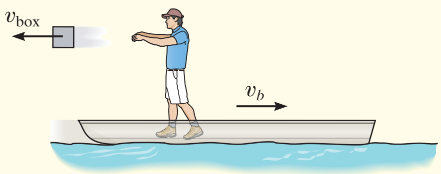
The 80-kg man can throw the 20-kg box horizontally at 4 m/s when standing on the ground. If instead he firmly stands in the 120-kg boat and throws the box, as shown in the photo, determine how far the boat will move in three seconds. Neglect water resistance.

SOLUTION

Free-Body Diagram.* If the man, boat, and box are considered as a single system, the horizontal forces between the man and the boat and the man and the box become internal to the system, Fig. 15–11*a*, and so linear momentum will be conserved along the x axis.



(a)



(b)

Fig. 15–11

Conservation of Momentum. When writing the conservation of momentum equation, it is *important* that the velocities be measured from the same inertial coordinate system, assumed here to be fixed. From this coordinate system, we will assume that the boat and man go to the right while the box goes to the left, as shown in Fig. 15–11*b*.

Applying the conservation of linear momentum to the man, boat, box system,

$$(\overset{+}{\rightarrow}) \qquad 0 + 0 + 0 = (m_m + m_b)v_b - m_{box}v_{box}$$

$$0 = (80\,\text{kg} + 120\,\text{kg})v_b - (20\,\text{kg})v_{box}$$

$$v_{box} = 10v_b \qquad (1)$$

Kinematics. Since the velocity of the box *relative* to the man (and boat), $v_{box/b}$, is known, then v_b can also be related to v_{box} using the relative velocity equation.

$$(\overset{+}{\rightarrow}) \qquad v_{box} = v_b + v_{box/b}$$

$$-v_{box} = v_b - 4\,\text{m/s} \qquad (2)$$

Solving Eqs. 1 and 2,

$$v_{box} = 3.64\,\text{m/s} \leftarrow$$

$$v_b = 0.3636\,\text{m/s} \rightarrow$$

The displacement of the boat in three seconds is therefore

$$s_b = v_b t = (0.3636\,\text{m/s})(3\,\text{s}) = 1.09\,\text{m} \qquad \textit{Ans.}$$

* Only horizontal forces are shown on the free-body diagram.

EXAMPLE 15.8

The 540-kg cannon shown in Fig. 15–12*a* fires a 3.6-kg projectile with a muzzle velocity of 450 m/s measured relative to the cannon. If firing takes place in 0.03 s, determine the recoil velocity of the cannon just after firing. The cannon support is fixed to the ground, and the horizontal recoil of the cannon is absorbed by two springs.

Fig. 15–12

SOLUTION

Part (a) Free-Body Diagram.* As shown in Fig. 15–12*b*, we will consider the projectile and cannon as a single system, since the impulsive forces, **F** and −**F**, between the cannon and projectile are *internal* to the system and will therefore cancel from the analysis. Furthermore, during the time $\Delta t = 0.03$ s, the two recoil springs which are attached to the support each exert a *nonimpulsive force* $\mathbf{F}_s$ on the cannon. This is because Δt is very short, so that during this time the cannon only moves through a *very small* distance x. Consequently, $F_s = kx \approx 0$, where k is the spring's stiffness, which is also considered to be relatively small. Hence it can be concluded that momentum for the system is conserved in the *horizontal direction*.

Conservation of Linear Momentum.

$$(\overset{+}{\rightarrow}) \qquad m_c(v_c)_1 + m_p(v_p)_1 = -m_c(v_c)_2 + m_p(v_p)_2$$

$$0 + 0 = -(540\text{ kg})(v_c)_2 + (3.6\text{ kg})(v_p)_2$$

$$(v_p)_2 = 150\,(v_c)_2 \qquad (1)$$

Kinematics. These two unknown velocities are measured by a *fixed* observer. As in Example 15.7, they can also be related using the relative velocity equation.

$$(\overset{+}{\rightarrow}) \qquad (v_p)_2 = (v_c)_2 + (v_{p/c})_2$$

$$(v_p)_2 = -(v_c)_2 + 450\text{ m/s} \qquad (2)$$

Solving Eqs. 1 and 2 yields

$$(v_c)_2 = 2.98\text{ m/s} \qquad \textit{Ans.}$$

$$(v_p)_2 = 447\text{ m/s}$$

Apply the principle of impulse and momentum to the projectile (or the cannon) and show that the average impulsive force on the projectile is 53.6 kN.

NOTE: If the cannon is firmly fixed to its support (no springs), the reactive force of the support on the cannon must be considered as an external impulse to the system, since the support would allow no movement of the cannon. In this case momentum is *not* conserved.

* Only horizontal forces are shown on the free-body diagram.

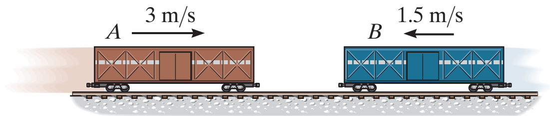
www

Refer to the companion website for a self quiz of these Example problems.

15

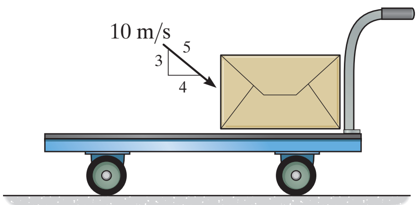
FUNDAMENTAL PROBLEMS

F15–7. The freight cars A and B have a mass of 20 Mg and 15 Mg, respectively. Determine the velocity of A after collision if the cars collide and rebound, such that B moves to the right with a speed of 2 m/s. If A and B are in contact for 0.5 s, find the average impulsive force which acts between them.



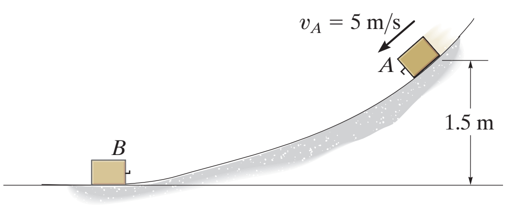
Prob. F15–7

F15–8. The cart and package have a mass of 20 kg and 5 kg, respectively. If the cart has a smooth surface and it is initially at rest, while the velocity of the package is as shown, determine the final common velocity of the cart and package after the impact.



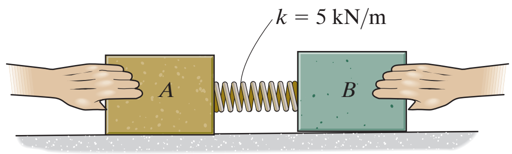
Prob. F15–8

F15–9. The 5-kg block A has an initial speed of 5 m/s as it slides down the smooth ramp, after which it collides with the stationary 8-kg block B. If the two blocks couple together after collision, determine their common velocity immediately after collision.



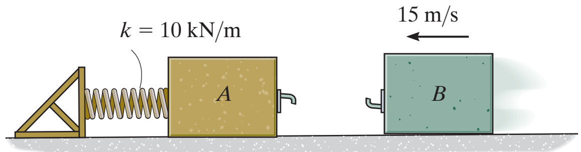
Prob. F15–9

F15–10. The spring is fixed to block A and block B is pressed against the spring. If the spring is compressed $s = 200$ mm and then the blocks are released, determine their velocity at the instant block B loses contact with the spring. The masses of blocks A and B are 10 kg and 15 kg, respectively.



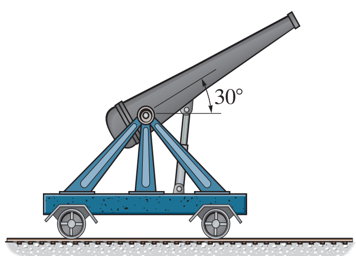
Prob. F15–10

F15–11. Blocks A and B have a mass of 15 kg and 10 kg, respectively. If A is stationary and B has a velocity of 15 m/s just before collision, and the blocks couple together after impact, determine the maximum compression of the spring.



Prob. F15–11

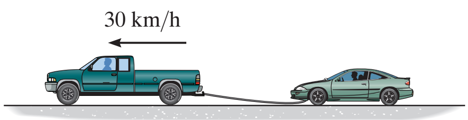
F15–12. The cannon and support without a projectile have a mass of 250 kg. If a 20-kg projectile is fired from the cannon with a velocity of 400 m/s, measured *relative* to the cannon, determine the speed of the projectile as it leaves the cannon barrel. Neglect rolling resistance.



Prob. F15–12

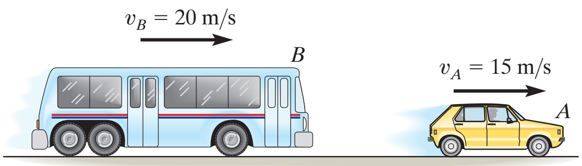
PROBLEMS

15–35. The 2.5-Mg pickup truck is towing the 1.5-Mg car using a cable as shown. If the car is initially at rest and the truck is coasting with a velocity of 30 km/h when the cable is slack, determine the common velocity of the truck and the car just after the cable becomes taut. Also, find the loss of energy.



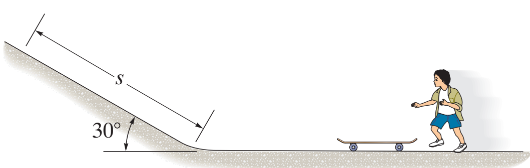
Prob. 15–35

***15–36.** The 5-Mg bus B is traveling to the right at 20 m/s. Meanwhile a 2-Mg car A is traveling at 15 m/s to the right. If the vehicles crash and become entangled, determine their common velocity just after the collision. Assume that the vehicles are free to roll during collision.



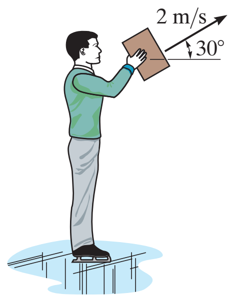
Prob. 15–36

15–37. The 50-kg boy jumps on the 5-kg skateboard with a horizontal velocity of 5 m/s. Determine the distance s the boy reaches up the inclined plane before momentarily coming to rest. Neglect the skateboard's rolling resistance.



Prob. 15–37

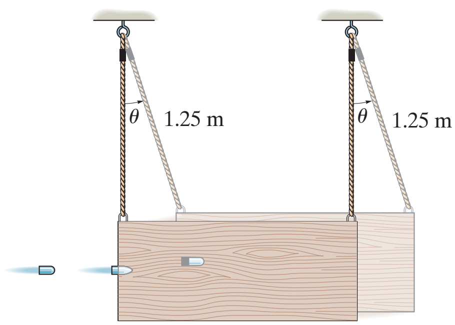
15–38. A man wearing ice skates throws an 8-kg block with an initial velocity of 2 m/s, measured relative to himself, in the direction shown. If he is originally at rest and completes the throw in 1.5 s while keeping his legs rigid, determine the horizontal velocity of the man just after releasing the block. What is the vertical reaction of both his skates on the ice during the throw? The man has a mass of 70 kg. Neglect friction and the motion of his arms.



Prob. 15–38

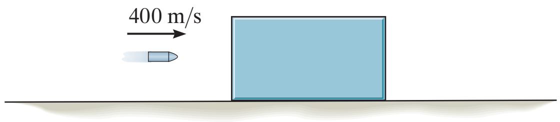
15–39. A railroad car having a mass of 15 Mg is coasting at 1.5 m/s on a horizontal track. At the same time another car having a mass of 12 Mg is coasting at 0.75 m/s in the opposite direction. If the cars meet and couple together, determine the speed of both cars just after the coupling. Find the difference between the total kinetic energy before and after coupling has occurred, and explain qualitatively what happened to this energy.

***15–40.** A ballistic pendulum consists of a 4-kg wooden block originally at rest, $\theta = 0°$. When a 2-g bullet strikes and becomes embedded in it, it is observed that the block swings upward to a maximum angle of $\theta = 6°$. Estimate the initial speed of the bullet.



Prob. 15–40

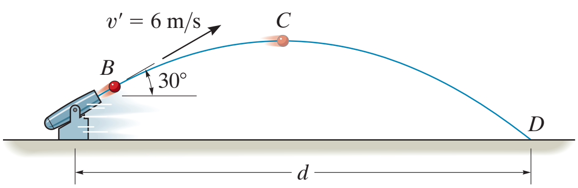
15–41. The 20-g bullet is traveling at 400 m/s when it becomes embedded in the 2-kg stationary block. Determine the distance the block will slide before it stops. The coefficient of kinetic friction between the block and the plane is $\mu_k = 0.2$.



Prob. 15–41

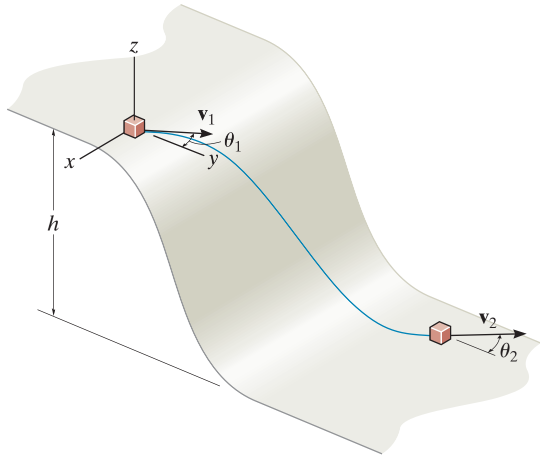
15–42. The 5-kg spring-loaded gun rests on the smooth surface. It fires a ball having a mass of 1 kg with a velocity of $v' = 6$ m/s relative to the gun in the direction shown. If the gun is originally at rest, determine the horizontal distance d the ball is from the initial position of the gun at the instant the ball strikes the ground at D. Neglect the size of the gun.

15–43. The 5-kg spring-loaded gun rests on the smooth surface. It fires a ball having a mass of 1 kg with a velocity of $v' = 6$ m/s relative to the gun in the direction shown. If the gun is originally at rest, determine the distance the ball is from the gun at the instant the ball reaches its highest elevation C. Neglect the size of the gun.



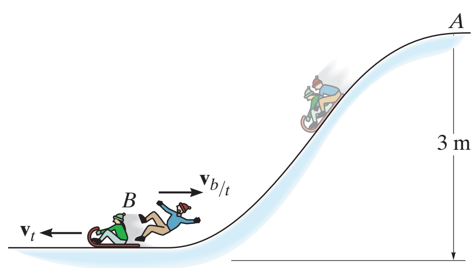
Probs. 15–42/43

***15–44.** The block of mass m travels at v_1 in the direction θ_1 shown at the top of the smooth slope. Determine its speed v_2 and its direction θ_2 when it reaches the bottom.



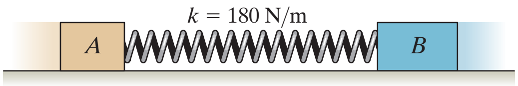
Prob. 15–44

15–45. A toboggan having a mass of 10 kg starts from rest at A and carries a girl and boy having a mass of 40 kg and 45 kg, respectively. When the toboggan reaches the bottom of the slope at B, the boy is pushed off from the back with a horizontal velocity of $v_{b/t} = 2$ m/s, measured relative to the toboggan. Determine the velocity of the toboggan afterwards. Neglect friction in the calculation.



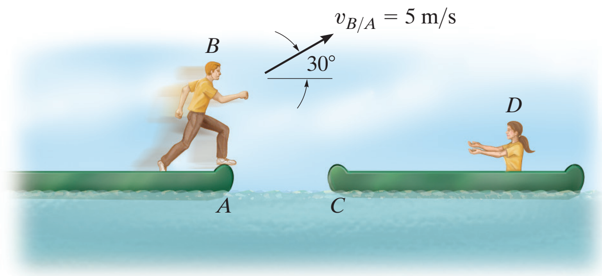
Prob. 15–45

15–46. Blocks A and B have masses of 40 kg and 60 kg, respectively. They are placed on a smooth surface and the spring connected between them is stretched 2 m. If they are released from rest, determine the speeds of both blocks the instant the spring becomes unstretched.



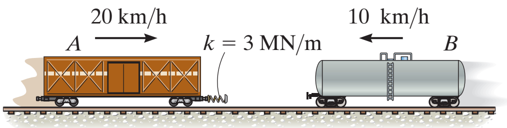
Prob. 15–46

15–47. The boy B jumps off the canoe at A with a velocity of 5 m/s relative to the canoe as shown. If he lands in the second canoe C, determine the final speed of both canoes after the motion. Each canoe has a mass of 40 kg. The boy's mass is 30 kg, and the girl D has a mass of 25 kg. Both canoes are originally at rest.



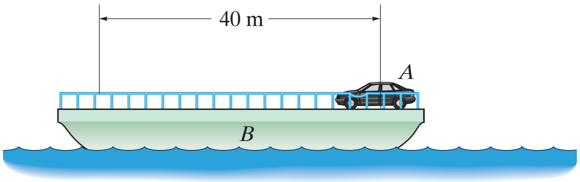
Prob. 15–47

***15–48.** The 30-Mg freight car A and 15-Mg freight car B are moving towards each other with the velocities shown. Determine the maximum compression of the spring mounted on car A. Neglect rolling resistance.



Prob. 15–48

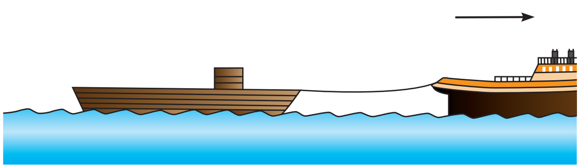
15–49. The 10-Mg barge B supports a 2-Mg automobile A. If someone drives the automobile to the other side of the barge, determine how far the barge moves. Neglect the resistance of the water.



Prob. 15–49

15–50. The crate has a mass m and rests on the barge which has a mass M and is initially at rest. The coefficient of kinetic friction between the crate and barge is μ. The rope is pulled (or jerked) such that it snaps and as a result the impulse gives the barge a sudden velocity $\mathbf{v}_0$. Determine the time the crate slides on the barge before coming to rest on the barge. Also, what is the final velocity of the barge?

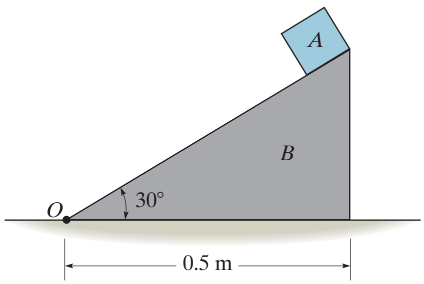
15–51. Using the data in Prob. 15–50, determine the distance the crate slides on the barge before coming to rest.



Probs. 15–50/51

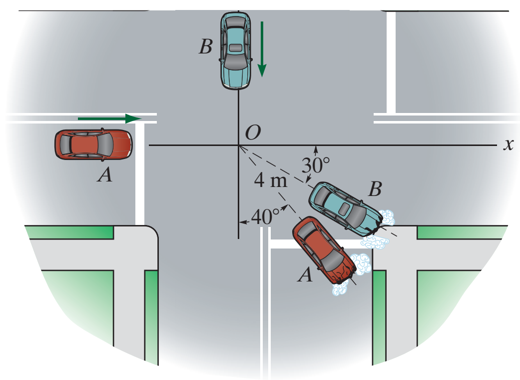
***15–52.** Block A has a mass of 5 kg and is placed on the smooth triangular block B having a mass of 30 kg. If the system is released from rest, determine the distance B moves from point O when A reaches the bottom. Neglect the size of block A.

15–53. Solve Prob. 15–52 if the coefficient of kinetic friction between A and B is $\mu_k = 0.3$. Neglect friction between block B and the horizontal plane.



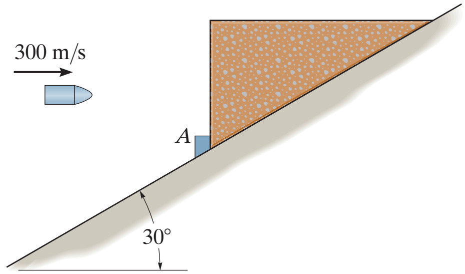
Probs. 15–52/53

15–54. Two cars A and B, each having a mass of 1.6 Mg, collide on the icy pavement of an intersection. The direction of motion of each car after collision is measured from snow tracks as shown. If the driver in car A states that he was going 50 km/h just before collision and that after collision he applied the brakes, so that his car skidded 4 m before stopping, determine the approximate speed of car B just before collision. Assume that the coefficient of kinetic friction between the car wheels and the pavement is $\mu_k = 0.15$. *Note*: The line of impact has not been defined; furthermore, this information is not needed for solution.



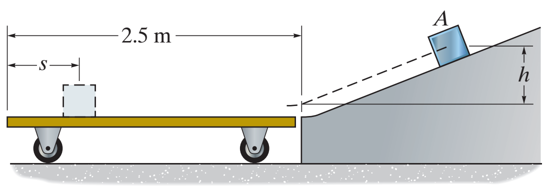
Prob. 15–54

15–55. The 10-kg block is held at rest on the smooth inclined plane by the stop block at A. If the 10-g bullet is traveling at 300 m/s when it becomes embedded in the 10-kg block, determine the distance the block will slide up along the plane before momentarily stopping.



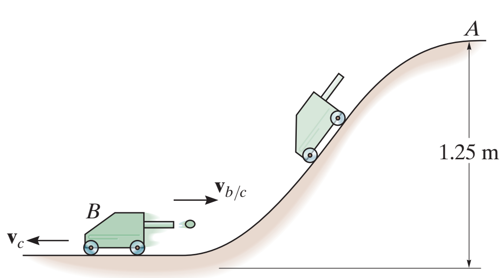
Prob. 15–55

***15–56.** A box A, having a mass of 20 kg, is released from rest at the position shown and slides freely down the smooth inclined ramp. When it reaches the bottom of the ramp, it slides horizontally onto the surface of a 10-kg cart for which the coefficient of kinetic friction between the cart and the box is $\mu_k = 0.6$. If $h = 0.2$ m, determine the final velocity of the cart once the block comes to rest on it. Also, determine the position s of the box on the cart after it comes to rest on the cart.



Prob. 15–56

15–57. The cart has a mass of 3 kg and rolls freely from A down the slope. When it reaches the bottom, a spring loaded gun fires a 0.5-kg ball out the back with a horizontal velocity of $v_{b/c} = 0.6$ m/s, measured relative to the cart. Determine the final velocity of the cart.



Prob. 15–57

15.4 IMPACT

Impact occurs when two bodies collide with each other during a very short period of time, causing relatively large (impulsive) forces to be exerted between the bodies. The striking of a hammer on a nail, or a golf club on a ball, are common examples of impact loadings.

In general, there are two types of impact. ***Central impact*** occurs when the direction of motion of the mass centers of the two colliding particles is along the ***line of impact*** that passes through the mass centers of the particles. This line is perpendicular to the ***plane of contact***, as shown in Fig. 15–13*a*. When the motion of one or both of the particles makes an angle with the line of impact, Fig. 15–13*b*, the impact is said to be ***oblique impact***.

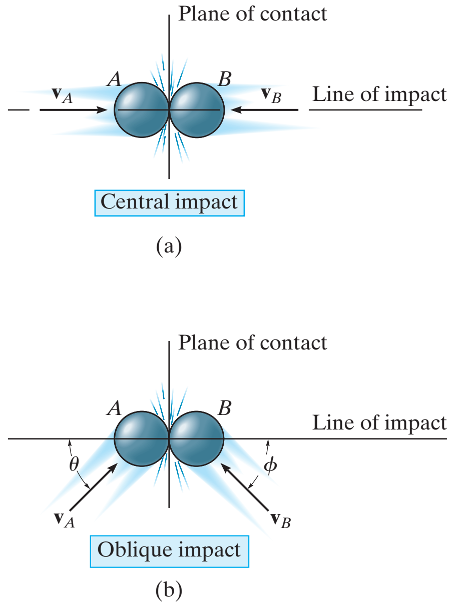
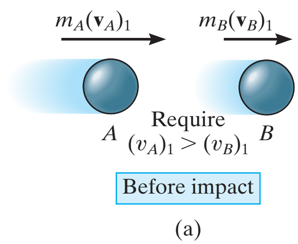


Fig. 15–13

Central Impact.

To illustrate the method for analyzing the mechanics of impact, consider the case involving the central impact of the two particles A and B shown in Fig. 15–14.

- The particles have the initial momenta shown in Fig. 15–14*a*. Provided $(v_A)_1 > (v_B)_1$, collision will eventually occur.
- During the collision the particles must be thought of as deformable or nonrigid. The particles will undergo a period of deformation such that they exert an equal but opposite ***deformation impulse*** $\int \mathbf{P}\,dt$ on each other, Fig. 15–14*b*.
- Only at the instant of maximum deformation will both particles move with a common velocity $\mathbf{v}$, since their relative motion is zero, Fig. 15–14*c*.
- Afterward a period of restitution occurs, in which case the particles will either return to their original shape or remain permanently deformed. The equal but opposite ***restitution impulse*** $\int \mathbf{R}\,dt$ pushes the particles apart from one another, Fig. 15–14*d*. In reality, the physical properties of any two bodies are such that the deformation impulse will always be greater than the restitution impulse, i.e., $\int P\,dt > \int R\,dt$.
- Just after separation the particles will have the final momenta shown in Fig. 15–14*e*, where $(v_B)_2 > (v_A)_2$.



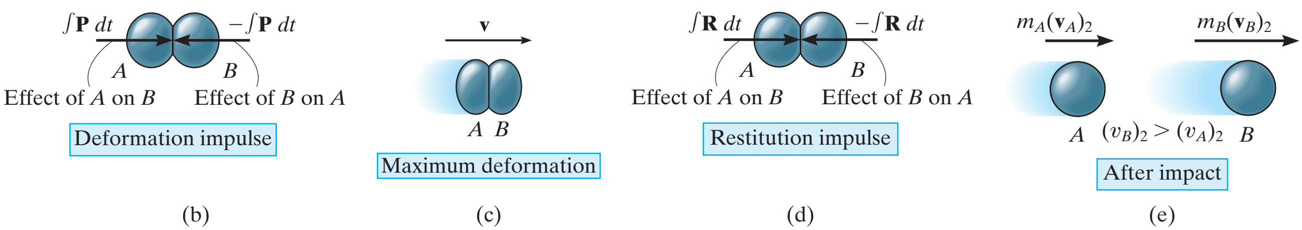


Fig. 15–14

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In most problems the initial velocities of the particles will be *known*, and it will be necessary to determine their final velocities $(v_A)_2$ and $(v_B)_2$. To do this we will need two equations that involve these unknowns. The first equation requires conservation of momentum for the two particle systems since during collision the internal impulses of deformation and restitution *cancel*. Hence, referring to Fig. 15–14*a* and Fig. 15–14*e* we require

$$(\overset{+}{\rightarrow}) \qquad m_A(v_A)_1 + m_B(v_B)_1 = m_A(v_A)_2 + m_B(v_B)_2 \qquad (15\text{–}10)$$

To obtain a second equation, we will apply the principle of impulse and momentum to *each particle*. For example, during the deformation phase for particle A, Figs. 15–14*a*, 15–14*b*, and 15–14*c*, we have

$$(\overset{+}{\rightarrow}) \qquad m_A(v_A)_1 - \int P\,dt = m_A v$$

For the restitution phase, Figs. 15–14*c*, 15–14*d*, and 15–14*e*,

$$(\overset{+}{\rightarrow}) \qquad m_A v - \int R\,dt = m_A(v_A)_2$$

15

If we define the ***coefficient of restitution***, e, as the ratio of the restitution impulse to the deformation impulse, then from the above equations, this value for particle A is

$$e = \frac{\int R\,dt}{\int P\,dt} = \frac{v - (v_A)_2}{(v_A)_1 - v}$$

In a similar manner, we can establish e by considering particle B, Fig. 15–14. This yields

$$e = \frac{\int R\,dt}{\int P\,dt} = \frac{(v_B)_2 - v}{v - (v_B)_1}$$

If the unknown v is eliminated from the above two equations, the coefficient of restitution can be expressed in terms of the particles' initial and final velocities as

$$(\overset{+}{\rightarrow}) \qquad \boxed{e = \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}} \qquad (15\text{–}11)$$

Provided a value for e is specified, Eqs. 15–10 and 15–11 can be solved simultaneously to obtain $(v_A)_2$ and $(v_B)_2$. In doing so, however, it is important to carefully establish a sign convention for defining the positive direction for both $\mathbf{v}_A$ and $\mathbf{v}_B$ and then use it *consistently* when writing *both* equations. For the case shown in Fig. 15–14, we have defined the positive direction to the right when referring to the motions of both A and B. Consequently, if a negative value results from the solution of either $(v_A)_2$ or $(v_B)_2$, it indicates motion is to the left.

The quality of a manufactured tennis ball is measured by the height of its bounce, which can be related to its coefficient of restitution. Using the mechanics of oblique impact, engineers can design a separation device to remove substandard tennis balls from a production line.

Coefficient of Restitution. From Figs. 15–14*a* and 15–14*e*, it is seen that Eq. 15–11 states that e is equal to the ratio of the relative velocity of the particles' separation just after impact, $(v_B)_2 - (v_A)_2$, to the relative velocity of the particles' approach just before impact, $(v_A)_1 - (v_B)_1$, and so we can also write Eq. 15–11 as

$$e = \frac{\text{rel. vel. after impact}}{\text{rel. vel. before impact}} \qquad (15\text{–}12)$$

By measuring these relative velocities experimentally, it has been found that e varies appreciably with impact velocity as well as with the size and shape of the colliding bodies. For these reasons the coefficient of restitution is reliable only when used with data which closely approximate the conditions which existed when measurements of e were made. In general e has a value between zero and one, and one should be aware of the physical meaning of these two limits.

Elastic Impact ($e = 1$). If the collision between the two particles is perfectly elastic, the deformation impulse $\left(\int \mathbf{P}\,dt\right)$ is equal and opposite to the restitution impulse $\left(\int \mathbf{R}\,dt\right)$. Although in reality this can never be achieved, $e = 1$ for an elastic collision.

Plastic Impact ($e = 0$). The impact is said to be inelastic or plastic when $e = 0$. In this case there is no restitution impulse $\left(\int \mathbf{R}\,dt = \mathbf{0}\right)$, so that after collision both particles couple or stick together and move with a common velocity.

From the above derivation it should be evident that the principle of work and energy cannot be used for the analysis of impact problems since it is not possible to know how the internal forces of deformation and restitution vary or displace during the collision. By knowing the particle's velocities before and after collision, however, the energy loss during collision can be calculated on the basis of the difference in the particle's kinetic energy. This energy loss, $\Sigma U_{1-2} = \Sigma T_2 - \Sigma T_1$, occurs because some of the initial kinetic energy of the particle is transformed into localized deformation of the material when the collision occurs. In particular, if the impact is perfectly elastic, no energy is lost in the collision; whereas if the collision is plastic, the energy lost during collision is a maximum.

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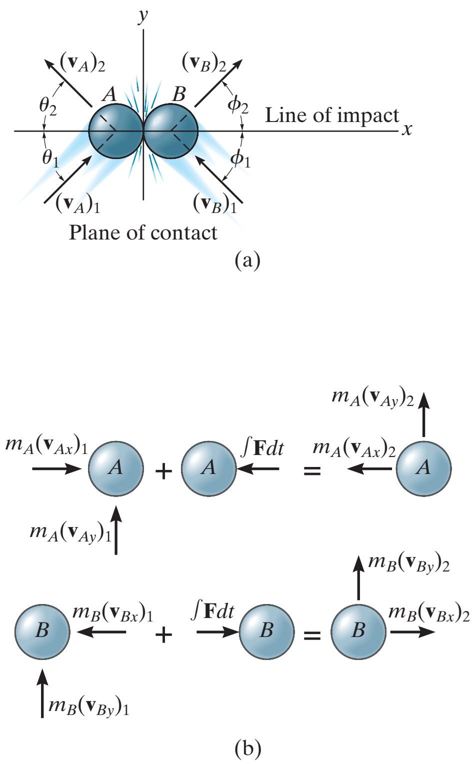


Fig. 15–15

PROCEDURE FOR ANALYSIS (CENTRAL IMPACT)

In most cases the final velocities of two smooth particles are to be determined just after they are subjected to direct central impact. Provided the coefficient of restitution, the mass of each particle, and each particle's initial velocity just before impact are known, the solution to this problem can be obtained using the following two equations.

- The conservation of momentum applies to the system of particles, $\Sigma mv_1 = \Sigma mv_2$.
- The coefficient of restitution, $e = [(v_B)_2 - (v_A)_2]/[(v_A)_1 - (v_B)_1]$, relates the relative velocities of the particles along the line of impact, just before and just after impact.

When applying these two equations, the sense of an unknown velocity can be assumed. If the solution yields a negative magnitude, the velocity acts in the opposite sense.

Oblique Impact. When oblique impact occurs between two smooth particles, the particles move away from each other with velocities having unknown directions as well as unknown magnitudes. Provided the initial velocities are known, then there are four unknowns in the problem. They are either $(v_A)_2$, $(v_B)_2$, θ_2, and ϕ_2, Fig. 15–15*a*, or the *x* and *y* components of the final velocities, Fig. 15–15*b*.

Refer to the companion website for Lecture Summary and Quiz videos.

PROCEDURE FOR ANALYSIS (OBLIQUE IMPACT)

If the *y* axis is established within the plane of contact and the *x* axis along the line of impact, then the impulsive forces of deformation and restitution act *only in the x direction*, Fig. 15–15. By resolving the velocity or momentum vectors into components along the *x* and *y* axes, it is then possible to write four independent scalar equations in order to determine $(v_{Ax})_2$, $(v_{Ay})_2$, $(v_{Bx})_2$, and $(v_{By})_2$.

- Momentum of the system is conserved *along the line of impact, x* axis, so that $\Sigma m(v_x)_1 = \Sigma m(v_x)_2$.
- The coefficient of restitution, $e = [(v_{Bx})_2 - (v_{Ax})_2]/[(v_{Ax})_1 - (v_{Bx})_1]$, relates the relative-velocity *components* of the particles *along the line of impact* (*x* axis).
- If these two equations are solved simultaneously, we obtain $(v_{Ax})_2$ and $(v_{Bx})_2$.
- Momentum of particle *A* is conserved along the *y* axis, perpendicular to the line of impact, since no impulse acts on particle *A* in this direction. As a result $m_A(v_{Ay})_1 = m_A(v_{Ay})_2$ or $(v_{Ay})_1 = (v_{Ay})_2$.
- Momentum of particle *B* is conserved along the *y* axis, perpendicular to the line of impact, since no impulse acts on particle *B* in this direction. Consequently $(v_{By})_1 = (v_{By})_2$.

Application of these four equations is illustrated in Example 15.11.

EXAMPLE 15.9

The bag A, having a mass of 2.7 kg, is released from rest at the position $\theta = 0°$, as shown in Fig. 15–16*a*. After falling to $\theta = 90°$, it strikes an 8.1-kg box B. If the coefficient of restitution between the bag and box is $e = 0.5$, determine the velocities of the bag and box just after impact. What is the loss of energy during collision?

SOLUTION

This problem involves central impact. Why? Before analyzing the mechanics of the impact, however, it is first necessary to obtain the velocity of the bag *just before* it strikes the box.

Conservation of Energy. With the datum at $\theta = 0°$, Fig. 15–16*b*, we have

$$T_0 + V_0 = T_1 + V_1$$

$$0 + 0 = \frac{1}{2}(2.7\,\text{kg})(v_A)_1^2 - 2.7\,\text{kg}(9.81\text{ m/s}^2)(0.9\text{ m});\;\; (v_A)_1 = 4.202\text{ m/s}$$

Conservation of Momentum. Just after impact we will assume both A and B travel to the left. Applying the conservation of momentum to the system, Fig. 15–16*c*, we have

$$(\overset{+}{\leftarrow}) \qquad m_B(v_B)_1 + m_A(v_A)_1 = m_B(v_B)_2 + m_A(v_A)_2$$

$$0 + (2.7\text{ kg})(4.202\text{ m/s}) = (8.1\text{ kg})(v_B)_2 + (2.7\text{ kg})(v_A)_2$$

$$(v_A)_2 = 4.202 - 3(v_B)_2 \qquad (1)$$

Coefficient of Restitution. Realizing that for separation to occur just after collision $(v_B)_2 > (v_A)_2$, Fig. 15–16*c*, we have

$$(\overset{+}{\leftarrow}) \qquad e = \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}; \qquad 0.5 = \frac{(v_B)_2 - (v_A)_2}{4.202\text{ m/s} - 0}$$

$$(v_A)_2 = (v_B)_2 - 2.101 \qquad (2)$$

Solving Eqs. 1 and 2 simultaneously yields

$$(v_A)_2 = -0.525\text{ m/s} = 0.525\text{ m/s} \rightarrow \quad \text{and} \quad (v_B)_2 = 1.58\text{ m/s} \leftarrow \qquad \textit{Ans.}$$

Loss of Energy. Applying the principle of work and energy to the bag and box just before and just after collision, we have

$$\Sigma U_{1-2} = T_2 - T_1;$$

$$\Sigma U_{1-2} = \left[\frac{1}{2}(8.1\text{ kg})(1.58\text{ m/s})^2 + \frac{1}{2}(2.7\text{ kg})(0.525\text{ m/s})^2\right] - \left[\frac{1}{2}(2.7\text{ kg})(4.202\text{ m/s})^2\right]$$

$$\Sigma U_{1-2} = -13.4\text{ J} \qquad \textit{Ans.}$$

NOTE: The energy loss occurs due to inelastic deformation during the collision.

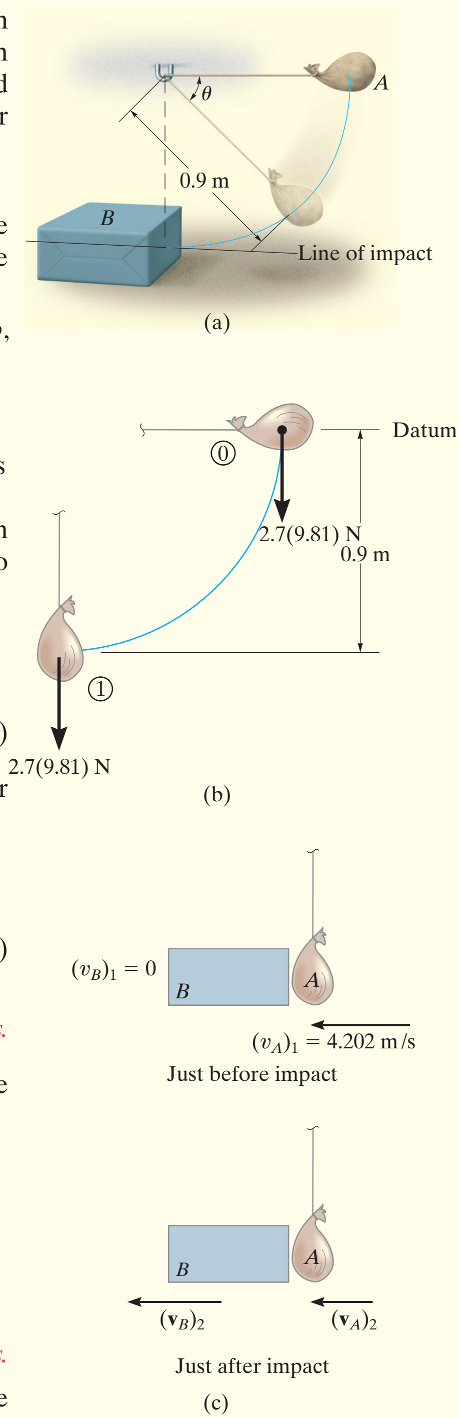


Fig. 15–16

EXAMPLE 15.10

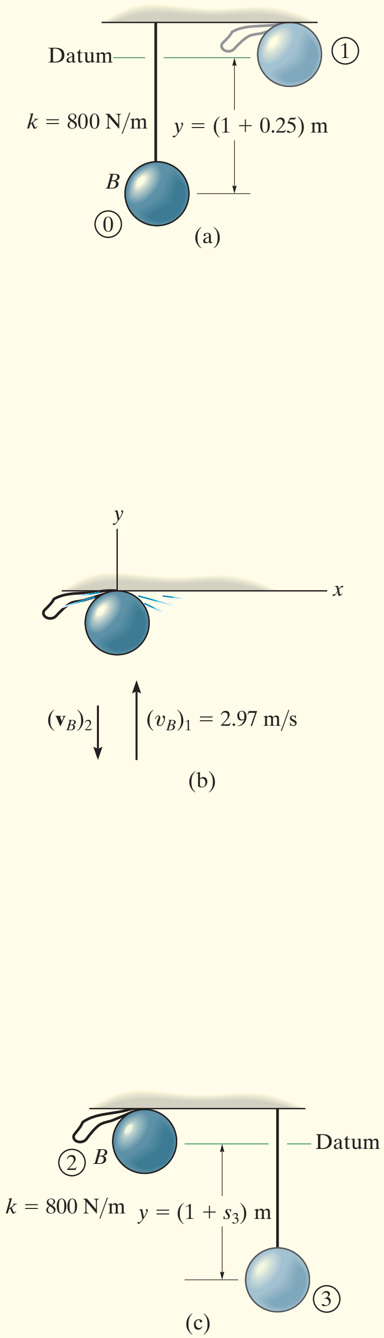


Fig. 15–17

Ball B shown in Fig. 15–17a has a mass of 1.5 kg and is suspended from the ceiling by a 1-m-long elastic cord. If the cord is *stretched* downward 0.25 m and the ball is released from rest, determine how far the cord stretches after the ball rebounds from the ceiling and falls downward. The stiffness of the cord is $k = 800$ N/m, and the coefficient of restitution between the ball and ceiling is $e = 0.8$. The ball makes a central impact with the ceiling.

SOLUTION

First we must obtain the velocity of the ball *just before* it strikes the ceiling using energy methods, then consider the impulse and momentum between the ball and ceiling, and finally again use energy methods to determine the stretch in the cord.

Conservation of Energy. With the datum located as shown in Fig. 15–17a, realizing that initially $y = y_0 = (1 + 0.25)$ m $= 1.25$ m, we have

$$T_0 + V_0 = T_1 + V_1$$

$$\tfrac{1}{2}m(v_B)_0^2 - W_B y_0 + \tfrac{1}{2}ks^2 = \tfrac{1}{2}m(v_B)_1^2 + 0$$

$$0 - 1.5(9.81)\text{N}(1.25\text{ m}) + \tfrac{1}{2}(800\text{ N/m})(0.25\text{ m})^2 = \tfrac{1}{2}(1.5\text{ kg})(v_B)_1^2$$

$$(v_B)_1 = 2.968\text{ m/s}\uparrow$$

The interaction of the ball with the ceiling will now be considered using the principles of impact.* Since an unknown portion of the mass of the ceiling is involved in the impact, the conservation of momentum for the ball–ceiling system will not be written. The "velocity" of this portion of ceiling is zero since it (or the earth) are assumed to remain at rest *both* before and after impact.

Coefficient of Restitution. Fig. 15–17b.

$$(+\uparrow) \quad e = \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}; \qquad 0.8 = \frac{(v_B)_2 - 0}{0 - 2.968\text{ m/s}}$$

$$(v_B)_2 = -2.374\text{ m/s} = 2.374\text{ m/s}\downarrow$$

Conservation of Energy. The maximum stretch s_3 in the cord can be determined by again applying the conservation of energy equation to the ball just after collision. Assuming that $y = y_3 = (1 + s_3)$ m, Fig. 15–17c, then

$$T_2 + V_2 = T_3 + V_3$$

$$\tfrac{1}{2}m(v_B)_2^2 + 0 = \tfrac{1}{2}m(v_B)_3^2 - W_B y_3 + \tfrac{1}{2}ks_3^2$$

$$\tfrac{1}{2}(1.5\text{ kg})(2.37\text{ m/s})^2 = 0 - 9.81(1.5)\text{ N}(1\text{ m} + s_3) + \tfrac{1}{2}(800\text{ N/m})s_3^2$$

$$400s_3^2 - 14.715s_3 - 18.94 = 0$$

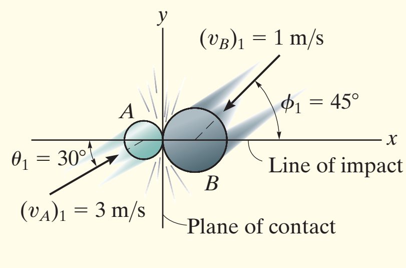
Solving this quadratic equation for the positive root yields

$$s_3 = 0.237\text{ m} = 237\text{ mm} \qquad \textit{Ans.}$$

* The weight of the ball is considered a nonimpulsive force.

EXAMPLE 15.11

Two smooth disks A and B, having a mass of 1 kg and 2 kg, respectively, slide on the horizontal plane and collide with the velocities shown in Fig. 15–18*a*. If the coefficient of restitution for the disks is $e = 0.75$, determine the x and y components of the final velocity of each disk just after collision.



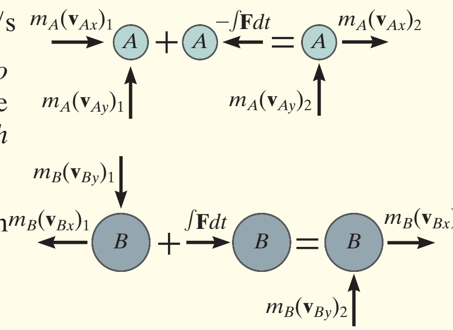
(a)

SOLUTION

This problem involves *oblique impact*. Why? In order to solve it, we have established the x and y axes along the line of impact and the plane of contact, respectively, Fig. 15–18*a*.

Resolving each of the initial velocities into x and y components, we have

$$(v_{Ax})_1 = 3\cos 30° = 2.598 \text{ m/s} \qquad (v_{Ay})_1 = 3\sin 30° = 1.50 \text{ m/s}$$
$$(v_{Bx})_1 = -1\cos 45° = -0.7071 \text{ m/s} \qquad (v_{By})_1 = -1\sin 45° = -0.7071 \text{ m/s}$$



(b)

The four unknown velocity components after collision are *assumed to act in the positive directions*, Fig. 15–18*b*. Since the impact occurs in the x direction (line of impact), the conservation of momentum for *both* disks can be applied in this direction. Why?

Conservation of "x" Momentum. In reference to the momentum diagrams, we have

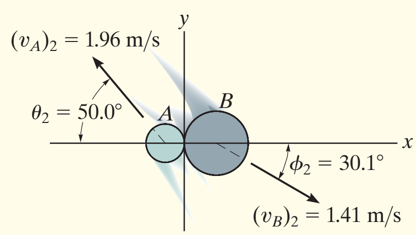
$$(\overset{+}{\rightarrow}) \qquad m_A(v_{Ax})_1 + m_B(v_{Bx})_1 = m_A(v_{Ax})_2 + m_B(v_{Bx})_2$$
$$1 \text{ kg}(2.598 \text{ m/s}) + 2 \text{ kg}(-0.707 \text{ m/s}) = 1 \text{ kg}(v_{Ax})_2 + 2 \text{ kg}(v_{Bx})_2$$
$$(v_{Ax})_2 + 2(v_{Bx})_2 = 1.184 \qquad (1)$$

Coefficient of Restitution (x).

$$(\overset{+}{\rightarrow}) \qquad e = \frac{(v_{Bx})_2 - (v_{Ax})_2}{(v_{Ax})_1 - (v_{Bx})_1}; \; 0.75 = \frac{(v_{Bx})_2 - (v_{Ax})_2}{2.598 \text{ m/s} - (-0.7071 \text{ m/s})}$$
$$(v_{Bx})_2 - (v_{Ax})_2 = 2.482 \qquad (2)$$

Solving Eqs. 1 and 2 for $(v_{Ax})_2$ and $(v_{Bx})_2$ yields

$$(v_{Ax})_2 = -1.26 \text{ m/s} = 1.26 \text{ m/s} \leftarrow \qquad (v_{Bx})_2 = 1.22 \text{ m/s} \rightarrow \qquad \textit{Ans.}$$



(c)

Fig. 15–18

Conservation of "y" Momentum. The momentum of *each disk* is *conserved* in the y direction (plane of contact), since the disks are smooth and therefore no frictional impulse acts in this direction. From Fig. 15–18*b*,

$$(+\uparrow) \; m_A(v_{Ay})_1 = m_A(v_{Ay})_2; \qquad (v_{Ay})_2 = 1.50 \text{ m/s} \uparrow \qquad \textit{Ans.}$$
$$(+\uparrow) \; m_B(v_{By})_1 = m_B(v_{By})_2; \; (v_{By})_2 = -0.707 \text{ m/s} = 0.707 \text{ m/s} \downarrow \qquad \textit{Ans.}$$

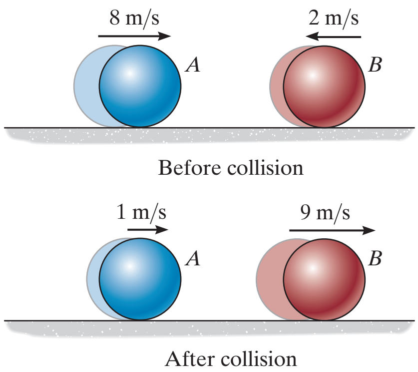
NOTE: Show that when the velocity components are summed vectorially, one obtains the results shown in Fig. 15–18*c*.

Refer to the companion website for a self quiz of these Example problems.

15

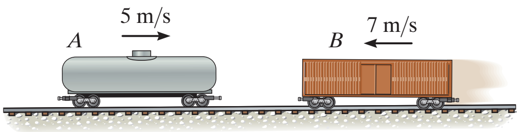
FUNDAMENTAL PROBLEMS

F15–13. Determine the coefficient of restitution e between ball A and ball B. The velocities of A and B before and after the collision are shown.



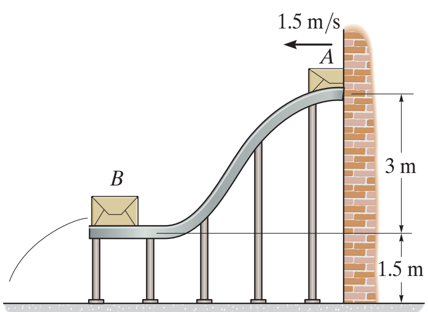
Prob. F15–13

F15–14. The 15-Mg tank car A and 25-Mg freight car B travel toward each other with the velocities shown. If the coefficient of restitution between the bumpers is $e = 0.6$, determine the velocity of each car just after the collision.



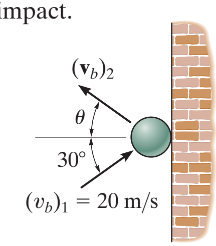
Prob. F15–14

F15–15. The 15-kg package A has a speed of 1.5 m/s when it enters the smooth ramp. As it slides down the ramp, it strikes the 40-kg package B which is initially at rest. If the coefficient of restitution between A and B is $e = 0.6$, determine the velocity of B just after the impact.



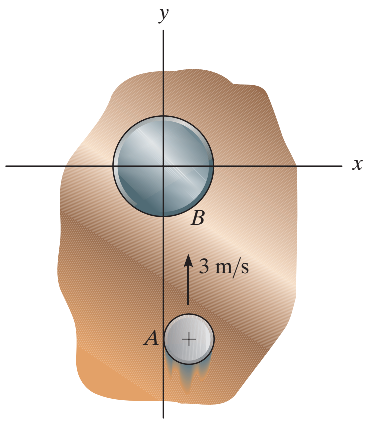
Prob. F15–15

F15–16. The ball strikes the smooth wall with a velocity of $(v_b)_1 = 20$ m/s. If the coefficient of restitution between the ball and the wall is $e = 0.75$, determine the velocity of the ball just after the impact.



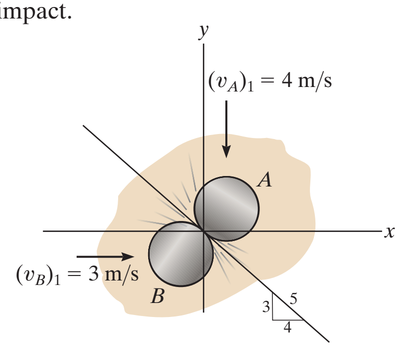
Prob. F15–16

F15–17. Disk A has a mass of 2 kg and slides on the smooth horizontal plane with a velocity of 3 m/s. Disk B has a mass of 11 kg and is initially at rest. If after impact A has a velocity of 1 m/s, parallel to the positive x axis, determine the speed of disk B after impact.



Prob. F15–17

F15–18. Two disks A and B each have a mass of 1 kg and the initial velocities shown just before they collide. If the coefficient of restitution is $e = 0.5$, determine their speeds just after impact.

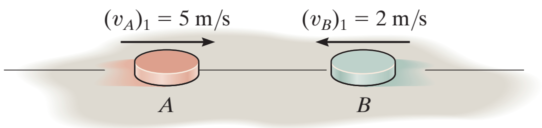


Prob. F15–18

15

PROBLEMS

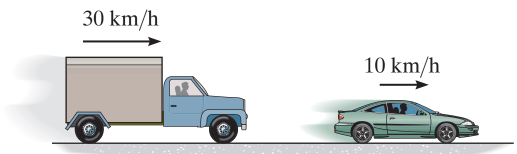
15–58. Disk A has a mass of 2 kg and is sliding forward on the *smooth* surface with a velocity $(v_A)_1 = 5$ m/s when it strikes the 4-kg disk B, which is sliding towards A at $(v_B)_1 = 2$ m/s, with direct central impact. If the coefficient of restitution between the disks is $e = 0.4$, compute the velocities of A and B just after collision.



Prob. 15–58

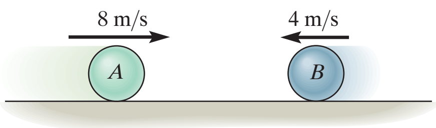
15–59. Disk A has a mass of 250 g and is sliding on a *smooth* horizontal surface with an initial velocity $(v_A)_1 = 2$ m/s. It makes a direct collision with disk B, which has a mass of 175 g and is originally at rest. If both disks are of the same size and the collision is perfectly elastic ($e = 1$), determine the velocity of each disk just after collision. Show that the kinetic energy of the disks before and after collision is the same.

***15–60.** The 5-Mg truck and 2-Mg car are traveling with the free-rolling velocities shown just before they collide. After the collision, the car moves with a velocity of 15 km/h to the right *relative* to the truck. Determine the coefficient of restitution between the truck and car and the loss of energy due to the collision.



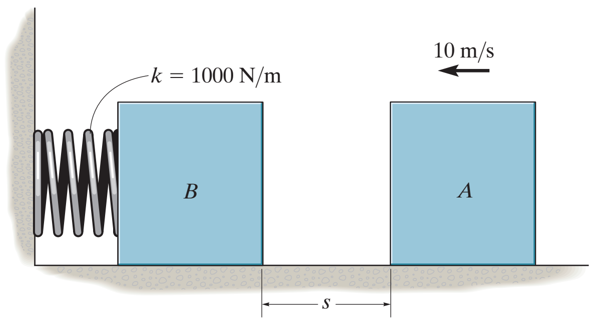
Prob. 15–60

15–61. Ball A has a mass of 3 kg and is moving with a velocity of 8 m/s when it makes a direct collision with ball B, which has a mass of 2 kg and is moving with a velocity of 4 m/s. If $e = 0.7$, determine the velocity of each ball just after the collision. Neglect the size of the balls.



Prob. 15–61

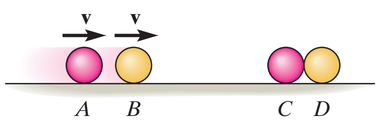
15–62. The 15-kg block A has a velocity $v = 10$ m/s when it is $s = 4$ m from the 10-kg block B. If the unstretched spring has a stiffness $k = 1000$ N/m, determine the maximum compression of the spring after the collision. Take $e = 0.6$, and for block A, $\mu_k = 0.3$. Assume block B is smooth.



Prob. 15–62

15–63. The four smooth balls each have the same mass m. If A and B are rolling forward with a velocity $\mathbf{v}$ and strike C, explain why after collision C and D each move off with a velocity $\mathbf{v}$. Why doesn't D move off with a velocity $2\mathbf{v}$? The collision is elastic, $e = 1$. Neglect the size of each ball.

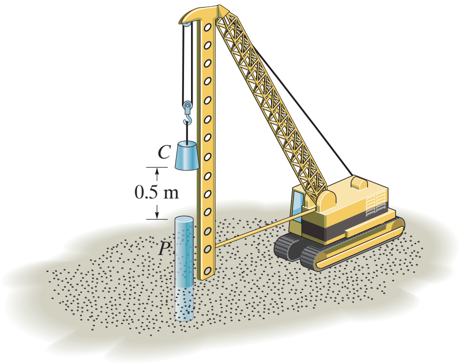
***15–64.** The four balls each have the same mass m. If A and B are rolling forward with a velocity $\mathbf{v}$ and strike C, determine the velocity of each ball after the first three collisions. Take $e = 0.5$ between each ball.



Probs. 15–63/64

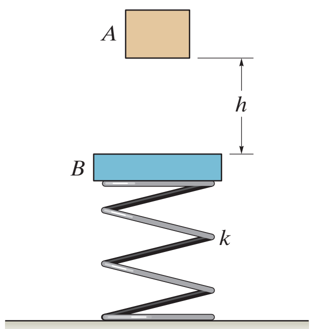
15–65. The pile P has a mass of 800 kg and is being driven into *loose sand* using the 300-kg hammer C which is dropped a distance of 0.5 m from the top of the pile. Determine the initial speed of the pile just after it is struck by the hammer. The coefficient of restitution between the hammer and the pile is $e = 0.1$. Neglect the impulses due to the weights of the pile and hammer and the impulse due to the sand during the impact.

15–66. The pile P has a mass of 800 kg and is being driven into *loose sand* using the 300-kg hammer C which is dropped a distance of 0.5 m from the top of the pile. Determine the distance the pile is driven into the sand after one blow if the sand offers a frictional resistance against the pile of 18 kN. The coefficient of restitution between the hammer and the pile is $e = 0.1$. Neglect the impulses due to the weights of the pile and hammer and the impulse due to the sand during the impact.



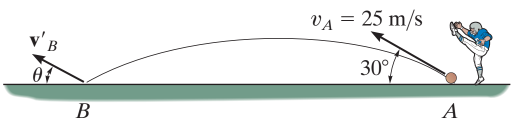
Probs. 15–65/66

15–67. Block A, having a mass m, is released from rest, falls a distance h and strikes the plate B having a mass $2m$. If the coefficient of restitution between A and B is e, determine the velocity of the plate just after collision. The spring has a stiffness k.



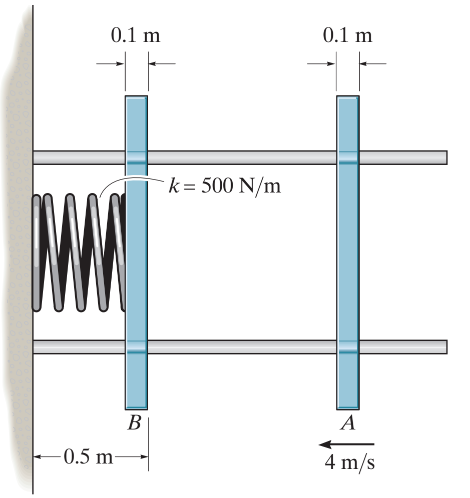
Prob. 15–67

***15–68.** A 300-g ball is kicked with a velocity of $v_A = 25$ m/s at point A as shown. If the coefficient of restitution between the ball and the field is $e = 0.4$, determine the magnitude and direction θ of the velocity of the rebounding ball at B.



Prob. 15–68

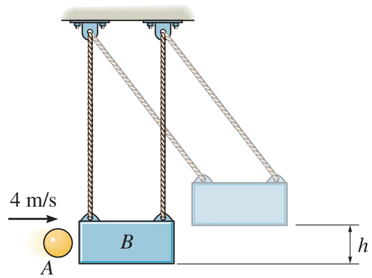
15–69. Plates A and B each have a mass of 4 kg and are restricted to move along the smooth guides. If the coefficient of restitution between the plates is $e = 0.7$, determine (a) the speed of both plates just after collision and (b) the maximum compression of the spring. Plate A has a velocity of 4 m/s just before striking B. Plate B is originally at rest and the spring is unstretched.



Prob. 15–69

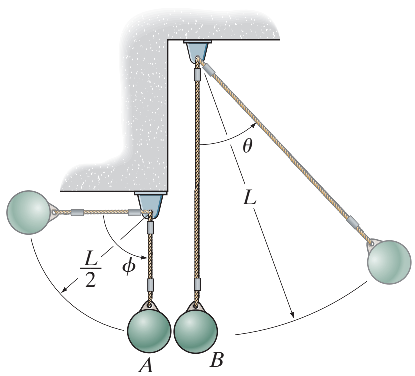
15–70. The 2-kg ball is thrown at the suspended 20-kg block with a velocity of 4 m/s. If the coefficient of restitution between the ball and the block is $e = 0.8$, determine the maximum height h to which the block will swing before it momentarily stops.

15–71. The 2-kg ball is thrown at the suspended 20-kg block with a velocity of 4 m/s. If the time of impact between the ball and the block is 0.005 s, determine the average normal force exerted on the block during this time. Take $e = 0.8$.



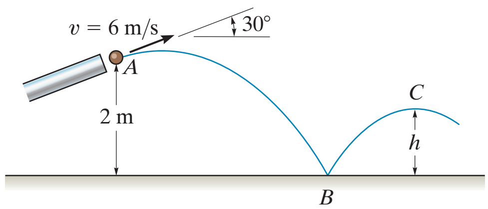
Probs. 15–70/71

***15–72.** Two identical balls A and B of mass m are suspended from cords of length $L/2$ and L, respectively. Ball A is released from rest when $\phi = 90°$ and swings down to $\phi = 0°$, where it strikes B. Determine the speed of each ball just after impact and the maximum angle θ through which B will swing. The coefficient of restitution between the balls is e.



Prob. 15–72

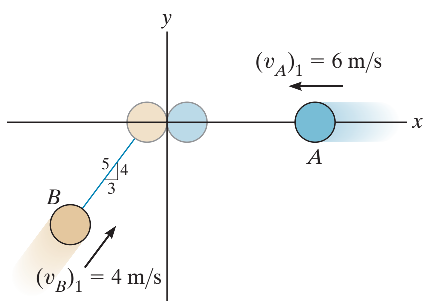
15–73. The 0.5-kg ball is fired from the tube at A with a velocity of $v = 6$ m/s. If the coefficient of restitution between the ball and the surface is $e = 0.8$, determine the height h after it bounces off the surface.



Prob. 15–73

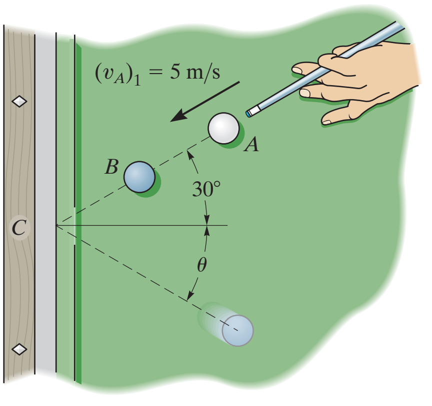
15–74. Two smooth disks A and B each have a mass of 0.5 kg. If both disks are moving with the velocities shown when they collide, determine their final velocities just after collision. The coefficient of restitution is $e = 0.75$.

15–75. Two smooth disks A and B each have a mass of 0.5 kg. If both disks are moving with the velocities shown when they collide, determine the coefficient of restitution between the disks if after collision B travels along a line 30° counterclockwise from the y axis.



Probs. 15–74/75

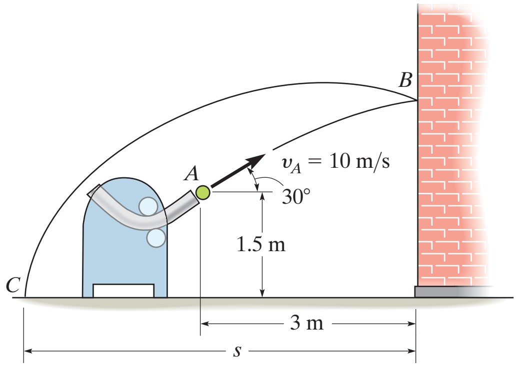
***15–76.** The cue ball A is given an initial velocity $(v_A)_1 = 5\ \text{m/s}$. If it makes a direct collision with ball B ($e = 0.8$), determine the velocity of B and the angle θ just after it rebounds from the cushion at C ($e' = 0.6$). Each ball has a mass of 0.4 kg. Neglect their size.



Prob. 15–76

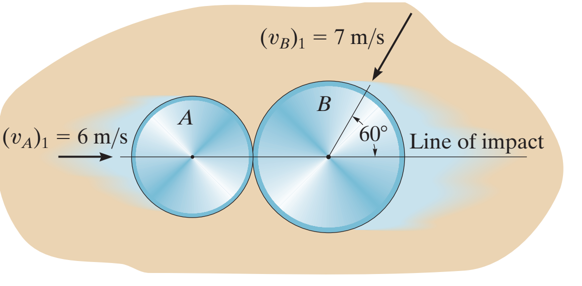
15

15–77. A pitching machine throws the 0.5-kg ball toward the wall with an initial velocity $v_A = 10\ \text{m/s}$ as shown. Determine (a) the velocity at which it strikes the wall at B, (b) the velocity at which it rebounds from the wall if $e = 0.5$, and (c) the distance s from the wall to where it strikes the ground at C.



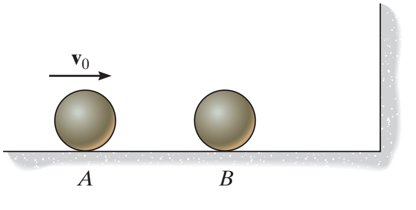
Prob. 15–77

15–78. The two disks A and B have a mass of 3 kg and 5 kg, respectively. If they collide with the initial velocities shown, determine their velocities just after impact. The coefficient of restitution is $e = 0.65$.



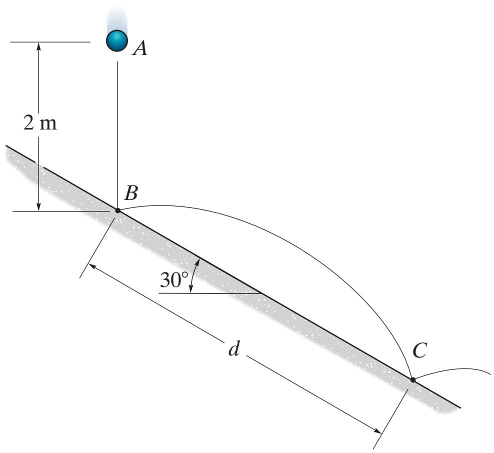
Prob. 15–78

15–79. Two smooth spheres A and B each have a mass m. If A is given a velocity of v_0, while sphere B is at rest, determine the velocity of B just after it strikes the wall. The coefficient of restitution for any collision is e.



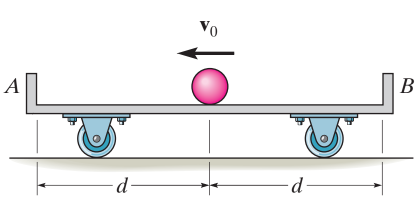
Prob. 15–79

***15–80.** The 1-kg ball is dropped from rest at point A, 2 m above the smooth plane. If the coefficient of restitution between the ball and the plane is $e = 0.6$, determine the distance d where the ball again strikes the plane.



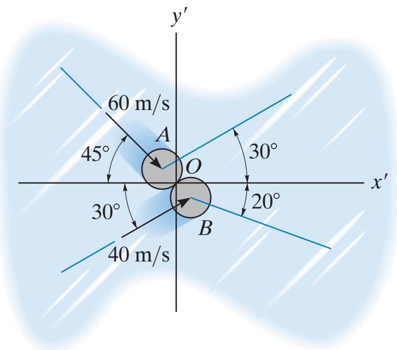
Prob. 15–80

15–81. A ball of negligible size and mass m is given a velocity of $\mathbf{v}_0$ on the center of the cart which has a mass M and is originally at rest. If the coefficient of restitution between the ball and walls A and B is e, determine the velocity of the ball and the cart just after the ball strikes A. Also, determine the total time needed for the ball to strike A, rebound, then strike B, and rebound and then return to the center of the cart. Neglect friction.



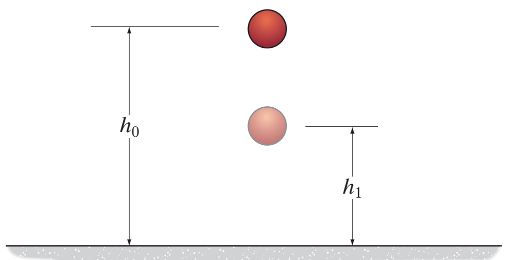
Prob. 15–81

15–82. The two hockey pucks A and B each have a mass of 250 g. If they collide at O and are deflected along the colored paths, determine their speeds just after impact. Assume that the icy surface over which they slide is smooth. *Hint:* Since the y' axis is *not* along the line of impact, apply the conservation of momentum along the x' and y' axes.



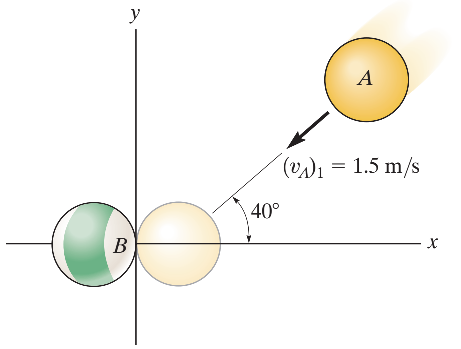
Prob. 15–82

15–83. A ball of mass m is dropped vertically from a height h_0 above the ground. If it rebounds to a height of h_1, determine the coefficient of restitution between the ball and the ground.



Prob. 15–83

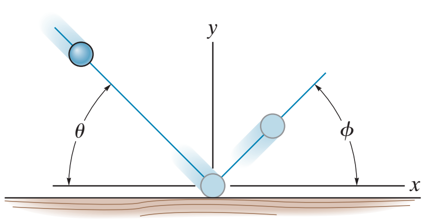
***15–84.** Two smooth billiard balls A and B each have a mass of 200 g. If A strikes B with a velocity $(v_A)_1 = 1.5$ m/s as shown, determine their final velocities just after collision. Ball B is originally at rest and the coefficient of restitution is $e = 0.85$. Neglect the size of each ball.



Prob. 15–84

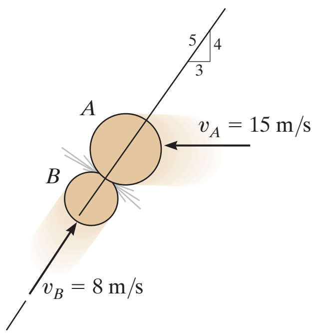
15–85. A ball is thrown onto a rough floor at an angle θ. If it rebounds at an angle ϕ and the coefficient of kinetic friction is μ, determine the coefficient of restitution e. Neglect the size of the ball. *Hint*: Show that during impact, the average impulses in the x and y directions are related by $I_x = \mu I_y$. Since the time of impact is the same, $F_x \Delta t = \mu F_y \Delta t$ or $F_x = \mu F_y$.

15–86. A ball is thrown onto a rough floor at an angle of $\theta = 45°$. If it rebounds at the same angle $\phi = 45°$, determine the coefficient of kinetic friction between the floor and the ball. The coefficient of restitution is $e = 0.6$. *Hint*: Show that during impact, the average impulses in the x and y directions are related by $I_x = \mu I_y$. Since the time of impact is the same, $F_x \Delta t = \mu F_y \Delta t$ or $F_x = \mu F_y$.



Probs. 15–85/86

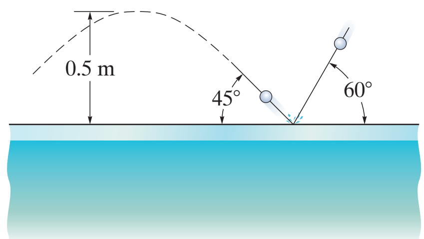
15–87. Two smooth disks A and B have the initial velocities shown just before they collide. If they have masses $m_A = 4$ kg and $m_B = 2$ kg, determine their speeds just after impact. The coefficient of restitution is $e = 0.8$.



Prob. 15–87

***15–88.** It is observed that hailstones strike the surface of a frozen lake at an angle of 60° with the horizontal and rebound at an angle of 45°. Determine the coefficient of restitution.

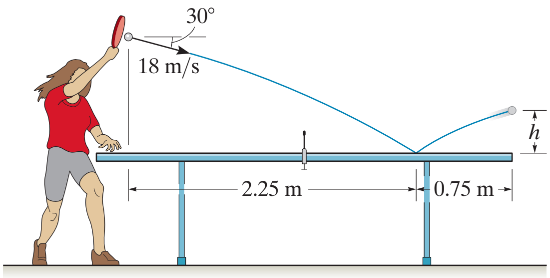
15–89. If the hailstones in Prob. 15–88 rise to a height of 0.5 m after rebounding, determine their velocity just before impact.



Probs. 15–88/89

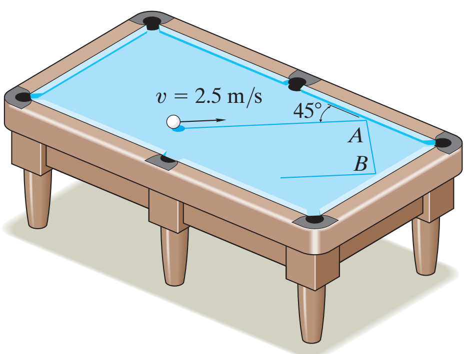
15

15–90. The ping-pong ball has a mass of 2 g. If it is struck with the velocity shown, determine how high h it rises above the end of the smooth table after the rebound. Take $e = 0.8$.



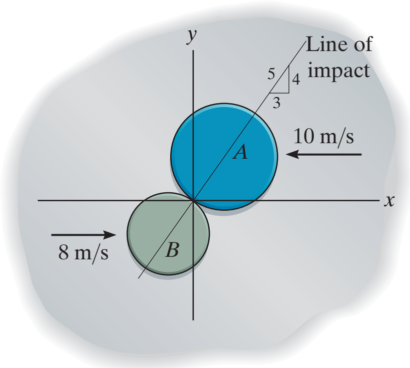
Prob. 15–90

15–91. The 200-g billiard ball is moving with a speed of 2.5 m/s when it strikes the side of the pool table at A. If the coefficient of restitution between the ball and the side of the table is $e = 0.6$, determine the speed of the ball just after striking the table twice, i.e., at A, then at B. Neglect the size of the ball.



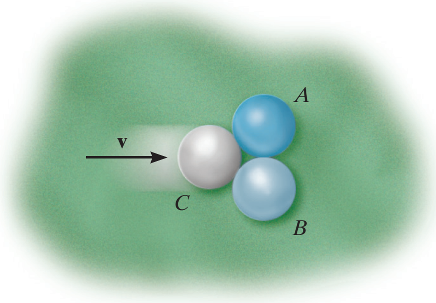
Prob. 15–91

***15–92.** Disks A and B have a mass of 15 kg and 10 kg, respectively. If they are sliding on a smooth horizontal plane with the velocities shown, determine their speeds just after impact. The coefficient of restitution between them is $e = 0.8$.



Prob. 15–92

15–93. The two billiard balls A and B are originally in contact with one another when a third ball C strikes each of them at the same time as shown. If ball C remains at rest after the collision, determine the coefficient of restitution. All the balls have the same mass. Neglect the size of each ball.



Prob. 15–93

15

15.5 ANGULAR MOMENTUM

The ***angular momentum*** of a particle about point O is defined as the "moment" of the particle's linear momentum about O. Since this concept is analogous to finding the moment of a force about a point, the angular momentum, $\mathbf{H}_O$, is sometimes referred to as the *moment of momentum*.

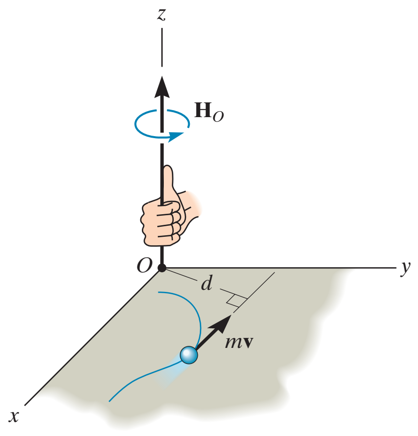


Fig. 15–19

Scalar Formulation.

If a particle moves along a curve lying in the x–y plane, Fig. 15–19, the angular momentum at any instant can be determined about point O (actually the z axis) by using a scalar formulation. The *magnitude* of $\mathbf{H}_O$ is

$$H_O = (d)(mv) \tag{15–13}$$

Here d is the moment arm or perpendicular distance from O to the line of action of $m\mathbf{v}$. Common units for H_O are $\text{kg} \cdot \text{m}^2/\text{s}$. The *direction* of $\mathbf{H}_O$ is defined by the right-hand rule. As shown, the curl of the fingers of the right hand indicates the sense of rotation of $m\mathbf{v}$ about O, so that in this case the thumb (or $\mathbf{H}_O$) is directed perpendicular to the x–y plane along the $+z$ axis.

15

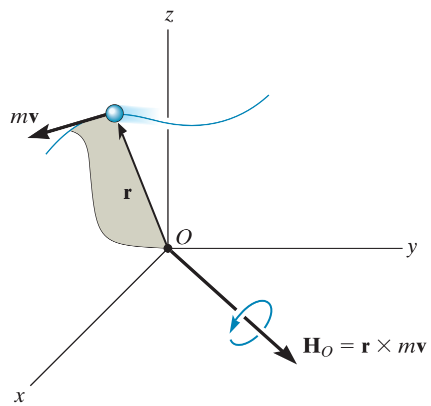


Fig. 15–20

Vector Formulation.

If the particle moves along a space curve, Fig. 15–20, the vector cross product can be used to determine the *angular momentum* about O. In this case

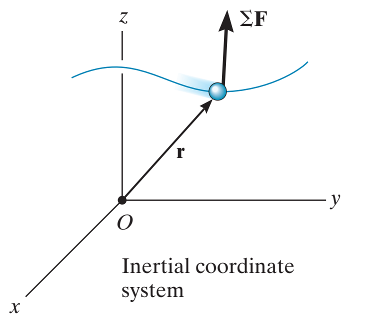
$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} \tag{15–14}$$

Here $\mathbf{r}$ denotes a position vector drawn from point O to the particle. As shown in the figure, $\mathbf{H}_O$ is *perpendicular* to the shaded plane containing $\mathbf{r}$ and $m\mathbf{v}$.

If $\mathbf{r}$ and $m\mathbf{v}$ are expressed in terms of their Cartesian components, then the angular momentum can be determined by evaluating the determinant:

$$\mathbf{H}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ mv_x & mv_y & mv_z \end{vmatrix} \tag{15–15}$$

15.6 RELATION BETWEEN THE MOMENT OF A FORCE AND ANGULAR MOMENTUM



(a)

Fig. 15–21

The moments about point O of all the forces acting on the particle in Fig. 15–21*a* can be related to the particle's angular momentum by applying the equation of motion. Since the mass of the particle is constant, we may write

$$\Sigma \mathbf{F} = m\dot{\mathbf{v}}$$

The moments of the forces about point O can be obtained by performing a cross-product multiplication of each side of this equation by the position vector $\mathbf{r}$, which is directed from O to the particle. We have

$$\Sigma \mathbf{M}_O = \mathbf{r} \times \Sigma \mathbf{F} = \mathbf{r} \times m\dot{\mathbf{v}}$$

From Appendix B, the derivative of $\mathbf{r} \times m\mathbf{v}$ can be written as

$$\dot{\mathbf{H}}_O = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \dot{\mathbf{r}} \times m\mathbf{v} + \mathbf{r} \times m\dot{\mathbf{v}}$$

The first term on the right side, $\dot{\mathbf{r}} \times m\mathbf{v} = m(\dot{\mathbf{r}} \times \dot{\mathbf{r}}) = \mathbf{0}$, since the cross product of a vector with itself is zero. Hence, the above equation becomes

$$\Sigma \mathbf{M}_O = \dot{\mathbf{H}}_O \qquad (15\text{–}16)$$

which states that *the resultant moment about point O of all the forces acting on the particle is equal to the time rate of change of the particle's angular momentum about point O*. This result is similar to Eq. 15–1, i.e.,

$$\Sigma \mathbf{F} = \dot{\mathbf{L}} \qquad (15\text{–}17)$$

Here *the resultant force acting on the particle is equal to the time rate of change of the particle's linear momentum,* $\mathbf{L} = m\mathbf{v}$.

From the derivations, it is seen that Eqs. 15–16 and 15–17 are actually another way of stating Newton's second law of motion. In other sections of this book it will be shown that these equations have many practical applications when extended and applied to problems involving either a system of particles or a rigid body.

System of Particles. An equation having the same form as Eq. 15–16 may be derived for the system of particles shown in Fig. 15–21*b*. The forces acting on the arbitrary *i*th particle of the system consist of a resultant *external force* $\mathbf{F}_i$ and a resultant *internal force* $\mathbf{f}_i$. Expressing the moments of these forces about point O, using the form of Eq. 15–16, we have

$$(\mathbf{r}_i \times \mathbf{F}_i) + (\mathbf{r}_i \times \mathbf{f}_i) = (\dot{\mathbf{H}}_i)_O$$

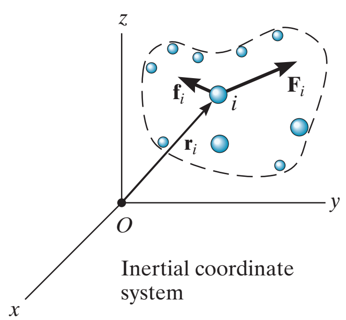
Here $(\dot{\mathbf{H}}_i)_O$ is the time rate of change of the angular momentum of the *i*th particle about O. Similar equations can be written for each of the other particles of the system. When the results are summed vectorially, the result is

$$\Sigma(\mathbf{r}_i \times \mathbf{F}_i) + \Sigma(\mathbf{r}_i \times \mathbf{f}_i) = \Sigma(\dot{\mathbf{H}}_i)_O$$

The second term is zero since the internal forces occur in equal but opposite collinear pairs, and hence the moment of each pair of forces about point O is zero. Dropping the index notation, the above equation can be written in a simplified form as

$$\Sigma\mathbf{M}_O = \dot{\mathbf{H}}_O \qquad (15\text{–}18)$$

which states that *the sum of the moments about point O of all the external forces acting on a system of particles is equal to the time rate of change of the total angular momentum of the system about point O*. Although O has been chosen here as the origin of coordinates, it actually can represent any *fixed point* in the inertial frame of reference.

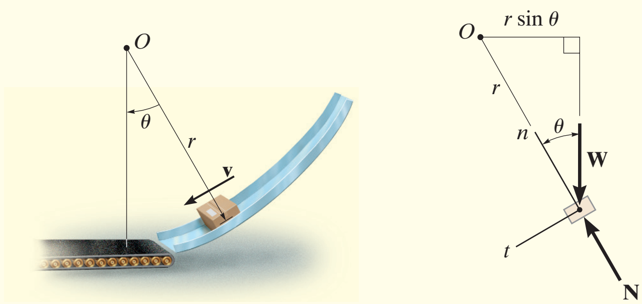


(b)

Fig. 15–21(cont.)

EXAMPLE 15.12

The box shown in Fig. 15–22*a* has a mass m and travels down the smooth circular ramp such that when it is at the angle θ it has a speed v. Determine its angular momentum about point O at this instant and the rate of increase in its speed, i.e., a_t.



(a) (b)

Fig. 15–22

SOLUTION

Since **v** is tangent to the path, applying Eq. 15–13 the angular momentum is

$$H_O = r\,mv \circlearrowright \qquad \textit{Ans.}$$

The rate of increase in its speed (dv/dt) can be found by applying Eq. 15–16. From the free-body diagram of the box, Fig. 15–22*b*, it can be seen that only the weight $W = mg$ contributes a moment about point O. We have

$$\circlearrowright + \Sigma M_O = \dot{H}_O; \qquad mg(r\sin\theta) = \frac{d}{dt}(r\,mv)$$

Since r and m are constant,

$$mgr\sin\theta = r\,m\frac{dv}{dt}$$

$$\frac{dv}{dt} = g\sin\theta \qquad \textit{Ans.}$$

NOTE: This same result can, of course, be obtained from the equation of motion applied in the tangential direction, Fig. 15–22*b*, i.e.,

$$+\swarrow \Sigma F_t = ma_t; \qquad mg\sin\theta = m\left(\frac{dv}{dt}\right)$$

$$\frac{dv}{dt} = g\sin\theta \qquad \textit{Ans.}$$

15

15.7 PRINCIPLE OF ANGULAR IMPULSE AND MOMENTUM

If Eq. 15–16 is rewritten in the form $\Sigma \mathbf{M}_O\,dt = d\mathbf{H}_O$ and integrated, assuming that at time $t = t_1$, $\mathbf{H}_O = (\mathbf{H}_O)_1$ and at time $t = t_2$, $\mathbf{H}_O = (\mathbf{H}_O)_2$, we have

$$\Sigma \int_{t_1}^{t_2} \mathbf{M}_O\,dt = (\mathbf{H}_O)_2 - (\mathbf{H}_O)_1$$

or

$$(\mathbf{H}_O)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{M}_O\,dt = (\mathbf{H}_O)_2 \qquad (15\text{–}19)$$

This equation is referred to as the ***principle of angular impulse and momentum***. The initial and final angular momenta $(\mathbf{H}_O)_1$ and $(\mathbf{H}_O)_2$ are defined as the moment of the linear momentum of the particle $(\mathbf{H}_O = \mathbf{r} \times m\mathbf{v})$ at the instants t_1 and t_2, respectively. The second term on the left side, $\Sigma \int \mathbf{M}_O\,dt$, is called the ***angular impulse***. It is determined by integrating, with respect to time, the moments of all the forces acting on the particle over the time period t_1 to t_2. Since the moment of a force about point O is $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$, the angular impulse may be expressed in vector form as

$$\text{angular impulse} = \int_{t_1}^{t_2} \mathbf{M}_O\,dt = \int_{t_1}^{t_2} (\mathbf{r} \times \mathbf{F})\,dt \qquad (15\text{–}20)$$

Here $\mathbf{r}$ is a position vector which extends from point O to any point on the line of action of $\mathbf{F}$.

In a similar manner, using Eq. 15–19, the principle of angular impulse and momentum for a system of particles may be written as

$$\Sigma(\mathbf{H}_O)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{M}_O\,dt = \Sigma(\mathbf{H}_O)_2 \qquad (15\text{–}21)$$

Here the first and third terms represent the angular momenta of all the particles $[\Sigma \mathbf{H}_O = \Sigma(\mathbf{r}_i \times m\mathbf{v}_i)]$ at the instants t_1 and t_2. The second term is the sum of the angular impulses given to all the particles from t_1 to t_2. Recall that these impulses are created only by the moments of the external forces acting on the system where, for the ith particle, $\mathbf{M}_O = \mathbf{r}_i \times \mathbf{F}_i$.

Vector Formulation. Using impulse and momentum principles, it is therefore possible to write two vector equations which define the particle's motion, namely, Eq. 15–3 and Eq. 15–19, restated as

$$m\mathbf{v}_1 + \Sigma \int_{t_1}^{t_2} \mathbf{F}\,dt = m\mathbf{v}_2$$
$$(\mathbf{H}_O)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{M}_O\,dt = (\mathbf{H}_O)_2 \qquad (15\text{–}22)$$

Scalar Formulation. In general, the above equations can be expressed in x, y, z component form. If the particle is confined to move in the x–y plane, then three scalar equations can be written to describe the motion, namely,

$$m(v_x)_1 + \Sigma \int_{t_1}^{t_2} F_x\,dt = m(v_x)_2$$
$$m(v_y)_1 + \Sigma \int_{t_1}^{t_2} F_y\,dt = m(v_y)_2 \qquad (15\text{–}23)$$
$$(H_O)_1 + \Sigma \int_{t_1}^{t_2} M_O\,dt = (H_O)_2$$

The first two of these equations represent the principle of linear impulse and momentum in the x and y directions, which has been discussed in Sec. 15.1, and the third equation represents the principle of angular impulse and momentum about the z axis.

15

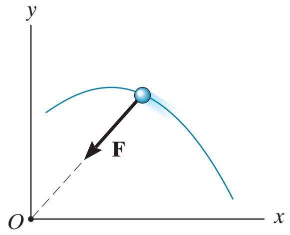


Fig. 15–23

Conservation of Angular Momentum. When the angular impulses acting on a particle are all zero during the time t_1 to t_2, Eq. 15–19 reduces to the following simplified form:

$$(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2 \tag{15–24}$$

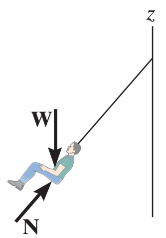
This equation is known as the ***conservation of angular momentum***. It states that from t_1 to t_2 the particle's angular momentum remains constant. Obviously, if no external impulse is applied to the particle, both linear and angular momentum will be conserved. In some cases, however, the particle's angular momentum will be conserved and linear momentum may not. An example of this occurs when the particle is subjected *only* to a *central force* (see Sec. 13.7). As shown in Fig. 15–23, the impulsive central force **F** is always directed toward point O as the particle moves along the path. Hence, the angular impulse (moment) created by **F** about the z axis is always zero, and therefore angular momentum of the particle is conserved about this axis.

From Eq. 15–21, we can also write the conservation of angular momentum for a system of particles as

$$\Sigma(\mathbf{H}_O)_1 = \Sigma(\mathbf{H}_O)_2 \tag{15–25}$$

In this case the summation must include the angular momenta of all particles in the system.

15



Provided air resistance is neglected, the passengers on this amusement-park ride are subjected to a conservation of angular momentum about the z axis of rotation. As shown on the free-body diagram, the line of action of the normal force **N** of the seat on the passenger passes through this axis, and the passenger's weight **W** is parallel to it. Thus, no angular impulse acts around the z axis.

PROCEDURE FOR ANALYSIS

When applying the principles of angular impulse and momentum, or the conservation of angular momentum, it is suggested that the following procedure be used.

Free-Body Diagram.

- Draw the particle's free-body diagram in order to determine any axis about which angular momentum may be conserved. For this to occur, the moments of all the forces (or impulses) must either be parallel or pass through the axis so as to create zero moment throughout the time period t_1 to t_2.
- The direction and sense of the particle's initial and final velocities should also be established.

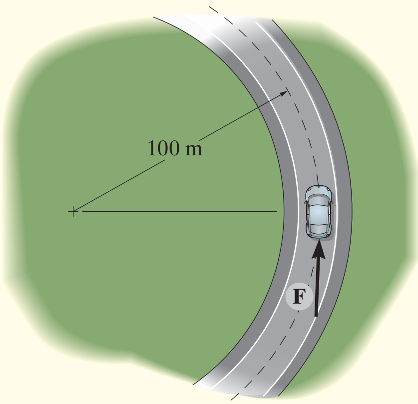
Momentum Equations.

- Apply the principle of angular impulse and momentum, $(\mathbf{H}_O)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{M}_O dt = (\mathbf{H}_O)_2$, or if appropriate, the conservation of angular momentum, $(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2$.

Refer to the companion website for Lecture Summary and Quiz videos.

EXAMPLE 15.13

The 1.5-Mg car travels along the circular road as shown in Fig. 15–24*a*. If the traction force of the wheels on the road is $F = (150t^2)$ N, where t is in seconds, determine the speed of the car when $t = 5$ s. The car initially travels with a speed of 5 m/s. Neglect the size of the car.



(a)

Free-Body Diagram. The free-body diagram of the car is shown in Fig. 15–24*b*. If we apply the principle of angular impulse and momentum about the z axis, then the angular impulse created by the weight, normal force, and radial frictional force will be eliminated since they act parallel to the axis or pass through it.

Principle of Angular Impulse and Momentum.

$$(H_z)_1 + \Sigma \int_{t_1}^{t_2} M_z \, dt = (H_z)_2$$

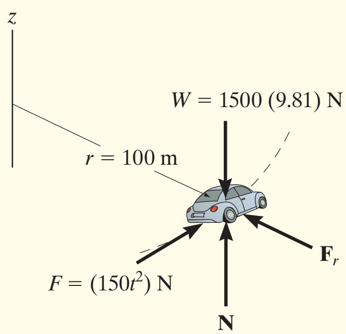
$$r\, m_c(v_c)_1 + \int_{t_1}^{t_2} r\, F\, dt = r\, m_c(v_c)_2$$

$$(100 \text{ m})(1500 \text{ kg})(5 \text{ m/s}) + \int_0^{5\text{ s}} (100 \text{ m})[(150t^2) \text{ N}]\, dt$$

$$= (100 \text{ m})(1500 \text{ kg})(v_c)_2$$

$$750(10^3) + 5000t^3 \Big|_0^{5\text{ s}} = 150(10^3)(v_c)_2$$

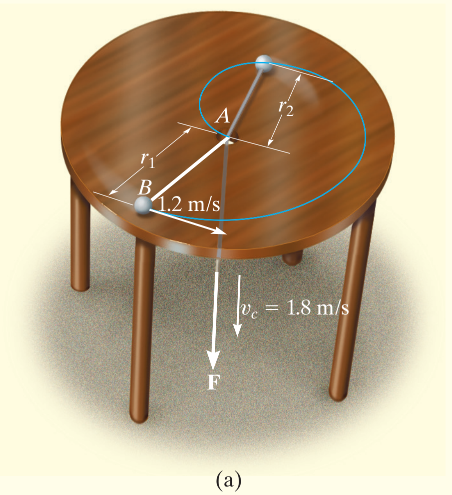
$$(v_c)_2 = 9.17 \text{ m/s} \qquad \textit{Ans.}$$



(b)

Fig. 15–24

EXAMPLE 15.14



(a)

The 0.36-kg ball B, shown in Fig. 15–25a, is attached to a cord which passes through a hole at A in a smooth table. When the ball is $r_1 = 0.525$ m from the hole, it is rotating around in a circle such that its speed is $v_1 = 1.2$ m/s. By applying the force **F** the cord is pulled downward through the hole with a constant speed $v_c = 1.8$ m/s. Determine (a) the speed of the ball at the instant it is $r_2 = 0.18$ m from the hole, and (b) the amount of work done by **F** in shortening the radial distance from r_1 to r_2. Neglect the size of the ball.

SOLUTION

Part (a) Free-Body Diagram. As the ball moves from r_1 to r_2, Fig. 15–25b, the cord force **F** on the ball always passes through the z axis, and the weight and $\mathbf{N}_B$ are parallel to it. Hence the moments, or angular impulses created by these forces, are all *zero* about this axis. Therefore, angular momentum is conserved about the z axis.

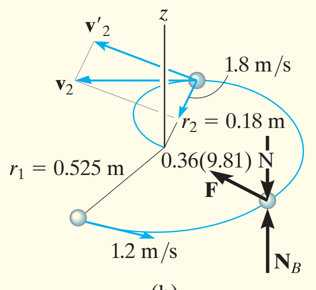
Conservation of Angular Momentum. The ball's velocity $\mathbf{v}_2$ is resolved into two components. The radial component, 1.8 m/s, is known; however, it produces zero angular momentum about the z axis. Thus,

$$\mathbf{H}_1 = \mathbf{H}_2$$

$$r_1 m_B v_1 = r_2 m_B v_2'$$

$$0.525\text{ m}(0.36\text{ kg})\,(1.2\text{ m/s}) = 0.18\text{ m}(0.36\text{ kg})v_2'$$

$$v_2' = 3.50\text{ m/s}$$



(b)

Fig. 15–25

The speed of the ball is thus

$$v_2 = \sqrt{(3.50\text{ m/s})^2 + (1.8\text{ m/s})^2}$$

$$= 3.94\text{ m/s} \qquad \textit{Ans.}$$

Part (b). The only force that does work on the ball is **F**. (The normal force and weight do not move vertically.) The initial and final kinetic energies of the ball can be determined so that from the principle of work and energy we have

$$T_1 + \Sigma U_{1-2} = T_2$$

$$\frac{1}{2}(0.36\text{ kg})(1.2\text{ m/s})^2 + U_F = \frac{1}{2}(0.36\text{ kg})(3.94\text{ m/s})^2$$

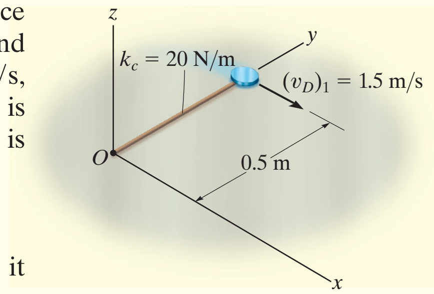
$$U_F = 2.53\text{ J} \qquad \textit{Ans.}$$

NOTE: The force F is not constant because the normal component of acceleration, $a_n = v^2/r$, changes as r changes.

15

EXAMPLE 15.15

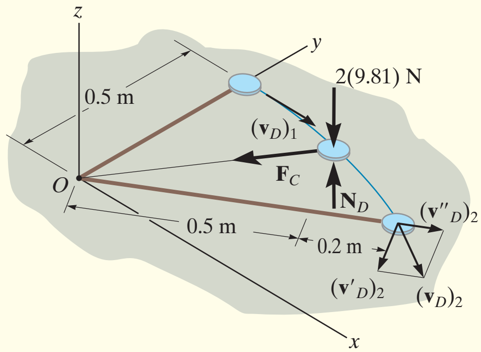
The 2-kg disk shown in Fig. 15–26*a* rests on a smooth horizontal surface and is attached to an elastic cord that has a stiffness $k_c = 20$ N/m and is initially unstretched. If the disk is given a velocity $(v_D)_1 = 1.5$ m/s, perpendicular to the cord, determine the rate at which the cord is being stretched and the speed of the disk at the instant the cord is stretched 0.2 m.



(a)

SOLUTION

Free-Body Diagram. After the disk has been launched, it slides along the path shown in Fig. 15–26*b*. By inspection, angular momentum about point O (or the z axis) is *conserved*, since none of the forces produce an angular impulse about this axis. Also, when the distance is 0.7 m, only the transverse component $(\mathbf{v}'_D)_2$ produces angular momentum of the disk about O.



(b)

Fig. 15–26

Conservation of Angular Momentum. To obtain $(\mathbf{v}'_D)_2$ we have

$$(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2$$

$$r_1 m_D (v_D)_1 = r_2 m_D (v'_D)_2$$

$$0.5 \text{ m } (2 \text{ kg})(1.5 \text{ m/s}) = 0.7 \text{ m}(2 \text{ kg})(v'_D)_2$$

$$(v'_D)_2 = 1.071 \text{ m/s}$$

Conservation of Energy. The speed of the disk can be obtained by applying the conservation of energy equation at the point where the disk was launched and at the point where the cord is stretched 0.2 m.

$$T_1 + V_1 = T_2 + V_2$$

$$\tfrac{1}{2}m_D(v_D)_1^2 + \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}m_D(v_D)_2^2 + \tfrac{1}{2}kx_2^2$$

$$\tfrac{1}{2}(2 \text{ kg})(1.5 \text{ m/s})^2 + 0 = \tfrac{1}{2}(2 \text{ kg})(v_D)_2^2 + \tfrac{1}{2}(20 \text{ N/m})(0.2 \text{ m})^2$$

$$(v_D)_2 = 1.360 \text{ m/s} = 1.36 \text{ m/s} \qquad \textit{Ans.}$$

Having determined $(v_D)_2$ and its component $(v'_D)_2$, the rate of stretch of the cord, or radial component, $(v''_D)_2$ is determined from the Pythagorean theorem,

$$(v''_D)_2 = \sqrt{(v_D)_2^2 - (v'_D)_2^2}$$

$$= \sqrt{(1.360 \text{ m/s})^2 - (1.071 \text{ m/s})^2}$$

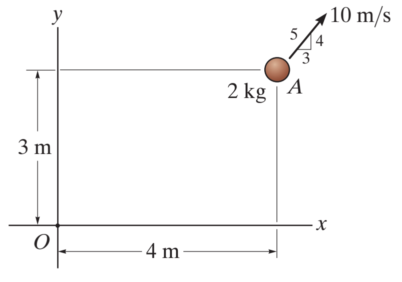
$$= 0.838 \text{ m/s} \qquad \textit{Ans.}$$

Refer to the companion website for a self quiz of these Example problems.

15

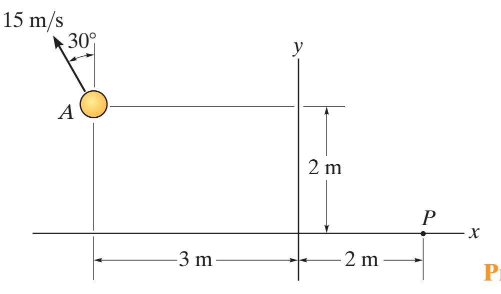
FUNDAMENTAL PROBLEMS

F15–19. The 2-kg particle A has the velocity shown. Determine its angular momentum $\mathbf{H}_O$ about point O.



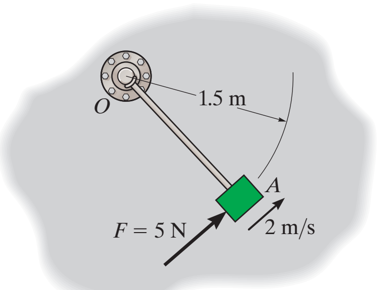
Prob. F15–19

F15–20. The 2-kg particle A has the velocity shown. Determine its angular momentum $\mathbf{H}_P$ about point P.



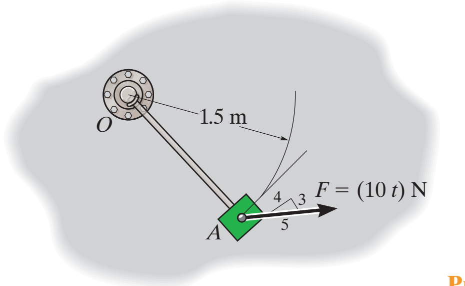
Prob. F15–20

F15–21. Initially the 5-kg block is moving with a constant speed of 2 m/s around the circular path centered at O on the smooth horizontal plane. If a constant tangential force $F = 5$ N is applied to the block, determine the block's speed when $t = 3$ s. Neglect the size of the block and the mass of the rod.



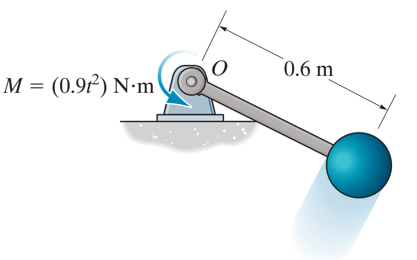
Prob. F15–21

F15–22. The 5-kg block is moving around the circular path centered at O on the smooth horizontal plane when it is subjected to the force $F = (10t)$ N, where t is in seconds. If the block starts from rest, determine its speed when $t = 4$ s. Neglect the size of the block and the mass of the rod. The force maintains the same constant angle tangent to the path.



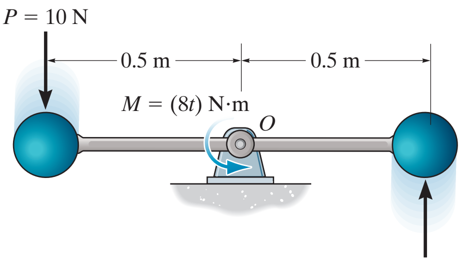
Prob. F15–22

F15–23. The 2-kg sphere is attached to the rod, which rotates in the *horizontal plane* centered at O. If the system is subjected to a couple moment $M = (0.9t^2)$ N·m, where t is in seconds, determine the speed of the sphere at the instant $t = 5$ s starting from rest. Neglect the mass of the rod.



Prob. F15–23

F15–24. Two identical 10-kg spheres are attached to the rod, which rotates in the horizontal plane centered at pin O. If the spheres are subjected to tangential forces of $P = 10$ N, and the rod is subjected to a couple moment $M = (8t)$ N·m, where t is in seconds, determine the speed of the spheres at the instant $t = 4$ s. The system starts from rest. Neglect the size of the spheres and the mass of the rod.



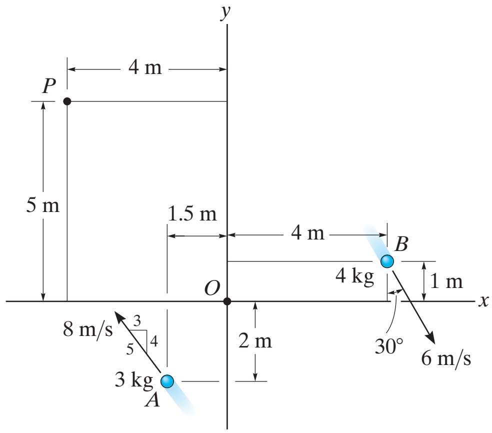
Prob. F15–24

PROBLEMS

All solutions, where necessary, must include a free-body diagram.

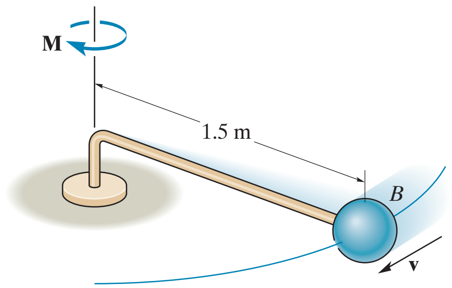
15–94. Determine the angular momentum $\mathbf{H}_O$ of each of the two particles about point O.

15–95. Determine the angular momentum $\mathbf{H}_P$ of each of the two particles about point P.



Probs. 15–94/95

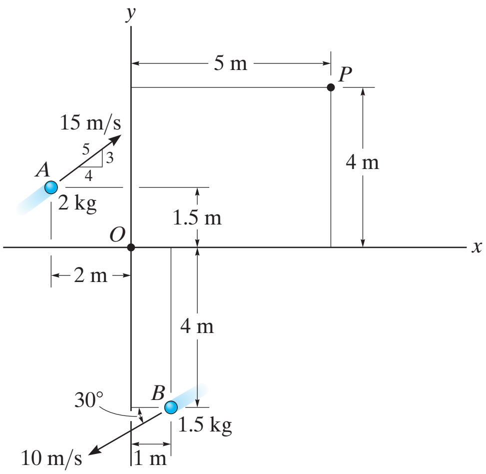
15–98. The ball B has a mass of 10 kg and is attached to the end of a rod whose mass may be neglected. If the rod is subjected to a torque $M = (3t^2 + 5t + 2)$ N·m, where t is in seconds, determine the speed of the ball when $t = 2$ s. The ball has a speed $v = 2$ m/s when $t = 0$.



Prob. 15–98

***15–96.** Determine the angular momentum $\mathbf{H}_O$ of each of the two particles about point O. Use a scalar solution.

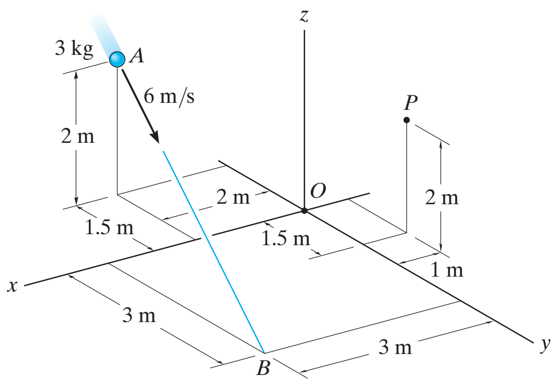
15–97. Determine the angular momentum $\mathbf{H}_P$ of each of the two particles about point P. Use a scalar solution.



Probs. 15–96/97

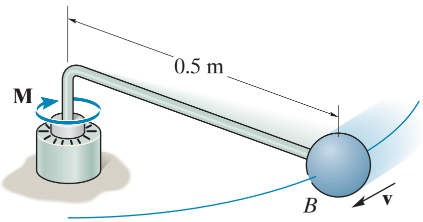
15–99. Determine the angular momentum $\mathbf{H}_O$ of the 3-kg particle about point O.

***15–100.** Determine the angular momentum $\mathbf{H}_P$ of the 3-kg particle about point P.



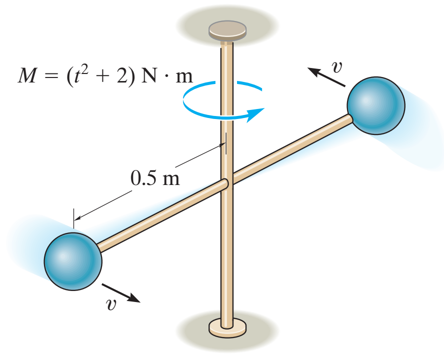
Probs. 15–99/100

15–101. The ball B has a mass of 10 kg and is attached to the end of a rod whose mass can be neglected. If the shaft is subjected to a torque $M = (2t^2 + 4)\ \text{N}\cdot\text{m}$, where t is in seconds, determine the speed of the ball when $t = 2$ s. The ball has a speed $v = 2$ m/s when $t = 0$.



Prob. 15–101

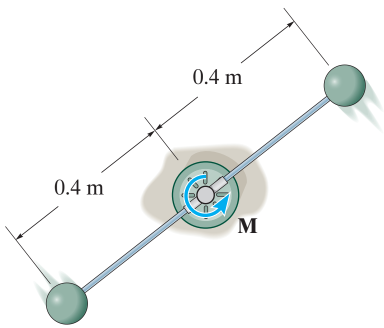
***15–104.** Each ball has a negligible size and a mass of 10 kg and is attached to the end of a rod whose mass may be neglected. If the rod is subjected to a torque $M = (t^2 + 2)\ \text{N}\cdot\text{m}$, where t is in seconds, determine the speed of each ball when $t = 3$ s. Each ball has a speed $v = 2$ m/s when $t = 0$.



Prob. 15–104

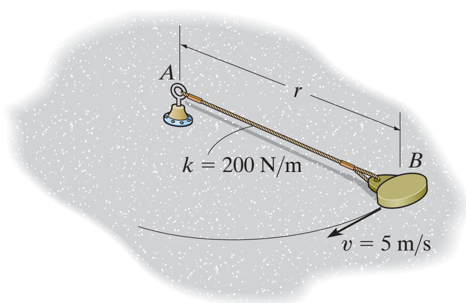
15–102. The two spheres each have a mass of 3 kg and are attached to the rod of negligible mass. If a torque $M = (6e^{0.2t})\ \text{N}\cdot\text{m}$, where t is in seconds, is applied to the rod as shown, determine the speed of each of the spheres in 2 s, starting from rest.

15–103. The two spheres each have a mass of 3 kg and are attached to the rod of negligible mass. Determine the time the torque $M = (8t)\ \text{N}\cdot\text{m}$, where t is in seconds, must be applied to the rod so that each sphere attains a speed of 3 m/s starting from rest.



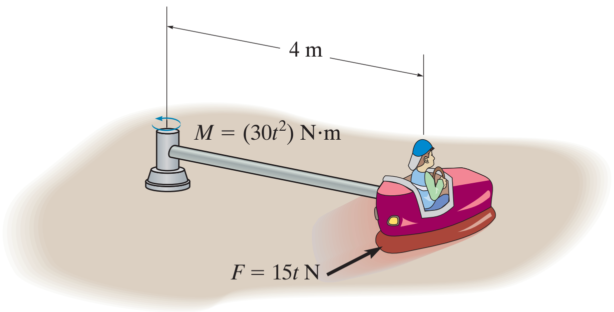
Probs. 15–102/103

15–105. At the instant $r = 1.5$ m, the 5-kg disk is given a speed of $v = 5$ m/s, perpendicular to the elastic cord. Determine the speed of the disk and the rate of shortening of the elastic cord at the instant $r = 1.2$ m. The disk slides on the smooth horizontal plane. Neglect its size. The cord has an unstretched length of 0.5 m.



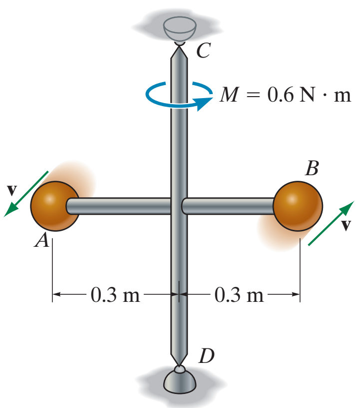
Prob. 15–105

15–106. If the rod of negligible mass is subjected to a couple moment of $M = (30t^2)$ N·m, and the engine of the car supplies a traction force of $F = (15t)$ N to the wheels, where t is in seconds, determine the speed of the car at the instant $t = 5$ s. The car starts from rest. The total mass of the car and rider is 150 kg. Neglect the size of the car.



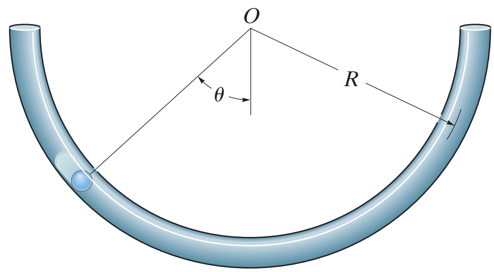
Prob. 15–106

15–107. The two spheres A and B each have a mass of 400 g. The spheres are fixed to the horizontal rods and their initial velocity is 2 m/s. If a couple moment of $M = 0.6$ N·m is applied to the frame, determine the speed of the spheres in 3 s. The mass of the supporting frame is negligible and it is free to rotate. Neglect the size of the spheres.



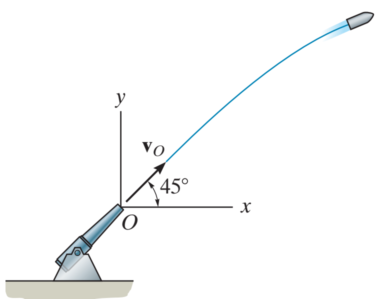
Prob. 15–107

***15–108.** A small particle having a mass m is placed inside the semicircular tube. The particle is placed at the position shown and released. Apply the principle of angular momentum about point O ($\Sigma M_O = \dot{H}_O$), and show that the motion of the particle is governed by the differential equation $\ddot{\theta} + (g/R)\sin\theta = 0$.



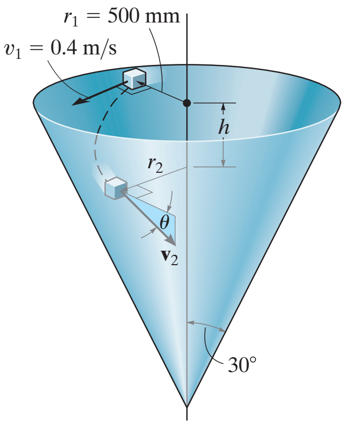
Prob. 15–108

15–109. The projectile having a mass of 3 kg is fired from a cannon with a muzzle velocity of $v_0 = 500$ m/s. Determine the projectile's angular momentum about point O at the instant it is at the maximum height of its trajectory.



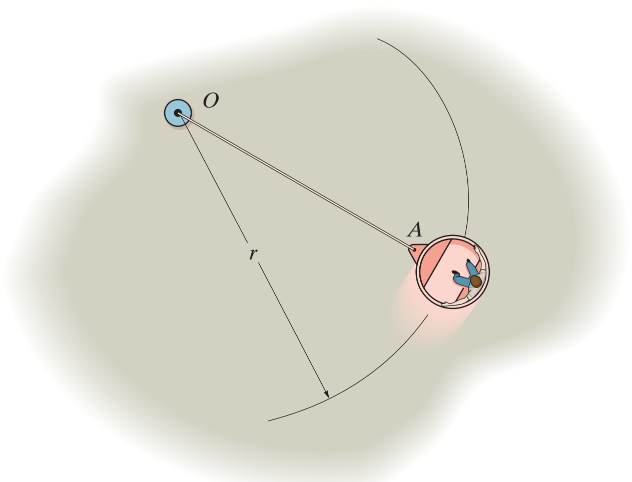
Prob. 15–109

15–110. A small block having a mass of 0.1 kg is given a horizontal velocity of $v_1 = 0.4$ m/s when $r_1 = 500$ mm. It slides along the smooth conical surface. Determine the distance h it must descend to reach a speed of $v_2 = 2$ m/s. Also, what is the angle of descent θ, that is, the angle measured from the horizontal to the tangent of the path?



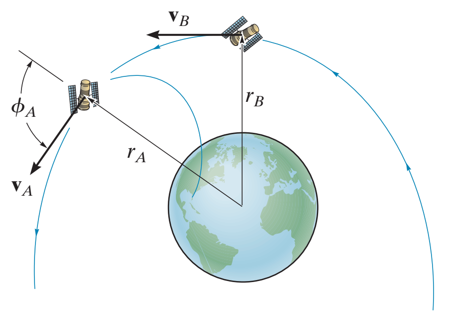
Prob. 15–110

15–111. The amusement park ride consists of a 200-kg car and passenger that are traveling at 3 m/s along a circular path having a radius of 8 m. If at $t = 0$, the cable OA is pulled in toward O at 0.5 m/s, determine the speed of the car when $t = 4$ s. Also, determine the work done to pull in the cable.



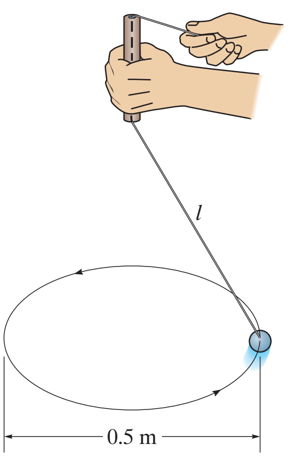
Prob. 15–111

***15–112.** An earth satellite of mass 700 kg is launched into a free-flight trajectory about the earth with an initial speed of $v_A = 10$ km/s when the distance from the center of the earth is $r_A = 15$ Mm. If the launch angle at this position is $\phi_A = 70°$, determine the speed v_B of the satellite and its closest distance r_B from the center of the earth. The earth has a mass $M_e = 5.976(10^{24})$ kg. *Hint*: Under these conditions, the satellite is subjected only to the earth's gravitational force, $F = GM_e m_s/r^2$, Eq. 13–1. For part of the solution, use the conservation of energy.



Prob. 15–112

15–113. The 2-kg ball rotates around a 0.5-m-diameter circular path with a constant speed. If the cord length is shortened from $l = 1$ m to $l' = 0.5$ m, by pulling the cord through the tube, determine the new diameter of the path d'. Also, what is the tension in the cord in each case?

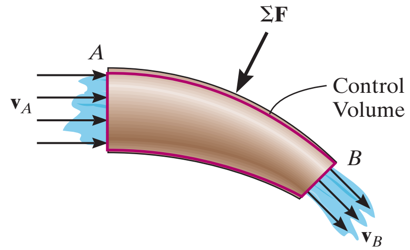


Prob. 15–113

15.8 BODIES SUBJECTED TO A MASS FLOW

In the previous sections we have considered application of impulse and momentum principles to a system of particles that compose a rigid body, having a closed boundary. In this section we will apply the principle of impulse and momentum to a system of particles that can flow either into or out of volume having an open boundary.

Normally the mass flow for these cases is analyzed using a control volume approach, and is discussed in the subject of fluid mechanics.* Here, we will use this method and consider two specific applications that have widespread application. Before we do this however, we will first derive a general equation that applies to both cases. The derivation can be developed by considering the flow of a system of particles, such as a fluid, that pass through a pipe, Fig. 15–27*a*. As shown in Fig. 15–27*b*, the initial momentum of the particles is $(m\mathbf{v})_1$. During the time Δt, impulsive forces $\Sigma\mathbf{F}$, such as caused by the wall of the pipe, or the pressure at the entrance and exit, act on the moving particles and change their velocity in both magnitude and direction, so the final momentum of the system is $(m\mathbf{v})_2$. If we apply the impulse and momentum principle to the particles, we have



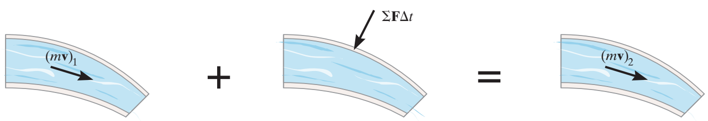
(a)

Fig. 15–27

$$(m\mathbf{v})_1 + \Sigma\mathbf{F}\,\Delta t = (m\mathbf{v})_2$$

$$\Sigma\mathbf{F}\,\Delta t = \Delta m\mathbf{v} \qquad (15\text{–}26)$$

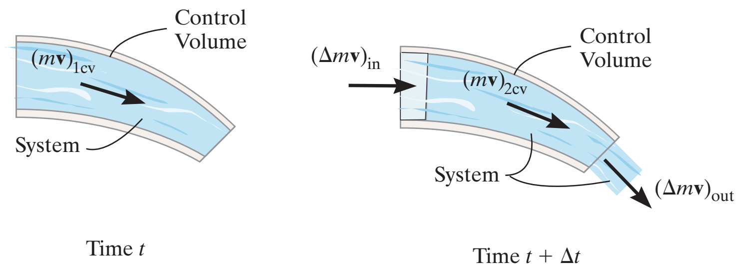
Finding this time rate of change in momentum, $\Delta m\mathbf{v}$, for all the particles is rather difficult in this case because of their random motion within the system. A better way to determine $\Delta m\mathbf{v}$ is to consider this time rate of change relative to a specific region of space, called a ***control volume***, Fig. 15–27*a*. For example, if the particles compose a fluid, then the control volume may define the size and shape of the solid boundaries and openings of a conduit or channel, a turbine, or a pump. A rocket or jet engine can also be selected as a control volume, and in this case we can consider it to be moving.



(b)

Fig. 15–27

* See *Fluid Mechanics*, R. C. Hibbeler, Pearson Education.



(c)

Fig. 15–27

For our purposes we will take the control volume to be the region in the pipe shown in Fig. 15–27*c*. At time t all the particles in the *system* are contained within the control volume, and so the total momentum of the system is $(m\mathbf{v})_1 = (m\mathbf{v})_{1\,\text{cv}}$. But at $t + \Delta t$, some of the particles of the system will flow out of the control volume, while other particles, not part of the system, will flow into it. The momenta of these particles are $(\Delta m\mathbf{v})_{\text{out}}$ and $(\Delta m\mathbf{v})_{\text{in}}$, respectively. Within the control volume the remaining momentum of the system's particles has changed to $(m\mathbf{v})_{2\text{cv}}$. As a result the change in momentum becomes

$$\Delta m\mathbf{v} = (\Delta m\mathbf{v})_{\text{cv}} + (\Delta m\mathbf{v})_{\text{out}} - (\Delta m\mathbf{v})_{\text{in}}$$

To summarize, the change of momentum of the system as seen relative to the control volume consists of two parts, namely, the change that occurs *within* the control volume, $(\Delta m\mathbf{v})_{\text{cv}} = (m\mathbf{v})_{2\text{cv}} - (m\mathbf{v})_{1\text{cv}}$, and the *net change* in the momentum of the system that is flowing out of the control volume, $(\Delta m\mathbf{v})_{\text{out}} - (\Delta m\mathbf{v})_{\text{in}}$.

If we substitute the above equation into Eq. 15–26 and divide by Δt, allowing $\Delta t \rightarrow 0$, we get the result

$$\Sigma\mathbf{F} = \left(\frac{d}{dt}(m\mathbf{v})\right)_{\text{cv}} + \dot{m}_{\text{out}}\mathbf{v}_{\text{out}} - \dot{m}_{\text{in}}\mathbf{v}_{\text{in}} \tag{15–27}$$

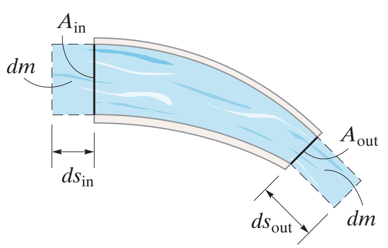


Fig. 15–28

The term $\dot{m} = dm/dt$ is called the ***mass flow***. It indicates the number of particles that flow into or out of the control volume per unit time. If the particles compose a fluid, and the open control surfaces have cross-sectional areas A_{in} and A_{out}, Fig. 15–28, then the conservation of mass through these surfaces requires $dm = \rho\, dV = \rho_{\text{in}}(ds_{\text{in}}A_{\text{in}}) = \rho_{\text{out}}(ds_{\text{out}}A_{\text{out}})$, where ρ is the density of the system (fluid) and $dV = ds\,A$ is an element of volume, entering or leaving a control surface. Hence, during the time dt, since $v_{\text{in}} = ds_{\text{in}}/dt$ and $v_{\text{out}} = ds_{\text{out}}/dt$, we have $\dot{m} = \rho_{\text{in}}v_{\text{in}}A_{\text{in}} = \rho_{\text{out}}v_{\text{out}}A_{\text{out}}$, or in general

$$\dot{m} = \rho v A = \rho Q \tag{15–28}$$

The term $Q = vA$ measures the volume of fluid flow per unit time, and is referred to as the ***volumetric flow***. In the next two sections we will consider two important applications of Eq. 15–27.

15.9 STEADY FLOW OF A FLUID STREAM

Some problems in engineering require finding the forces acting on a fluid stream in order to divert it as the fluid passes through a conduit or over a blade. When this is the case, provided we consider the fluid to be *incompressible* and the *flow* to be *steady*, that is, independent of time, then no change in momentum will occur within the control volume. As a result, the first term on the right of Eq. 15–27 will be equal to zero. Also, because of the conservation of mass, $\dot{m} = \dot{m}_{\text{in}} = \dot{m}_{\text{out}}$. Therefore, our result becomes

$$\Sigma \mathbf{F} = \dot{m}(\mathbf{v}_{\text{out}} - \mathbf{v}_{\text{in}}) \qquad (15\text{–}29)$$

The following procedure can be used whenever this equation is applied to determine the force acting on a device, such as a pipe or blade, that diverts a fluid stream having steady flow.

The conveyor belt must supply frictional forces to the gravel that falls upon it in order to change the momentum of the gravel stream, so that it begins to travel along the belt.

The air on one side of this fan is essentially at rest, and as it passes through the blades its momentum is increased. To change the momentum of the air flow in this manner, the blades must exert a horizontal thrust on the air stream. As the blades turn faster, the equal but opposite thrust of the air on the blades could overcome the rolling resistance of the wheels on the ground and begin to move the frame of the fan.

15

PROCEDURE FOR ANALYSIS

Control Volume. Select the control volume so its free-body diagram includes the unknown forces that are to be determined. This volume can contain both solid and fluid parts.

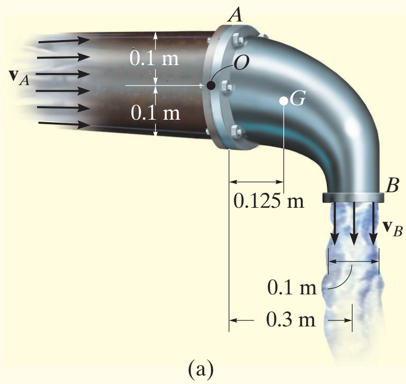
Free-Body Diagram. Draw the free-body diagram of the control volume in order to identify the forces $\Sigma \mathbf{F}$ that act on it. These forces may include the support reactions, the weight of the control volume, and the static gage pressure forces acting on the fluid at the open control surfaces, or on a closed surface or boundary.* This ***gage pressure*** is the pressure measured above or below atmospheric pressure. As a result, if an opening is exposed to the atmosphere, the gage pressure will be zero.

Momentum Equation. The mass flow is determined from Eq. 15–28. Establish an inertial x, y coordinate system and apply Eq. 15–29 along the axes using the components of velocity at the open control surfaces, and the components of the forces shown on the free-body diagram. If the control volume is moving with a velocity, then apply the equation *relative to the control volume* so that the steady flow will then occur relative to the open control surfaces.

* In the SI system, a unit of pressure is measured using the pascal (Pa), where $1 \text{ Pa} = 1 \text{ N/m}^2$.

EXAMPLE 15.16

Determine the components of force reaction which the fixed pipe joint at A exerts on the elbow in Fig. 15–29a, if water flowing through the pipe is subjected to a static gage pressure of 100 kPa at A. The volumetric flow at B is $Q_B = 0.2\ \text{m}^3/\text{s}$. Water has a density $\rho_w = 1000\ \text{kg/m}^3$, and the water-filled elbow has a mass of 20 kg and center of mass at G.



(a)

SOLUTION

We will consider the control volume to be the elbow and the water within it. Using a *fixed* inertial coordinate system, the velocity of flow at A and B and the mass flow can be obtained from Eq. 15–28. Since the density of water is constant, $Q_B = Q_A = Q$. Hence,

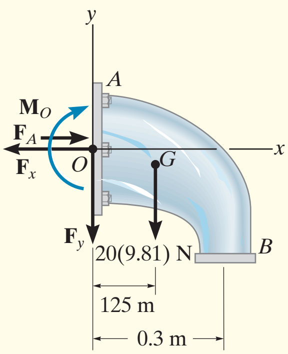
$$\dot{m} = \rho_w Q = (1000\ \text{kg/m}^3)(0.2\ \text{m}^3/\text{s}) = 200\ \text{kg/s}$$

$$v_B = \frac{Q}{A_B} = \frac{0.2\ \text{m}^3/\text{s}}{\pi(0.05\ \text{m})^2} = 25.46\ \text{m/s} \downarrow$$

$$v_A = \frac{Q}{A_A} = \frac{0.2\ \text{m}^3/\text{s}}{\pi(0.1\ \text{m})^2} = 6.37\ \text{m/s} \rightarrow$$

Free-Body Diagram. As shown on the free-body diagram of the control volume (elbow) Fig. 15–29b, the *fixed* connection at A exerts a resultant couple moment $\mathbf{M}_O$ and force components $\mathbf{F}_x$ and $\mathbf{F}_y$ on the elbow. Due to the static pressure of water in the pipe, the pressure force acting on the open control surface at A is $F_A = p_A A_A$. Since $1\ \text{kPa} = 1000\ \text{N/m}^2$,

$$F_A = p_A A_A = [100(10^3)\ \text{N/m}^2][\pi(0.1\ \text{m})^2] = 3141.6\ \text{N}$$



(b)

Fig. 15–29

There is no static pressure acting at B, since the water is discharged at atmospheric pressure; i.e., the pressure measured by a gage at B is equal to zero, $p_B = 0$.

Equations of Steady Flow.

$$\overset{+}{\rightarrow} \Sigma F_x = \dot{m}(v_{Bx} - v_{Ax});\ -F_x + 3141.6\ \text{N} = 200\ \text{kg/s}(0 - 6.37\ \text{m/s})$$

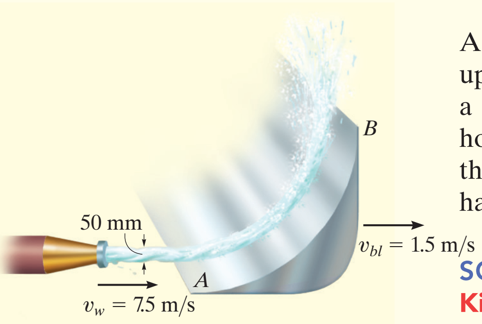
$$F_x = 4.41\ \text{kN} \qquad \textit{Ans.}$$

$$+\uparrow \Sigma F_y = \dot{m}(v_{By} - v_{Ay});\ -F_y - 20(9.81)\ \text{N} = 200\ \text{kg/s}(-25.46\ \text{m/s} - 0)$$

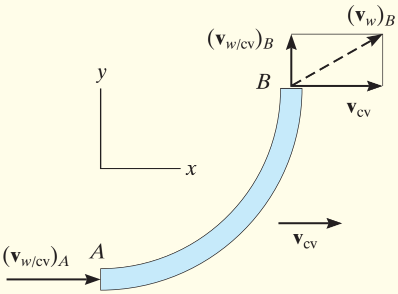
$$F_y = 4.90\ \text{kN} \qquad \textit{Ans.}$$

NOTE: The couple moment $\mathbf{M}_O$ can be determined using an angular impulse and momentum equation, which is discussed in fluid mechanics.

EXAMPLE 15.17



(a)



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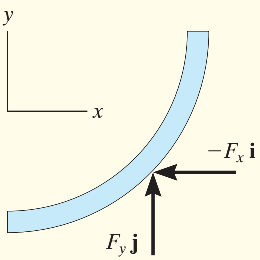


(c)

Fig. 15–30

A 50-mm-diameter water jet having a velocity of 7.5 m/s impinges upon a single moving blade, Fig. 15–30*a*. If the blade moves with a constant velocity of 1.5 m/s away from the jet, determine the horizontal and vertical components of force that the blade exerts on the water. What power does the water generate on the blade? Water has a density of $\rho_w = 1000\ \text{kg/m}^3$.

SOLUTION

Kinematic Diagram. Here the control volume will be the stream of water on the blade. The inertial coordinate system is moving at 1.5 m/s, so that the flow onto the blade appears as steady flow, Fig. 15–30*b*. The rate at which water enters the control volume at *A* is, therefore

$$(\mathbf{v}_{w/cv})_A = (\mathbf{v}_w)_A - \mathbf{v}_{cv} = 7.5\mathbf{i} - 1.5\mathbf{i} = \{6.0\mathbf{i}\}\ \text{m/s}$$

Also, the velocity of flow at *B* measured from the control volume is

$$(\mathbf{v}_{w/cv})_B = \{6.0\mathbf{j}\}\ \text{m/s}$$

The mass flow of water through the control volume is

$$\dot{m} = \rho_w (v_{w/cv})_A A_A = (1000)(6.0)\left[\pi\left(\frac{25}{1000}\right)^2\right] = 11.78\ \text{kg/s}$$

Free-Body Diagram. The free-body diagram of the control volume is shown in Fig. 15–30*c*. The weight of the water will be neglected in the calculation, since this force will be small compared to the reactive components $\mathbf{F}_x$ and $\mathbf{F}_y$.

Equations of Steady Flow.

$$\Sigma\mathbf{F} = \dot{m}[(\mathbf{v}_{w/cv})_B - (\mathbf{v}_{w/cv})_A]$$

$$-F_x\mathbf{i} + F_y\mathbf{j} = 11.78(6.0\mathbf{j} - 6.0\mathbf{i})$$

Equating the respective **i** and **j** components gives

$$F_x = 11.78(6.0) = 70.7\ \text{N} \leftarrow \qquad \textit{Ans.}$$

$$F_y = 11.78(6.0) = 70.7\ \text{N} \uparrow \qquad \textit{Ans.}$$

The water exerts equal but opposite forces on the blade.

Since the water force that causes the blade to move forward horizontally with a velocity of 1.5 m/s is $F_x = 70.7$ N, then from Eq. 14–10 the power is

$$P = \mathbf{F}\cdot\mathbf{v}; \qquad P = 70.7(1.5) = 106\ \text{W} \qquad \textit{Ans.}$$

15.10 BODIES THAT LOSE OR GAIN MASS

In this section we will apply Eq. 15–27 to cases where a body will lose or gain mass. Typical examples include an open ended balloon or a moving shovel. In both of these cases it is important to recognize that the loss or gain of mass will produce a force or impulse on the inner boundary of the control volume. Whatever the case, we will consider the control volume to be the body and its contents, which at a given instant is moving with a velocity $\mathbf{v}_{cv}$. Here the time rate of change of the momentum *within* the control volume must now account for the changes in both its mass and velocity, that is, $(d(mv)/dt)_{cv} = (\dot{m}\mathbf{v})_{cv} + (m\dot{\mathbf{v}})_{cv}$. Substituting into Eq. 15–27, we have

$$\Sigma \mathbf{F} = (\dot{m}\mathbf{v})_{cv} + (m\dot{\mathbf{v}})_{cv} + \dot{m}_{out}\mathbf{v}_{out} - \dot{m}_{in}\mathbf{v}_{in} \qquad (15\text{–}30)$$

To illustrate an application, consider a balloon (control volume) which loses mass (air) as it moves upward with a velocity $\mathbf{v}$, Fig. 15–31*a*. The last term in the above equation is zero because the air is self contained within the balloon. If the rate at which the air is expelled is $\dot{m}_{out} = \dot{m}_e$, then there is a corresponding *loss* of mass within the control volume, so $\dot{m}_{cv} = -\dot{m}_e$. Also, if the velocity of the ejected air is $(\mathbf{v}_e)_{rel}$, as measured relative to the balloon, then the velocity of this air as seen by a fixed observer is $\mathbf{v}_{out} = \mathbf{v} - (\mathbf{v}_e)_{rel}$. Substituting these terms into the above equation and simplifying, we obtain

$$\Sigma \mathbf{F} = (m\dot{\mathbf{v}})_{cv} - \dot{m}_e(\mathbf{v}_e)_{rel} \qquad (15\text{–}31)$$

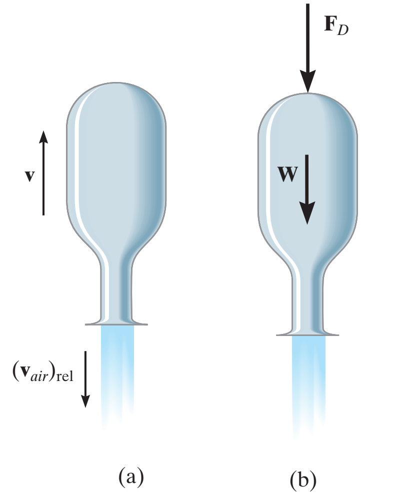


Fig. 15–31

As shown on the free-body diagram for the balloon, Fig. 15–31*b*, the only forces acting are its weight $\mathbf{W}$ and the atmospheric drag force $\mathbf{F}_D$. The last term on the right of Eq. 15–31 represents the thrust which the exhaust exerts on the balloon and propels it upwards, giving it an acceleration $\dot{\mathbf{v}}$.

In a similar manner, Eq. 15–30 can also be applied to the scoop in Fig. 15–32, which is moving to the right with a velocity $\mathbf{v}$ so that mass is being injected at the rate of $(\dot{m})_{cv} = \dot{m}_i$. Here, $\dot{m}_{out} = 0$ and $\dot{m}_{in} = \dot{m}_i$. If the relative velocity of the injected mass is $(\mathbf{v}_i)_{rel}$, then $\mathbf{v}_{in} = \mathbf{v} + (\mathbf{v}_i)_{rel}$, and Eq. 15–30 now becomes

$$\Sigma \mathbf{F} = (m\dot{\mathbf{v}})_{cv} - \dot{m}_i(\mathbf{v}_i)_{rel} \qquad (15\text{–}32)$$

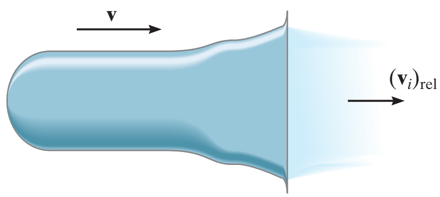


Fig. 15–32

The last term represents the magnitude of the force that the injected mass exerts on the scoop.

As noted by these two examples, when applying Eq. 15–30, whatever the situation, it is important to first define the control volume and then draw its free-body diagram in order to identify the forces exerted on the body and identify the thrust or retardation caused by the fluid or particle stream.

The scraper box behind this tractor gains mass at the rate of $\dot{m}$. If the tractor maintains a constant velocity v, then $\dot{v} = 0$ and, because the soil is originally at rest, $(v_i)_{rel} = v$. Applying Eq. 15–32, the horizontal towing force on the scraper box is then $T = 0 + \dot{m}v$.

EXAMPLE 15.18

(© NASA)

The initial combined mass of a rocket and its fuel is m_0. The mass m_f of fuel within the rocket is expelled at a constant rate of $\dot{m}_e = c$ and at a constant speed of u measured relative to the rocket. Determine the maximum velocity of the rocket, i.e., at the instant the fuel runs out. Neglect the change in the rocket's weight with altitude and the drag resistance of the air. The rocket is fired vertically from rest.

SOLUTION

Since the rocket loses mass as it moves upward, $\dot{m}_{out} = -c$. Also, $\dot{m}_{in} = 0$, and Eq. 15–31 applies. The only *external force* acting on the *control volume* (rocket) is its weight **W**, Fig. 15–33. Hence,

$$+\uparrow \Sigma F = (m\dot{v})_{cv} - \dot{m}_e(v_e)_{rel}; \qquad -W = m\dot{v} - cu \qquad (1)$$

The rocket's velocity is obtained by integrating this equation.

At any given instant t during the flight, the mass of the rocket is $m = m_0 - (\dot{m}_e)t = m_0 - ct$. Since $W = mg$, Eq. 1 then becomes

$$-(m_0 - ct)g = (m_0 - ct)\frac{dv}{dt} - cu$$

Separating the variables and integrating, realizing that $v = 0$ when $t = 0$, we have

$$\int_0^v dv = \int_0^t \left(\frac{cu}{m_0 - ct} - g\right) dt$$

$$v = -u\ln(m_0 - ct) - gt\Big|_0^t = u\ln\left(\frac{m_0}{m_0 - ct}\right) - gt \qquad (2)$$

Note that liftoff requires the first term on the right to be greater than the second term during the initial phase of motion. To find the time t' needed to consume all the fuel, we have

$$m_f = ct'$$

or

$$t' = m_f/c$$

Substituting into Eq. 2 yields

$$v_{max} = u\ln\left(\frac{m_0}{m_0 - m_f}\right) - \frac{gm_f}{c} \qquad \textit{Ans.}$$

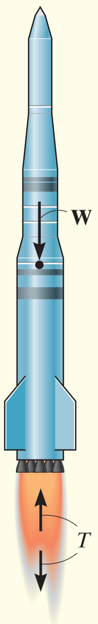
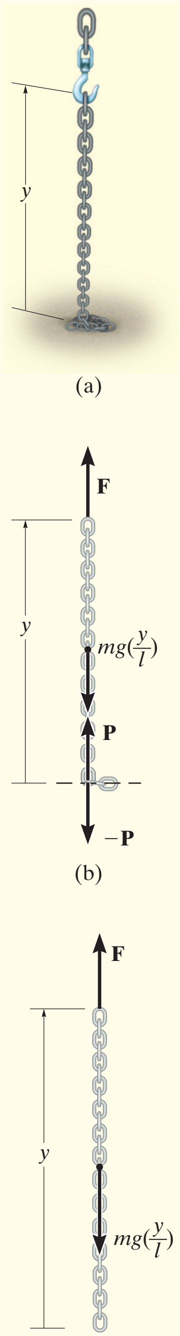


Fig. 15–33

15

EXAMPLE 15.19

A chain of length l, Fig. 15–34a, has a mass m. Determine the magnitude of force $\mathbf{F}$ required to (a) raise the chain with a constant speed v_c, starting from rest when $y = 0$; and (b) lower the chain with a constant speed v_c, starting from rest when $y = l$.



(a) (b) (c)

Fig. 15–34

SOLUTION

Part (a). As the chain is raised, all the suspended links are given a sudden downward impulse by each added link which is lifted off the ground. Thus, the *suspended portion* of the chain may be considered as a device which is *gaining mass*. The control volume to be considered is the length of chain y which is suspended by $\mathbf{F}$ at any instant, including the next link which is about to be added but is still at rest, Fig. 15–34b. The forces acting on the control volume *exclude* the internal forces $\mathbf{P}$ and $-\mathbf{P}$, which act between the added link and the suspended portion of the chain. Hence, $\Sigma F = F - mg(y/l)$.

Since v_c is constant, $y = v_c t$. Thus, the mass of the control volume at any instant is $m = m(y/l) = m(v_c t/l)$, and therefore the *rate* at which mass is *added* to the control volume is

$$\dot{m}_i = m\left(\frac{v_c}{l}\right)$$

Applying Eq. 15–32, we have

$$+\uparrow \Sigma F = (m\dot{v})_{cv} + \dot{m}_i (v_i)_{rel}$$

$$F - mg\left(\frac{y}{l}\right) = 0 + m\left(\frac{v_c}{l}\right)v_c$$

or

$$F = (m/l)(gy + v_c^2) \qquad \textit{Ans.}$$

Part (b). When the chain is being lowered at v_c the (loose) links which are expelled (given zero velocity) *do not* impart an impulse or a force on the *remaining* suspended links, that is, they do not create an impulse or force on the control volume, instead, they simply "fall off." And so the control volume in Part (a) will not be considered. Rather, we can apply the equation of motion to obtain the solution. At time t the length of chain still off the floor is y. Using the free-body diagram for a suspended portion of the chain shown in Fig. 15–34c, we have

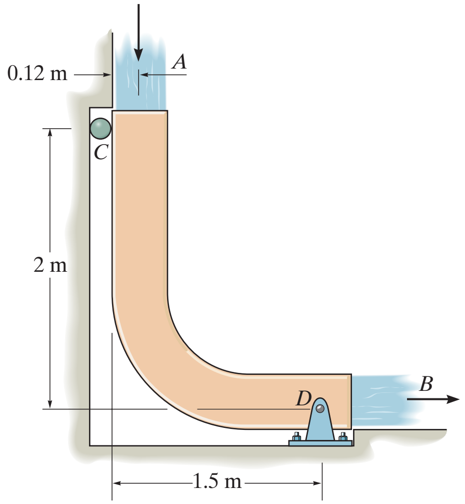
$$+\uparrow \Sigma F = ma; \qquad F - mg\left(\frac{y}{l}\right) = 0$$

$$F = mg\left(\frac{y}{l}\right) \qquad \textit{Ans.}$$

15

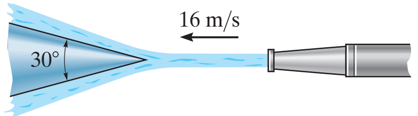
PROBLEMS

15–114. The chute is used to divert the flow of water, $Q = 0.6\,\text{m}^3/\text{s}$. If the water has a cross-sectional area of $0.05\ \text{m}^2$, determine the force components at the pin D and roller C necessary for equilibrium. Neglect the weight of the chute and weight of the water on the chute. $\rho_w = 1\,\text{Mg/m}^3$.



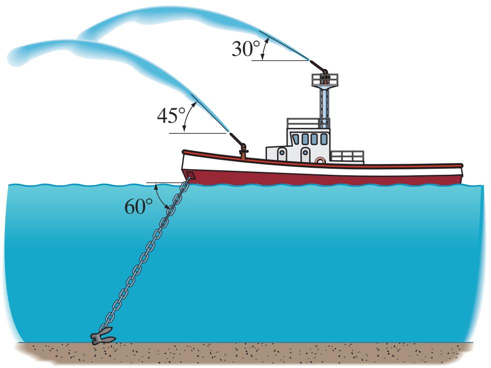
Prob. 15–114

15–115. Water is discharged at 16 m/s against the fixed cone diffuser. If the opening diameter of the nozzle is 40 mm, determine the horizontal force exerted by the water on the diffuser. $\rho_w = 1\,\text{Mg/m}^3$.



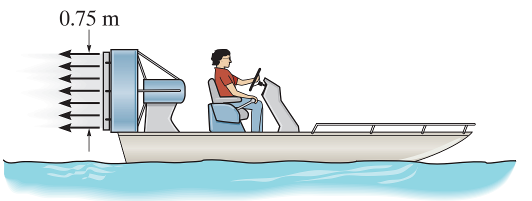
Prob. 15–115

***15–116.** The fire boat discharges two streams of seawater, each at a flow of $0.25\ \text{m}^3/\text{s}$ and with a nozzle velocity of 50 m/s. Determine the tension developed in the anchor chain, needed to secure the boat. The density of seawater is $\rho_{s\omega} = 1020\ \text{kg/m}^3$.



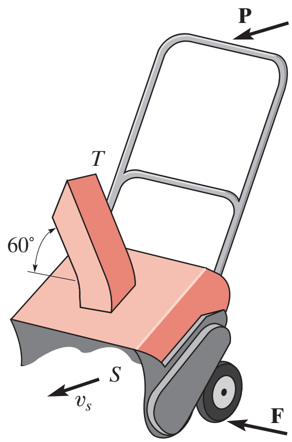
Prob. 15–116

15–117. The 200-kg boat is powered by the fan which develops a slipstream having a diameter of 0.75 m. If the fan ejects air with a speed of 14 m/s, measured relative to the boat, determine the initial acceleration of the boat if it is initially at rest. Assume that air has a constant density of $\rho_a = 1.22\ \text{kg/m}^3$ and that the entering air is essentially at rest. Neglect the drag resistance of the water.



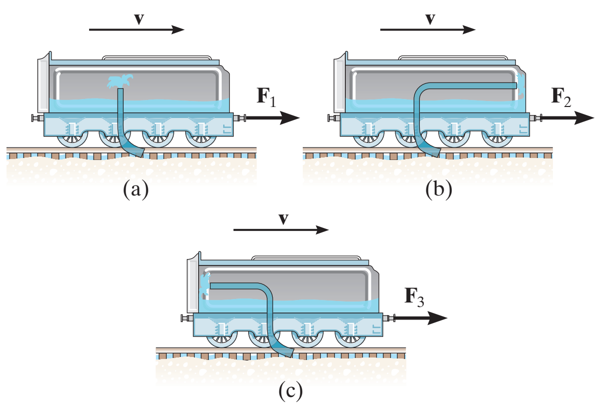
Prob. 15–117

15–118. A snowblower having a scoop S with a cross-sectional area of $A_s = 0.12\ \text{m}^3$ is pushed into snow with a speed of $v_s = 0.5\ \text{m/s}$. The machine discharges the snow through a tube T that has a cross-sectional area of $A_t = 0.03\ \text{m}^2$ and is directed 60° from the horizontal. If the density of snow is $\rho_s = 104\ \text{kg/m}^3$, determine the horizontal force P required to push the blower forward, and the resultant frictional force F of the wheels on the ground, necessary to prevent the blower from moving sideways. The wheels roll freely.



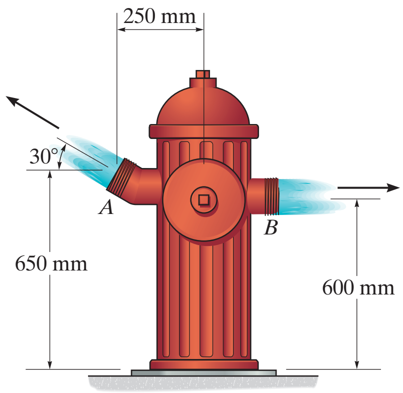
Prob. 15–118

15–119. The car is used to scoop up water that is lying in a trough at the tracks. Determine the force needed to pull the car forward at constant velocity **v** for each of the three cases. The scoop has a cross-sectional area A and the density of water is ρ_w.



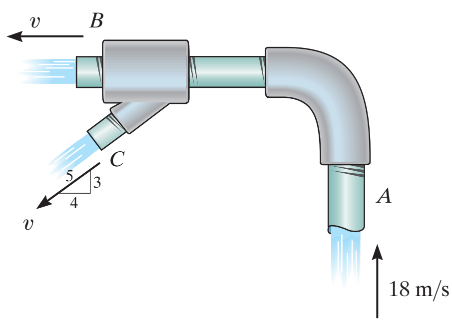
Prob. 15–119

***15–120.** The water flow enters below the hydrant at C at the rate of $0.75\ \text{m}^3/\text{s}$. It is then divided equally between the two outlets at A and B. If the gauge pressure at C is 300 kPa, determine the horizontal and vertical force reactions and the moment reaction on the fixed support at C. The diameter of the two outlets at A and B is 75 mm, and the diameter of the inlet pipe at C is 150 mm. The density of water is $\rho_w = 1000\ \text{kg/m}^3$. Neglect the mass of the contained water and the hydrant.



Prob. 15–120

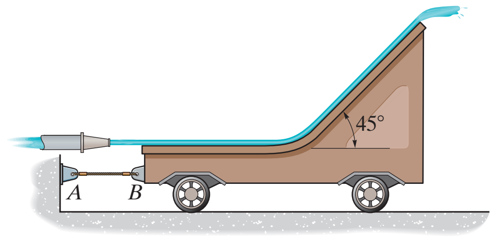
15–121. The gage pressure of water at A is 150.5 kPa. Water flows through the pipe at A with a velocity of 18 m/s, and out the pipe at B and C with the same velocity v. Determine the horizontal and vertical components of force exerted on the elbow necessary to hold the pipe assembly in equilibrium. Neglect the weight of water within the pipe and the weight of the pipe. The pipe has a diameter of 50 mm at A, and at B and C the diameter is 30 mm. $\rho_w = 1000\ \text{kg/m}^3$.



Prob. 15–121

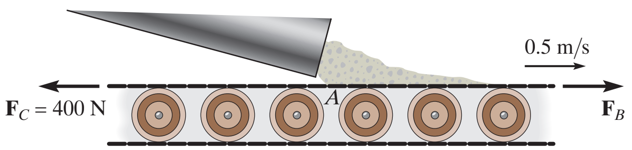
15

15–122. Water is discharged from a nozzle with a velocity of 12 m/s and strikes the blade mounted on the 20-kg cart. Determine the tension developed in the cord, needed to hold the cart stationary, and the normal reaction of the wheels on the cart. The nozzle has a diameter of 50 mm and the density of water is $\rho_w = 1000 \text{ kg/m}^3$.



Prob. 15–122

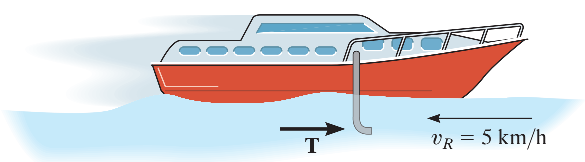
15–123. Sand is deposited from a chute onto a conveyor belt which is moving at 0.5 m/s. If the sand is assumed to fall vertically onto the belt at A at the rate of 4 kg/s, determine the belt tension F_B to the right of A. The belt is free to move over the conveyor rollers and its tension to the left of A is $F_C = 400$ N.



Prob. 15–123

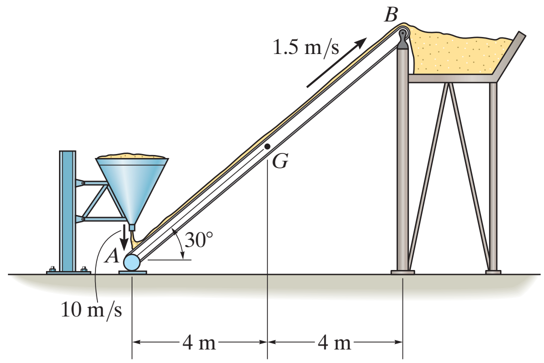
***15–124.** A scoop in front of the tractor collects snow at a rate of 200 kg/s. Determine the resultant traction force **T** that must be developed on all the wheels as it moves forward on level ground at a constant speed of 5 km/h. The tractor has a mass of 5 Mg.

15–125. The boat has a mass of 180 kg and is traveling forward on a river with a constant velocity of 70 km/h, measured *relative* to the river. The river is flowing in the opposite direction at 5 km/h. If a tube is placed in the water, as shown, and it collects 40 kg of water in the boat in 80 s, determine the horizontal thrust T on the tube that is required to overcome the resistance due to the water collection and yet maintain the constant speed of the boat. $\rho_w = 1 \text{ Mg/m}^3$.



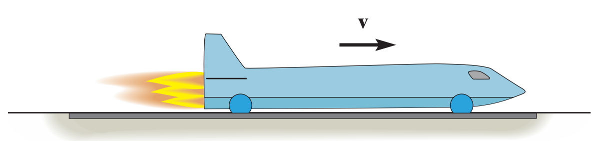
Prob. 15–125

15–126. Sand is discharged from the silo at A at a rate of 50 kg/s with a vertical velocity of 10 m/s onto the conveyor belt, which is moving with a constant velocity of 1.5 m/s. If the conveyor system and the sand on it have a total mass of 750 kg and center of mass at point G, determine the horizontal and vertical components of reaction at the pin support B and roller support A. Neglect the thickness of the conveyor.



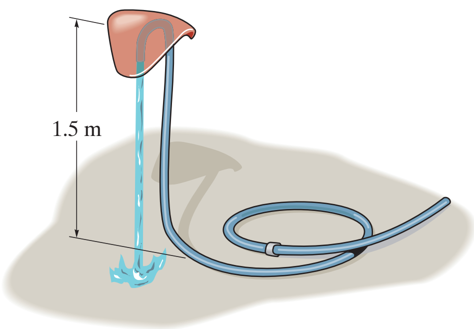
Prob. 15–126

15–127. The rocket car has a mass of 2 Mg (empty) and carries 120 kg of fuel. If the fuel is consumed at a constant rate of 6 kg/s and ejected from the car with a relative velocity of 800 m/s, determine the maximum speed attained by the car starting from rest. The drag resistance due to the atmosphere is $F_D = (6.8v^2)$ N, where v is the speed in m/s.



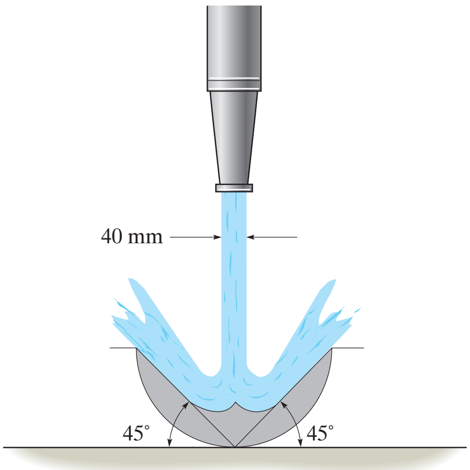
Prob. 15–127

***15–128.** The toy sprinkler for children consists of a 0.2-kg cap and a hose that has a mass per length of 30 g/m. Determine the required rate of flow of water through the 5-mm-diameter tube so that the sprinkler will lift 1.5 m from the ground and hover from this position. Neglect the weight of the water in the tube. $\rho_w = 1\,\text{Mg/m}^3$.



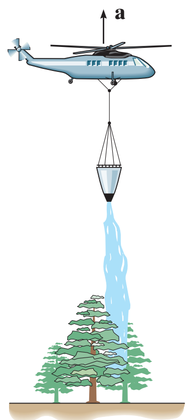
Prob. 15–128

15–129. The nozzle has a diameter of 40 mm. If it discharges water uniformly with a downward velocity of 20 m/s against the fixed blade, determine the vertical force exerted by the water on the blade. $\rho_w = 1\,\text{Mg/m}^3$.



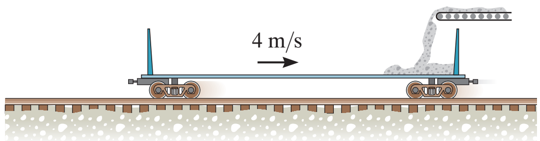
Prob. 15–129

15–130. The 10-Mg helicopter carries a bucket containing 500 kg of water, which is used to fight fires. If it hovers over the land in a fixed position and then releases 50 kg/s of water at 10 m/s, measured relative to the helicopter, determine the initial upward acceleration the helicopter experiences as the water is being released.



Prob. 15–130

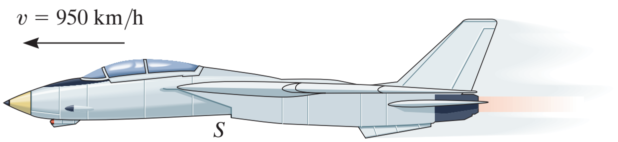
15–131. Sand drops onto the 2-Mg empty rail car at 50 kg/s from a conveyor belt. If the car is initially coasting at 4 m/s, determine the speed of the car as a function of time.



Prob. 15–131

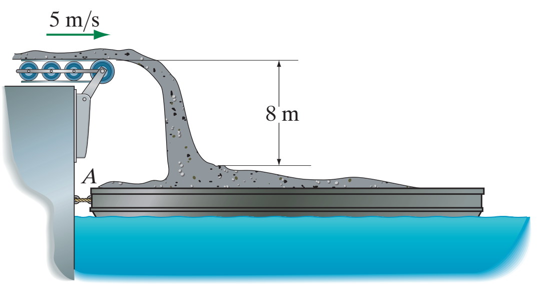
***15–132.** The 12-Mg jet airplane has a constant speed of 950 km/h when it is flying along a horizontal straight line. Air enters the intake scoops S at the rate of 50 m^3/s. If the engine burns fuel at the rate of 0.4 kg/s and the gas (air and fuel) is exhausted relative to the plane with a speed of 450 m/s, determine the resultant drag force exerted on the plane by air resistance. Assume that air has a constant density of 1.22 kg/m^3. *Hint:* Since mass both enters and exits the plane, Eqs. 15–28 and 15–29 must be combined to yield

$$\Sigma F_s = m\frac{dv}{dt} - v_{D/e}\frac{dm_e}{dt} + v_{D/i}\frac{dm_i}{dt}.$$



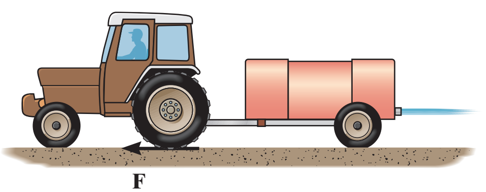
Prob. 15–132

15–133. Aggregates fall off the conveyor belt at 500 kg/s and into the barge. Determine the force the rope at A exerts on the barge to hold it in place.



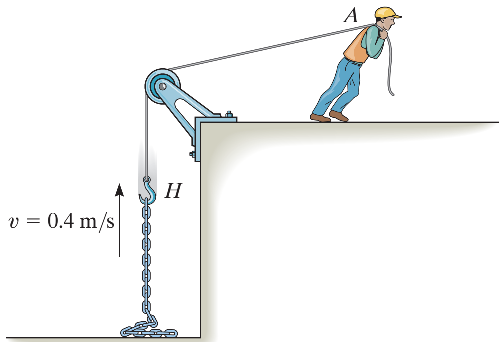
Prob. 15–133

15–134. The tractor together with the empty tank has a total mass of 4 Mg. The tank is filled with 2 Mg of water. The water is discharged at a constant rate of 50 kg/s with a constant velocity of 5 m/s, measured relative to the tractor. If the tractor starts from rest, and the rear wheels provide a resultant traction force of 250 N, determine the velocity and acceleration of the tractor at the instant the tank becomes empty.



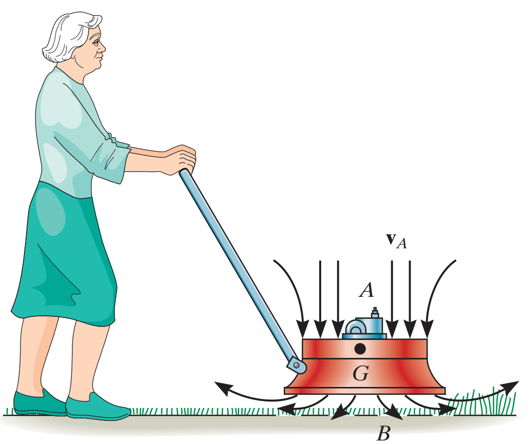
Prob. 15–134

15–135. Determine the magnitude of force **F** as a function of time, which must be applied to the end of the cord at A to raise the hook H with a constant speed $v = 0.4$ m/s. Initially the chain is at rest on the ground. Neglect the mass of the cord and the hook. The chain has a mass of 2 kg/m.



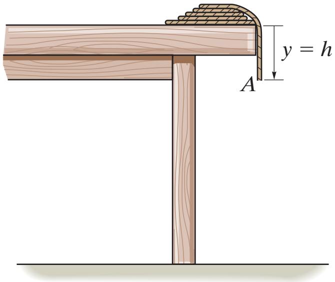
Prob. 15–135

***15–136.** A power lawn mower hovers very close over the ground. This is done by drawing air in at a speed of 6 m/s through an intake unit A, which has a cross-sectional area of $A_A = 0.25$ m^2, and then discharging it at the ground, B, where the cross-sectional area is $A_B = 0.35$ m^2. If air at A is subjected only to atmospheric pressure, determine the air pressure which the lawn mower exerts on the ground when the weight of the mower is freely supported and no load is placed on the handle. The mower has a mass of 15 kg with center of mass at G. Assume that air has a constant density of $\rho_a = 1.22$ kg/m^3.



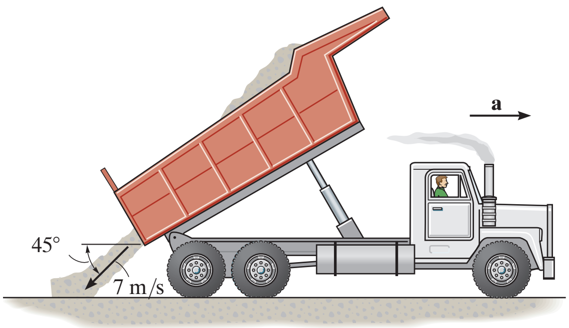
Prob. 15–136

15–137. The rope has a mass m' per unit length. If the end length $y = h$ is draped off the edge of the table, and released, determine the velocity of its end A for any position y, as the rope uncoils and begins to fall.



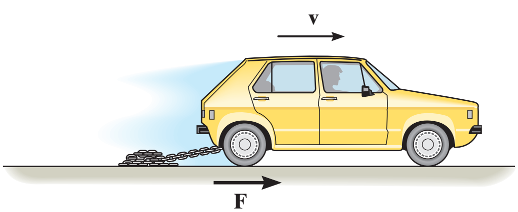
Prob. 15–137

15–138. The truck has a mass of 50 Mg when empty. When it is unloading 5 m^3 of sand at a constant rate of 0.8 m^3/s, the sand flows out the back at a speed of 7 m/s, measured relative to the truck, in the direction shown. If the truck is free to roll, determine its initial acceleration just as the load begins to empty. Neglect the mass of the wheels and any frictional resistance to motion. The density of sand is $\rho_s = 1520\ kg/m^3$.



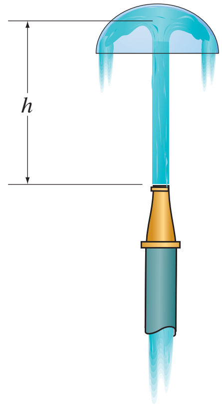
Prob. 15–138

15–139. The car has a mass m_0 and is used to tow the smooth chain having a total length l and a mass per unit of length m'. If the chain is originally piled up, determine the tractive force F that must be supplied by the rear wheels of the car, necessary to maintain a constant speed v while the chain is being drawn out.



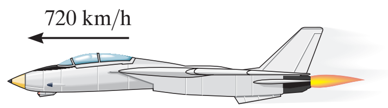
Prob. 15–139

***15–140.** The hemispherical bowl of mass m is held in equilibrium by the vertical jet of water discharged through a nozzle of diameter d. If the discharge of the water through the nozzle is Q, determine the height h at which the bowl is suspended. The water density is ρ_w. Neglect the weight of the water jet.



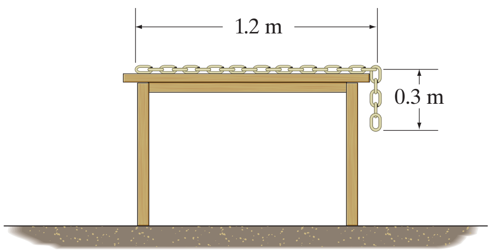
Prob. 15–140

15–141. The jet is traveling at a speed of 720 km/h. If the fuel is being spent at 0.8 kg/s, and the engine takes in air at 200 kg/s, whereas the exhaust gas (air and fuel) has a relative speed of 12 km/s, determine the acceleration of the plane at this instant. The drag resistance of the air is $F_D = (55\ v^2)$, where the speed is measured in m/s. The jet has a mass of 7 Mg.



Prob. 15–141

15–142. The 1.5-m-long smooth chain hangs over the edge of the table. If the chain is released from rest, determine its speed just after the last link clears the table. The chain has a mass per unit length of 6 kg/m.



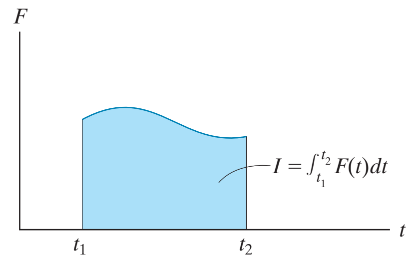
Prob. 15–142

15

CHAPTER REVIEW

Impulse

An impulse is defined as the product of force and time. Graphically it represents the area under the F–t diagram. If the force is constant, then the impulse becomes $I = F_c(t_2 - t_1)$.



Principle of Impulse and Momentum

When the equation of motion, $\Sigma \mathbf{F} = m\mathbf{a}$, and the kinematic equation, $a = dv/dt$, are combined, we obtain the principle of impulse and momentum. This is a vector equation that can be resolved into rectangular components and used to solve problems that involve force, velocity, and time. For application, the free-body diagram should be drawn in order to account for all the impulses that act on the particle.

$$m\mathbf{v}_1 + \Sigma \int_{t_1}^{t_2} \mathbf{F}\, dt = m\mathbf{v}_2$$

15

Conservation of Linear Momentum

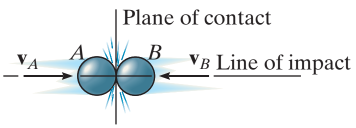
If the principle of impulse and momentum is applied to a *system of particles*, then the collisions between the particles produce internal impulses that are equal, opposite, and collinear, and therefore cancel from the equation. Furthermore, if an external impulse is small, that is, the force is small and the time is short, then the impulse can be classified as nonimpulsive and can be neglected. Consequently, momentum for the system of particles is conserved.

The conservation-of-momentum equation is useful for finding the final velocity of a particle when internal impulses are exerted between two particles and the initial velocities of the particles are known. If the internal impulse is to be determined, then one of the particles is isolated and the principle of impulse and momentum is applied to this particle.

$$\Sigma m_i(\mathbf{v}_i)_1 = \Sigma m_i(\mathbf{v}_i)_2$$

Impact

When two particles A and B have a direct impact, the internal impulse between them is equal, opposite, and collinear. Consequently the conservation of momentum for this system applies along the line of impact.

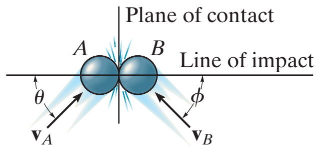


$$m_A(v_A)_1 + m_B(v_B)_1 = m_A(v_A)_2 + m_B(v_B)_2$$

If the final velocities are unknown, a second equation is needed for solution. We must use the coefficient of restitution, e. This experimentally determined coefficient depends upon the physical properties of the colliding particles. It can be expressed as the ratio of their relative velocity after collision to their relative velocity before collision. If the collision is elastic, no energy is lost and $e = 1$. For a plastic collision $e = 0$.

$$e = \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$$

If the impact is oblique, then the conservation of momentum for the system and the coefficient-of-restitution equation apply along the line of impact. Also, conservation of momentum for each particle applies perpendicular to this line (plane of contact) because no impulse acts on the particles in this direction.



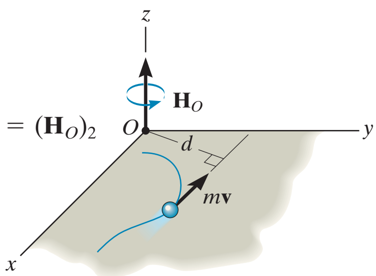
Principle of Angular Impulse and Momentum

The moment of the linear momentum about an axis (z) is called the angular momentum.

The principle of angular impulse and momentum is often used to eliminate unknown impulses by summing the moments about an axis through which the lines of action of these impulses produce no moment. For this reason, a free-body diagram should accompany the solution.

$$(H_O)_z = (d)(mv)$$

$$(\mathbf{H}_O)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{M}_O \, dt = (\mathbf{H}_O)_2$$



Steady Fluid Streams

Impulse-and-momentum methods are often used to determine the forces that a device exerts on the mass flow of a fluid—liquid or gas. To do so, a free-body diagram of the fluid mass in contact with the device is drawn in order to identify these forces. Also, the velocity of the fluid as it flows into and out of a control volume for the device is calculated. The equations of steady flow involve summing the forces and the moments to determine these reactions.

$$\Sigma \mathbf{F} = \frac{dm}{dt}(\mathbf{v}_B - \mathbf{v}_A)$$

$$\Sigma \mathbf{M}_O = \frac{dm}{dt}(\mathbf{r}_B \times \mathbf{v}_B - \mathbf{r}_A \times \mathbf{v}_A)$$

Propulsion with Variable Mass

Some devices, such as a rocket, lose mass as they are propelled forward. Others gain mass, such as a shovel. We can account for this mass loss or gain by applying the principle of impulse and momentum to a control volume for the device. From this equation, the force exerted on the device by the mass flow can then be determined.

$$\Sigma F_{cv} = m\frac{dv}{dt} - v_{D/e}\frac{dm_e}{dt}$$

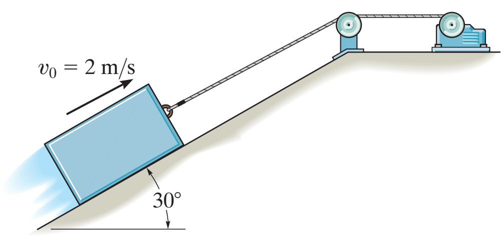
Loses Mass

$$\Sigma F_{cv} = m\frac{dv}{dt} + v_{D/i}\frac{dm_i}{dt}$$

Gains Mass

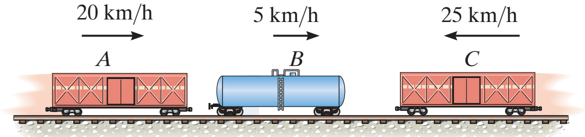
REVIEW PROBLEMS

R15–1. The 50-kg block is hoisted up the incline using the cable and motor arrangement shown. The coefficient of kinetic friction between the block and the surface is $\mu_k = 0.4$. If the block is initially moving up the plane at $v_0 = 2$ m/s, and at this instant ($t = 0$) the motor develops a tension in the cord of $T = (300 + 120\sqrt{t})$ N, where t is in seconds, determine the velocity of the block when $t = 2$ s.



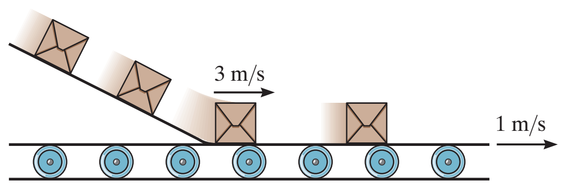
Prob. R15–1

R15–2. The three freight cars A, B, and C have masses of 10 Mg, 5 Mg, and 20 Mg, respectively. They are traveling along the track with the velocities shown. Car A collides with car B first, followed by car C. If the three cars couple together in sequence after collision, determine the common velocity of the cars after the two collisions have taken place.



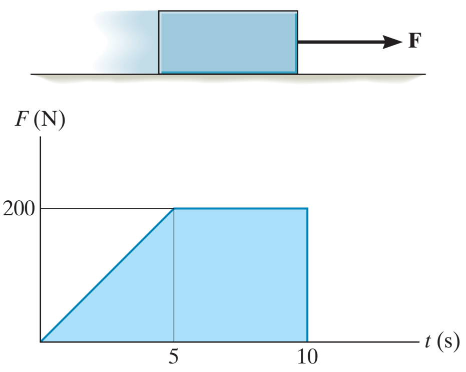
Prob. R15–2

R15–3. Packages having a mass of 6 kg slide down a smooth chute and land horizontally with a speed of 3 m/s on the surface of a conveyor belt. If the coefficient of kinetic friction between the belt and a package is $\mu_k = 0.2$, determine the time needed to bring the package to rest on the belt if the belt is moving in the same direction as the package with a speed $v = 1$ m/s.



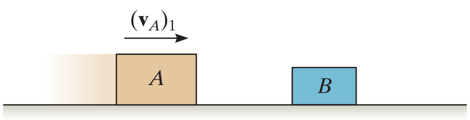
Prob. R15–3

R15–4. A 20-kg block is originally at rest on a horizontal surface for which the coefficient of static friction is $\mu_s = 0.6$ and the coefficient of kinetic friction is $\mu_k = 0.5$. If a horizontal force F is applied such that it varies with time as shown, determine the speed of the block in 10 s. *Hint*: First determine the time needed to overcome friction and start the block moving.



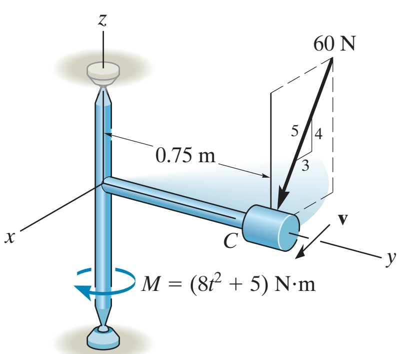
Prob. R15–4

R15–5. Block A has a mass of 3 kg and is sliding on a rough horizontal surface with a velocity $(v_A)_1 = 2$ m/s when it makes a direct collision with block B, which has a mass of 2 kg and is originally at rest. If the collision is perfectly elastic ($e = 1$), determine the velocity of each block just after collision and the distance between the blocks when they stop sliding. The coefficient of kinetic friction between the blocks and the plane is $\mu_k = 0.3$.



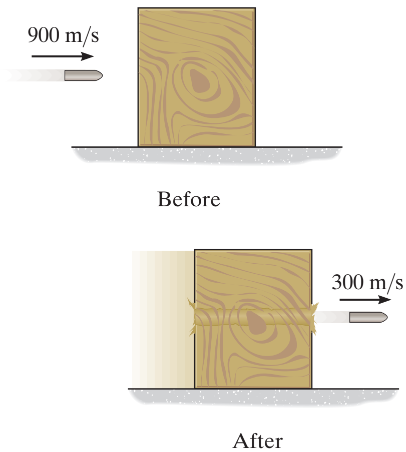
Prob. R15–5

R15–6. The small cylinder C has a mass of 10 kg and is attached to the end of a rod whose mass may be neglected. If the frame is subjected to a couple $M = (8t^2 + 5)$ N·m, where t is in seconds, and the cylinder is subjected to a force of 60 N, which is always directed as shown, determine the speed of the cylinder when $t = 2$ s. The cylinder has a speed $v_0 = 2$ m/s when $t = 0$.



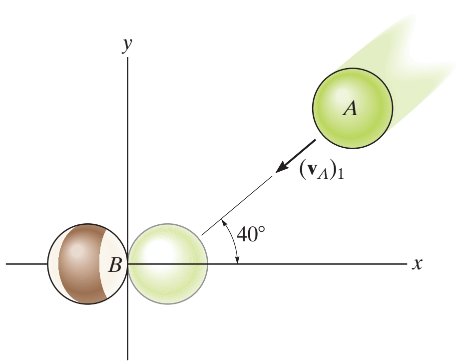
Prob. R15–6

R15–7. The 200-g projectile is fired with a velocity of 900 m/s towards the center of the 15-kg wooden block, which rests on a rough surface. If the projectile penetrates and emerges from the block with a velocity of 300 m/s, determine the velocity of the block just after the projectile emerges. How long does the block slide on the rough surface, after the projectile emerges, before it comes to rest again? The coefficient of kinetic friction between the surface and the block is $\mu_k = 0.2$.



Prob. R15–7

R15–8. Two smooth billiard balls A and B have an equal mass of $m = 200$ g. If A strikes B with a velocity of $(v_A)_1 = 2$ m/s as shown, determine their final velocities just after collision. Ball B is originally at rest and the coefficient of restitution is $e = 0.75$.



Prob. R15–8

CHAPTER 16

Kinematics is important for the design of gear trains used in different mechanisms

Lecture Summary and Quiz, Example, and Problem-solving videos are available where this icon appears.

PLANAR KINEMATICS OF A RIGID BODY

CHAPTER OBJECTIVES

- To classify the various types of rigid-body planar motion.
- To investigate rigid-body translation and angular motion about a fixed axis.
- To study planar motion using an absolute motion analysis.
- To provide a relative-motion analysis of velocity and acceleration using a translating frame of reference.
- To show how to find the instantaneous center of zero velocity and determine the velocity of a point on a body using this method.
- To provide a relative-motion analysis of velocity and acceleration using a rotating frame of reference.

16.1 PLANAR RIGID-BODY MOTION

In this chapter, the planar kinematics of a rigid body will be discussed. This study is important for the design of gears, cams, and mechanisms used for many structural and mechanical operations. Once the kinematics is thoroughly understood, then we can apply the equations of motion, which relate the forces on the body to the body's motion.

The *planar motion* of a body occurs when all the particles of a rigid body move along paths which are equidistant from a fixed plane. There are three types of rigid-body planar motion. In order of increasing complexity, they are

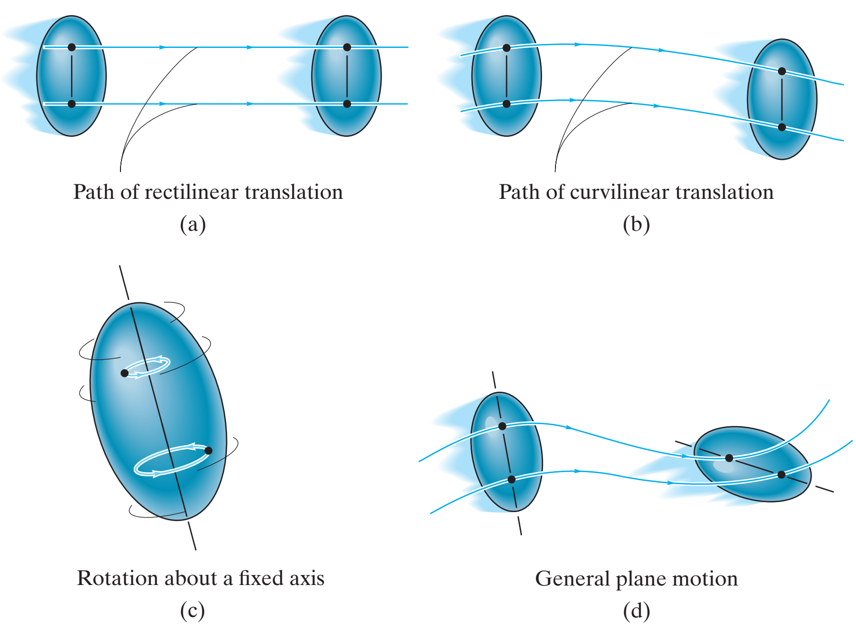


Fig. 16–1

- *Translation.* This type of motion occurs when a line in the body remains parallel to its original orientation throughout the motion. When the paths of motion for any two points on the body are parallel straight lines, the motion is called ***rectilinear translation***, Fig. 16–1*a*. If the paths of motion are along curved lines, the motion is called ***curvilinear translation***, Fig. 16–1*b*.
- *Rotation about a fixed axis.* When a rigid body rotates about a fixed axis, all the particles of the body, except those which lie on the axis of rotation, move along circular paths, Fig. 16–1*c*.
- *General plane motion.* When a body is subjected to general plane motion, it undergoes a simultaneous translation *and* rotation, Fig. 16–1*d*. The translation occurs within a reference plane, and the rotation occurs about an axis perpendicular to the reference plane.

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In the following sections we will consider each of these motions in detail. Examples of bodies undergoing each type of motion are shown in Fig. 16–2.

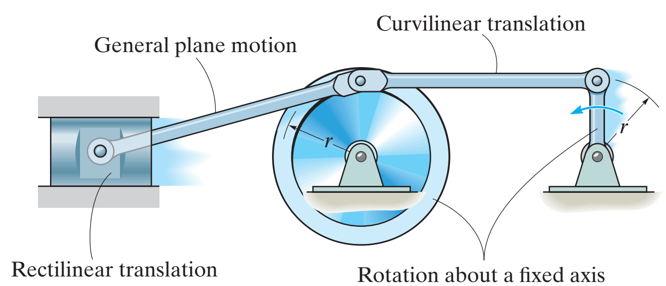

Fig. 16–2

16.2 TRANSLATION

Consider a rigid body which is subjected to either rectilinear or curvilinear translation in the x–y plane, Fig. 16–3.

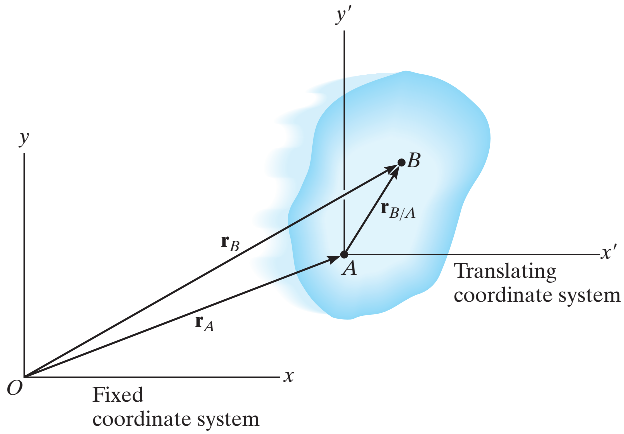

Fig. 16–3

Position. The locations of points A and B on the body are defined with respect to the fixed x, y reference frame using position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$. The *translating* x', y' coordinate system is fixed in the body and has its origin at A, hereafter referred to as the ***base point***. The position of B with respect to A is denoted by the relative-position vector $\mathbf{r}_{B/A}$ ("**r** of B with respect to A"). By vector addition,

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$$

Velocity. A relation between the instantaneous velocities of A and B is obtained by taking the time derivative of this equation, which yields $\mathbf{v}_B = \mathbf{v}_A + d\mathbf{r}_{B/A}/dt$. Here $\mathbf{v}_A$ and $\mathbf{v}_B$ denote absolute velocities since they are measured with respect to the x, y axes. The term $d\mathbf{r}_{B/A}/dt = \mathbf{0}$, since the magnitude of $\mathbf{r}_{B/A}$ is constant by definition of a rigid body, and because the body is translating, the direction of $\mathbf{r}_{B/A}$ is also constant. Therefore,

$$\mathbf{v}_B = \mathbf{v}_A$$

Acceleration. Taking the time derivative of the velocity equation yields a similar relationship between the instantaneous accelerations of A and B:

$$\mathbf{a}_B = \mathbf{a}_A$$

The above two equations indicate that *all points in a rigid body subjected to either rectilinear or curvilinear translation move with the same velocity and acceleration*. As a result, the kinematics of points located in a translating rigid body can be studied using the kinematics of particle motion studied in Chapter 12.

Riders on this Ferris wheel are subjected to curvilinear translation, since the gondolas move in a circular path, yet they always remain in the upright position.

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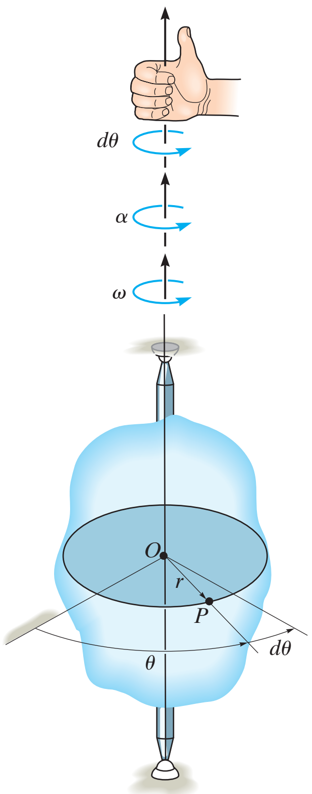
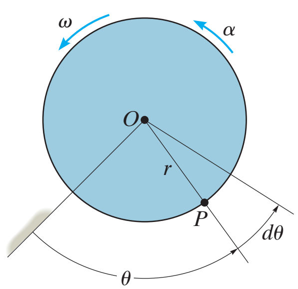


(a)



(b)

Fig. 16–4

16.3 ROTATION ABOUT A FIXED AXIS

When a body rotates about a fixed axis, any point P located in the body travels along a circular path. To study the motion of P it is first necessary to discuss the angular motion of the body about the axis.

Angular Motion. Since a point is without dimension, it cannot have angular motion. *Only lines or bodies undergo angular motion*. Here we will consider the angular motion of a radial line r located within the shaded plane of the body shown in Fig. 16–4*a*.

Angular Position. At the instant shown, the ***angular position*** of r is defined by the angle θ, measured from a *fixed* reference line to r.

Angular Displacement. The change in the angular position, which can be measured as a differential $d\boldsymbol{\theta}$, is called the ***angular displacement***.* This vector has a *magnitude* of $d\theta$ measured in degrees, radians, or revolutions, where $1 \text{ rev} = 2\pi \text{ rad}$. Since motion is about a *fixed axis*, the direction of $d\boldsymbol{\theta}$ is *always* along this axis. Specifically, the *direction* is determined by the right-hand rule; that is, the fingers of the right hand are curled with the sense of rotation, so that in this case the thumb, or $d\boldsymbol{\theta}$, points upward, Fig. 16–4*a*. In two dimensions, as shown by the top view of the shaded plane, Fig. 16–4*b*, both $\boldsymbol{\theta}$ and $d\boldsymbol{\theta}$ are counterclockwise, and so the thumb points outward from the page.

Angular Velocity. The time rate of change in the angular position is called the ***angular velocity*** $\boldsymbol{\omega}$ (omega). Since $d\boldsymbol{\theta}$ occurs during an instant of time dt, and maintains its fixed position, then

$$(\circlearrowleft +) \qquad \omega = \frac{d\theta}{dt} \qquad (16\text{–}1)$$

The vector $\boldsymbol{\omega}$ has a *magnitude* which is often measured in rad/s. Its *direction* is always along the axis of rotation, Fig. 16–4*a*. When indicating the angular motion in the shaded plane, Fig. 16–4*b*, we can refer to the sense of rotation as clockwise or counterclockwise. Here we have *arbitrarily* chosen counterclockwise rotations as *positive* and indicated this by the curl shown in parentheses next to Eq. 16–1. Realize, however, that the directional sense of $\boldsymbol{\omega}$ is actually outward from the page.

* It is shown in Sec. 20.1 that finite rotations or finite angular displacements are *not* vector quantities, although differential rotations $d\boldsymbol{\theta}$ are vectors.

Angular Acceleration. The ***angular acceleration*** $\boldsymbol{\alpha}$ (alpha) measures the time rate of change of the angular velocity. The *magnitude* of this vector is

$$(\circlearrowleft +) \qquad \alpha = \frac{d\omega}{dt} \qquad (16\text{–}2)$$

Using Eq. 16–1, it is also possible to express α as

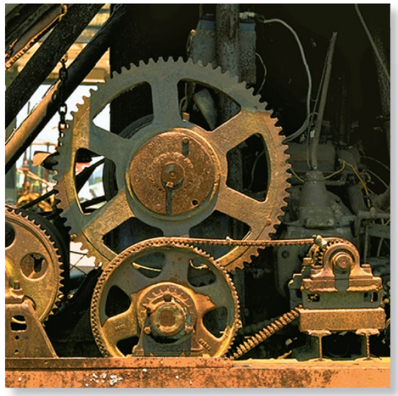
$$(\circlearrowleft +) \qquad \alpha = \frac{d^2\theta}{dt^2} \qquad (16\text{–}3)$$

The line of action of $\boldsymbol{\alpha}$ is the same as that for $\boldsymbol{\omega}$, Fig. 16–4*a*; however, its sense of *direction* depends on whether $\boldsymbol{\omega}$ is increasing or decreasing. If $\boldsymbol{\omega}$ is decreasing, then $\boldsymbol{\alpha}$ is called an ***angular deceleration***, and therefore it has a sense of direction which is opposite to $\boldsymbol{\omega}$.

By eliminating dt from Eqs. 16–1 and 16–2, we obtain a differential relation between the angular acceleration, angular velocity, and angular displacement, namely,

$$(\circlearrowleft +) \qquad \alpha \, d\theta = \omega \, d\omega \qquad (16\text{–}4)$$

The similarity between the differential relations for angular motion and those developed for rectilinear motion of a particle ($v = ds/dt$, $a = dv/dt$, and $a\,ds = v\,dv$) should be apparent.



The gears used in the operation of a crane all rotate about fixed axes. Engineers must be able to relate their angular motions in order to properly design this gear system.

Constant Angular Acceleration. If the angular acceleration of the body is *constant*, $\boldsymbol{\alpha} = \boldsymbol{\alpha}_c$, then Eqs. 16–1, 16–2, and 16–4, when integrated, yield the following set of formulas.

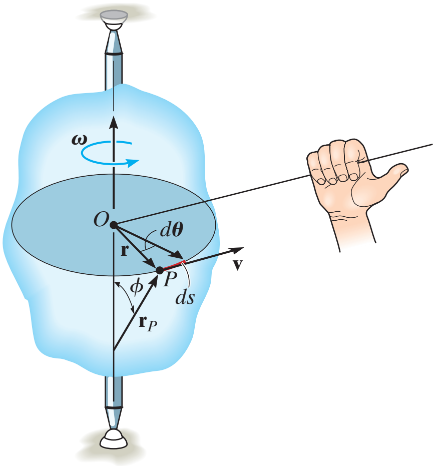
$$(\circlearrowleft +) \qquad \omega = \omega_0 + \alpha_c t \qquad (16\text{–}5)$$

$$(\circlearrowleft +) \qquad \theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha_c t^2 \qquad (16\text{–}6)$$

$$(\circlearrowleft +) \qquad \omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0) \qquad (16\text{–}7)$$

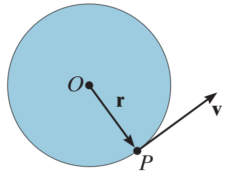
Constant Angular Acceleration

Here θ_0 and ω_0 are the initial values of the body's angular position and angular velocity, respectively. Note that these equations are similar to Eqs. 12–4 to 12–6 used for rectilinear motion.



(c)

Fig. 16–4 (cont.)



(d)

Motion of Point *P*. As the rigid body in Fig. 16–4*c* rotates, point *P* travels along a *circular path* of radius *r* with center at point *O*. This path is contained within the shaded plane shown in top view, Fig. 16–4*d*.

Position and Displacement. The position of *P* is defined by the position vector **r**, which extends from *O* to *P*. If the body rotates $d\theta$ then *P* will displace $ds = r\,d\theta$.

Velocity. The velocity of *P* has a magnitude which can be found by dividing $ds = r\,d\theta$ by dt. This yields

$$\boxed{v = \omega r} \qquad (16\text{–}8)$$

As shown in Figs. 16–4*c* and 16–4*d*, the *direction* of **v** is *tangent* to the circular path.

Both the magnitude and direction of **v** can also be accounted for by using the cross product of $\boldsymbol{\omega}$ and $\mathbf{r}_P$ (see Appendix B). Here, $\mathbf{r}_P$ is directed from *any point* on the axis of rotation to point *P*, Fig. 16–4*c*. We have

$$\boxed{\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_P} \qquad (16\text{–}9)$$

The order of the vectors in this formulation is important, since the cross product is not commutative, i.e., $\boldsymbol{\omega} \times \mathbf{r}_P \neq \mathbf{r}_P \times \boldsymbol{\omega}$. Notice in Fig. 16–4*c* how the correct direction of **v** is established by the right-hand rule. The fingers of the right hand are curled from $\boldsymbol{\omega}$ toward $\mathbf{r}_P$ ($\boldsymbol{\omega}$ "cross" $\mathbf{r}_P$). The thumb indicates the correct direction of **v**, which is tangent to the path in the direction of motion. From Eq. B–8, the magnitude of **v** in Eq. 16–9 is $v = \omega r_P \sin\phi$, and since $r = r_P \sin\phi$, Fig. 16–4*c*, then $v = \omega r$, which agrees with Eq. 16–8. As a special case, the position vector **r** can be chosen for $\mathbf{r}_P$. Here **r** lies in the plane of motion and again the velocity of point *P* is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \qquad (16\text{–}10)$$

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Acceleration. The acceleration of P can be expressed in terms of its normal and tangential components. For the tangential component, applying Eq. 12–19 and Eq. 16–8, we have $a_t = dv/dt = d(\omega r)/dt = (d\omega/dt)r$, and since $\alpha = d\omega/dt$, we get

$$a_t = \alpha r \qquad (16\text{–}11)$$

The normal component is determined from Eq. 12–20 and Eq. 16–8. We have $a_n = v^2/\rho = (\omega r)^2/r$ or

$$a_n = \omega^2 r \qquad (16\text{–}12)$$

The *tangential component of acceleration*, Figs. 16–4*e* and 16–4*f*, represents the time rate of change in the velocity's magnitude. If the speed of P is increasing, then $\mathbf{a}_t$ acts in the same direction as $\mathbf{v}$; if the speed is decreasing, $\mathbf{a}_t$ acts in the opposite direction of $\mathbf{v}$; and finally, if the speed is constant, $\mathbf{a}_t$ is zero.

The *normal component of acceleration* represents the time rate of change in the velocity's direction. The *direction* of $\mathbf{a}_n$ is always toward O, the center of the circular path, Figs. 16–4*e* and 16–4*f*.

Like the velocity, the acceleration of point P can be expressed in terms of the vector cross product. Taking the time derivative of Eq. 16–9 we have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}_P + \boldsymbol{\omega} \times \frac{d\mathbf{r}_P}{dt}$$

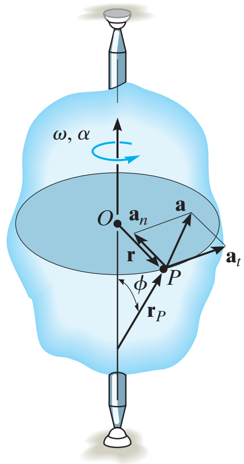
Since $\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt$, and using Eq. 16–9 ($d\mathbf{r}_P/dt = \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_P$), then

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r}_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_P) \qquad (16\text{–}13)$$

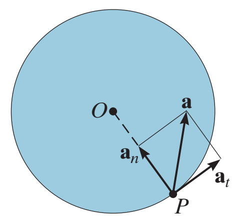
From the definition of the cross product, the first term on the right has a magnitude $a_t = \alpha r_P \sin\phi = \alpha r$, and by the right-hand rule, $\boldsymbol{\alpha} \times \mathbf{r}_P$ is in the direction of $\mathbf{a}_t$, Fig. 16–4*e*. Likewise, the second term has a magnitude $a_n = \omega^2 r_P \sin\phi = \omega^2 r$, and applying the right-hand rule twice, first to determine the result $\mathbf{v}_P = \boldsymbol{\omega} \times \mathbf{r}_P$ then $\boldsymbol{\omega} \times \mathbf{v}_P$, it can be seen that this result is in the same direction as $\mathbf{a}_n$, shown in Fig. 16–4*e*. Noting that this is also the *same* direction as $-\mathbf{r}$, which lies in the plane of motion, we can express $\mathbf{a}_n$ in a much simpler form as $\mathbf{a}_n = -\omega^2\mathbf{r}$. Hence, Eq. 16–13 can be identified by its two components as

$$\begin{aligned}\mathbf{a} &= \mathbf{a}_t + \mathbf{a}_n \\ &= \boldsymbol{\alpha} \times \mathbf{r} - \omega^2\mathbf{r}\end{aligned} \qquad (16\text{–}14)$$

Since $\mathbf{a}_t$ and $\mathbf{a}_n$ are perpendicular to one another, if needed the magnitude of acceleration can be determined from the Pythagorean theorem; namely, $a = \sqrt{a_n^2 + a_t^2}$, Fig. 16–4*f*.

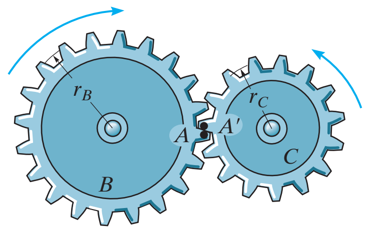


(e)



(f)

Fig. 16–4 (cont.)



(a)

If two rotating bodies are in contact, as in Fig. 16–5*a*, then the *points in contact* A and A' move along *different circular paths*, and the velocity and *tangential components* of acceleration of the points will be the *same*, $\mathbf{v}_A = \mathbf{v}_{A'}$, Fig. 16–5*b*, and $(\mathbf{a}_A)_t = (\mathbf{a}_{A'})_t$, Fig. 16–5*c*. However, the *normal components* of acceleration will *not* be the same, $(\mathbf{a}_A)_n \neq (\mathbf{a}_{A'})_n$, and therefore, as shown in Fig. 16–5*c*, $\mathbf{a}_A \neq \mathbf{a}_{A'}$.

(b)

(c)

Fig. 16–5

IMPORTANT POINTS

- A body can undergo two types of translation. During rectilinear translation all points follow parallel straight-line paths, and during curvilinear translation the points follow curved paths that are the same shape.
- All the points on a translating body move with the same velocity and acceleration.
- Points located on a body that rotates about a fixed axis follow circular paths.
- The relation $\alpha\, d\theta = \omega\, d\omega$ is derived from $\alpha = d\omega/dt$ and $\omega = d\theta/dt$ by eliminating dt.
- Once the angular motions ω and α are known, the velocity and acceleration of any point on the body can be determined.
- The velocity always acts tangent to the path of motion.
- The acceleration has two components. The tangential acceleration measures the time rate of change in the magnitude of the velocity and can be determined from $a_t = \alpha r$. The normal acceleration measures the time rate of change in the direction of the velocity and can be determined from $a_n = \omega^2 r$.

PROCEDURE FOR ANALYSIS

The velocity and acceleration of a point located on a rigid body that is rotating about a fixed axis can be determined using the following procedure.

Angular Motion.

- Establish the positive sense of rotation about the axis of rotation and show it alongside each kinematic equation as it is applied.

- If a relation is known between any *two* of the four variables α, ω, θ, and t, then a third variable can be obtained by using one of the following kinematic equations which relates all three variables.

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} \qquad \alpha\, d\theta = \omega\, d\omega$$

- If the body's angular acceleration is *constant*, then the following equations can be used:

$$\omega = \omega_0 + \alpha_c t$$
$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha_c t^2$$
$$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$$

- Once the solution is obtained, the sense of θ, ω, and α is determined from the algebraic signs of their numerical quantities.

Motion of Point *P*.

- In most cases the velocity of P and its two components of acceleration can be determined from the scalar equations

$$v = \omega r$$
$$a_t = \alpha r$$
$$a_n = \omega^2 r$$

- If the geometry of the problem is difficult to visualize, the following vector equations should be used:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_P = \boldsymbol{\omega} \times \mathbf{r}$$
$$\mathbf{a}_t = \boldsymbol{\alpha} \times \mathbf{r}_P = \boldsymbol{\alpha} \times \mathbf{r}$$
$$\mathbf{a}_n = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_P) = -\omega^2\mathbf{r}$$

Here $\mathbf{r}_P$ is directed from any point on the axis of rotation to point P, whereas $\mathbf{r}$ lies in the plane of motion of P. Either of these vectors, along with $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$, should be expressed in terms of its **i**, **j**, **k** components, and, if necessary, the cross products determined using a determinant expansion (see Eq. B–12).

Refer to the companion website for Lecture Summary and Quiz videos.

EXAMPLE 16.1

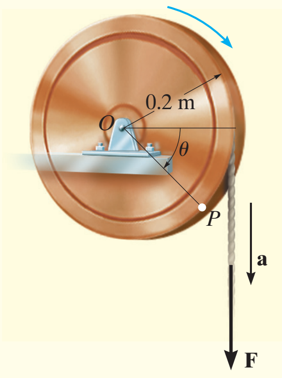


Fig. 16–6

A cord is wrapped around the wheel in Fig. 16–6, which is initially at rest when $\theta = 0$. If a force is applied to the cord and gives it an acceleration $a = (4t)\ \text{m/s}^2$, where t is in seconds, determine, as a function of time, (a) the angular velocity of the wheel, and (b) the angular position of line OP.

SOLUTION

Part (a). The wheel is subjected to rotation about a fixed axis passing through point O. Thus, point P on the wheel has motion about a circular path, and the acceleration of this point has *both* tangential and normal components. The tangential component is $(a_P)_t = (4t)\ \text{m/s}^2$, since the cord is wrapped around the wheel and moves *tangent* to it. Hence the angular acceleration of the wheel is

$$(\curvearrowleft +) \qquad (a_P)_t = \alpha r$$

$$(4t)\ \text{m/s}^2 = \alpha(0.2\ \text{m})$$

$$\alpha = (20t)\ \text{rad/s}^2 \circlearrowright$$

Using this result, the wheel's angular velocity ω can now be determined from $\alpha = d\omega/dt$, since this equation relates α, t, and ω. Integrating, with the initial condition that $\omega = 0$ when $t = 0$, yields

$$(\curvearrowleft +) \qquad \alpha = \frac{d\omega}{dt} = (20t)\ \text{rad/s}^2$$

$$\int_0^\omega d\omega = \int_0^t 20t\, dt$$

$$\omega = 10t^2\ \text{rad/s} \circlearrowright \qquad \textit{Ans.}$$

Part (b). Using this result, the angular position θ of OP can be found from $\omega = d\theta/dt$, since this equation relates θ, ω, and t. Integrating, with the initial condition $\theta = 0$ when $t = 0$, we have

$$(\curvearrowleft +) \qquad \frac{d\theta}{dt} = \omega = (10t^2)\ \text{rad/s}$$

$$\int_0^\theta d\theta = \int_0^t 10t^2\, dt$$

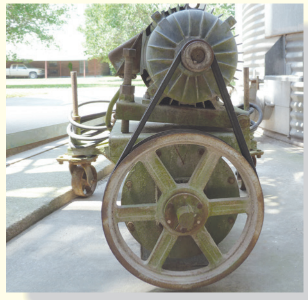
$$\theta = 3.33t^3\ \text{rad} \qquad \textit{Ans.}$$

NOTE: We cannot use the equation of constant angular acceleration, since α is a function of time.

16

EXAMPLE 16.2

The motor shown in the photo is used to turn the wheel. The details are shown in Fig. 16–7*a*. If the pulley *A* connected to the motor begins to rotate from rest with a constant angular acceleration of $\alpha_A = 2 \text{ rad/s}^2$, determine the magnitudes of the velocity and acceleration of point *P* on the wheel, after the pulley has turned two revolutions. Assume the transmission belt does not slip on the pulley and wheel.



SOLUTION

Angular Motion. First we will convert the two revolutions to radians. Since there are 2π rad in one revolution, then

$$\theta_A = 2 \text{ rev}\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right) = 12.57 \text{ rad}$$

Since α_A is constant, the angular velocity of pulley *A* is therefore

$$(\circlearrowright +) \qquad \omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$$

$$\omega_A^2 = 0 + 2(2 \text{ rad/s}^2)(12.57 \text{ rad} - 0)$$

$$\omega_A = 7.090 \text{ rad/s}$$

The belt has the same speed and tangential component of acceleration as it passes over the pulley and wheel. Thus,

$$v = \omega_A r_A = \omega_B r_B; \qquad 7.090 \text{ rad/s}\,(0.15 \text{ m}) = \omega_B(0.4 \text{ m})$$

$$\omega_B = 2.659 \text{ rad/s}$$

$$a_t = \alpha_A r_A = \alpha_B r_B; \qquad 2 \text{ rad/s}^2\,(0.15 \text{ m}) = \alpha_B(0.4 \text{ m})$$

$$\alpha_B = 0.750 \text{ rad/s}^2$$

Motion of *P*. As shown on the kinematic diagram in Fig. 16–7*b*, we have

$$v_P = \omega_B r_B = 2.659 \text{ rad/s}\,(0.4 \text{ m}) = 1.06 \text{ m/s} \qquad \textit{Ans.}$$

$$(a_P)_t = \alpha_B r_B = 0.750 \text{ rad/s}^2\,(0.4 \text{ m}) = 0.3 \text{ m/s}^2$$

$$(a_P)_n = \omega_B^2 r_B = (2.659 \text{ rad/s})^2(0.4 \text{ m}) = 2.827 \text{ m/s}^2$$

Thus

$$a_P = \sqrt{(0.3 \text{ m/s}^2)^2 + (2.827 \text{ m/s}^2)^2} = 2.84 \text{ m/s}^2 \qquad \textit{Ans.}$$

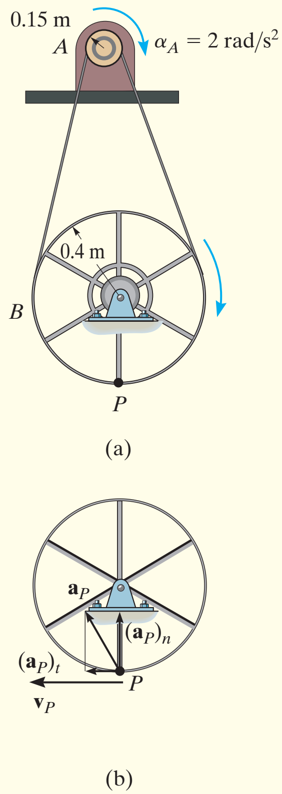
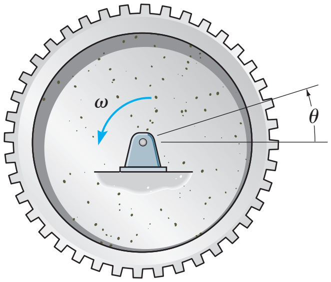


Fig. 16–7

Refer to the companion website for a self quiz of these Example problems.

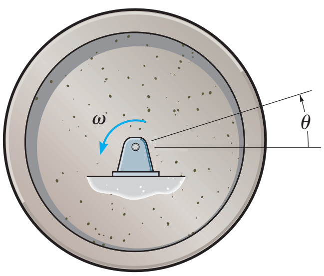
FUNDAMENTAL PROBLEMS

F16–1. When the gear rotates 20 revolutions, it achieves an angular velocity of $\omega = 30\ \text{rad/s}$, starting from rest. Determine its constant angular acceleration and the time required.



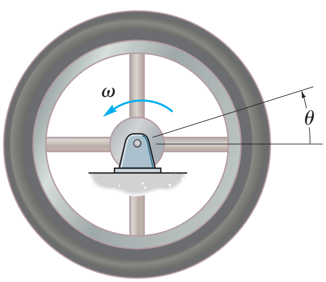
Prob. F16–1

F16–2. The flywheel rotates with an angular velocity of $\omega = (0.005\theta^2)\ \text{rad/s}$, where θ is in radians. Determine the angular acceleration when the wheel has rotated 20 revolutions.



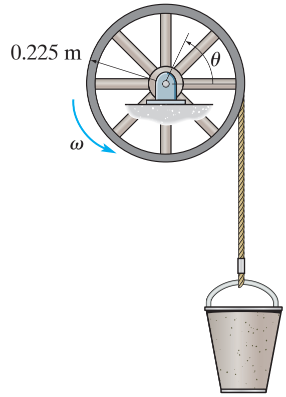
Prob. F16–2

F16–3. The flywheel rotates with an angular velocity of $\omega = (4\,\theta^{1/2})\ \text{rad/s}$, where θ is in radians. Determine the time it takes to achieve an angular velocity of $\omega = 150\ \text{rad/s}$. When $t = 0$, $\theta = 1\ \text{rad}$.



Prob. F16–3

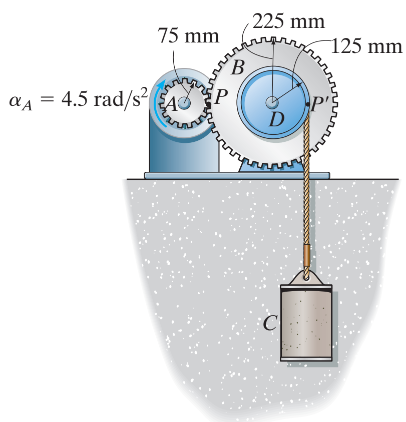
F16–4. The bucket is hoisted by the rope that wraps around a drum wheel. If the angular displacement of the wheel is $\theta = (0.5t^3 + 15t)\ \text{rad}$, where t is in seconds, determine the velocity and acceleration of the bucket when $t = 3\ \text{s}$.



Prob. F16–4

F16–5. A wheel has an angular acceleration of $\alpha = (0.5\,\theta)\ \text{rad/s}^2$, where θ is in radians. Determine the magnitude of the velocity and acceleration of a point P located on its rim after the wheel has rotated 2 revolutions. The wheel has a radius of 0.2 m and starts when $\theta = 0$, $\omega_0 = 2\ \text{rad/s}$.

F16–6. For a short period of time, the motor turns gear A with a constant angular acceleration of $\alpha_A = 4.5\ \text{rad/s}^2$, starting from rest. Determine the velocity of the cylinder and the distance it travels in three seconds. The cord is wrapped around the drum D which is rigidly attached to gear B.



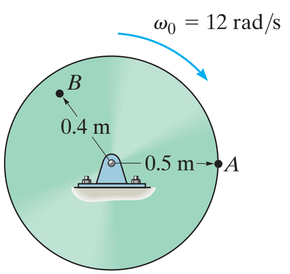
Prob. F16–6

PROBLEMS

16–1. The angular acceleration of the disk is defined by $\alpha = (3t^2 + 12)\ \text{rad/s}^2$, where t is in seconds. If the disk is originally rotating at $\omega_0 = 12\ \text{rad/s}$, determine the magnitude of the velocity and the n and t components of acceleration of point A on the disk when $t = 2$ s.

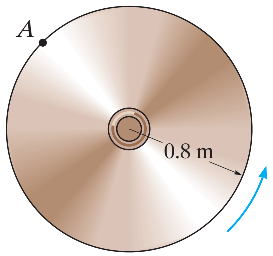
16–2. The disk is originally rotating at $\omega_0 = 12\ \text{rad/s}$. If it is subjected to a constant angular acceleration of $\alpha = 20\ \text{rad/s}^2$, determine the magnitudes of the velocity and the n and t components of acceleration of point A at the instant $t = 2$ s.

16–3. The disk is originally rotating at $\omega_0 = 12\ \text{rad/s}$. If it is subjected to a constant angular acceleration of $\alpha = 20\ \text{rad/s}^2$, determine the magnitudes of the velocity and the n and t components of acceleration of point B when the disk undergoes 2 revolutions.



Probs. 16–1/2/3

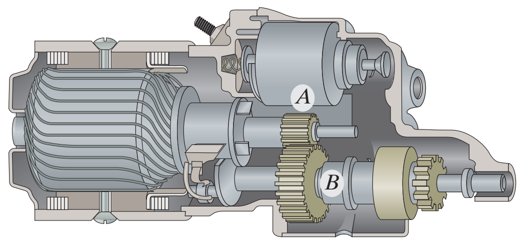
***16–4.** The angular velocity of the disk is defined by $\omega = (5t^2 + 2)\ \text{rad/s}$, where t is in seconds. Determine the magnitudes of the velocity and acceleration of point A on the disk when $t = 0.5$ s.



Prob. 16–4

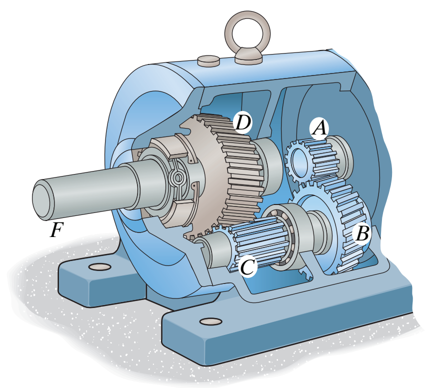
16–5. For a short time, gear A of the automobile starter rotates with an angular acceleration of $\alpha_A = (450t^2 + 60)\ \text{rad/s}^2$, where t is in seconds. Determine the angular velocity and angular displacement of gear B when $t = 2$ s, starting from rest. The radii of gears A and B are 10 mm and 25 mm, respectively.

16–6. For a short time, gear A of the automobile starter rotates with an angular acceleration of $\alpha_A = (50\omega^{1/2})\ \text{rad/s}^2$, where ω is in rad/s. Determine the angular velocity of gear B after gear A has rotated 50 rev, starting from rest. The radii of gears A and B are 10 mm and 25 mm, respectively.



Probs. 16–5/6

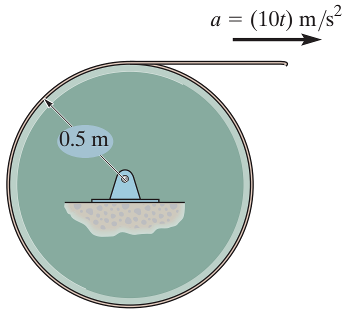
16–7. If gear A rotates with a constant angular acceleration of $\alpha_A = 90\ \text{rad/s}^2$, starting from rest, determine the time required for gear D to attain an angular velocity of 600 rpm. Also, find the number of revolutions of gear D to attain this angular velocity. Gears A, B, C, and D have radii of 15 mm, 50 mm, 25 mm, and 75 mm, respectively.



Prob. 16–7

***16–8.** A wheel has an initial angular velocity of 10 rev/s and a constant angular acceleration of 3 rad/s². Determine the number of revolutions required for the wheel to have an angular velocity of 20 rev/s. What time is required?

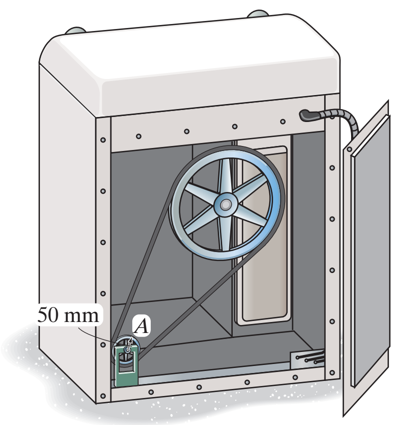
16–9. The cord, which is wrapped around the disk, is given an acceleration of $a = (10t)\ \mathrm{m/s^2}$, where t is in seconds. Starting from rest, determine the angular displacement, angular velocity, and angular acceleration of the disk when $t = 3$ s.



Prob. 16–9

16–10. The 50-mm-radius pulley A of the clothes dryer rotates with an angular acceleration of $\alpha_A = (27\theta_A^{1/2})\ \mathrm{rad/s^2}$, where θ_A is in radians. Determine its angular acceleration when $t = 1$ s, starting from rest.

16–11. If the 50-mm-radius motor pulley A of the clothes dryer rotates with an angular acceleration of $\alpha_A = (10 + 50t)\ \mathrm{rad/s^2}$, where t is in seconds, determine its angular velocity when $t = 3$ s, starting from rest.

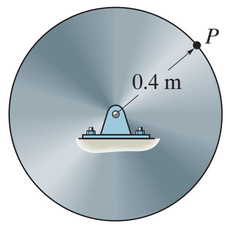


Probs. 16–10/11

***16–12.** The disk starts from rest and is given an angular acceleration $\alpha = (2t^2)\ \mathrm{rad/s^2}$, where t is in seconds. Determine the angular velocity of the disk and its angular displacement when $t = 4$ s.

16–13. The disk starts from rest and is given an angular acceleration $\alpha = (5\,t^{1/2})\ \mathrm{rad/s^2}$, where t is in seconds. Determine the magnitudes of the normal and tangential components of acceleration of a point P on the rim of the disk when $t = 2$ s.

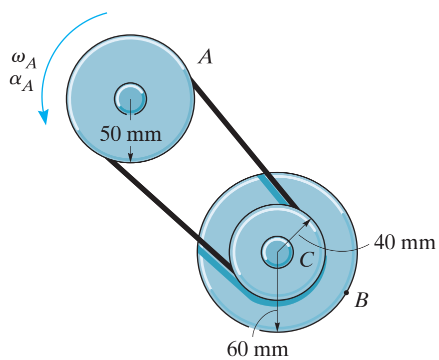
16–14. The disk starts at $\omega_0 = 1$ rad/s when $\theta = 0$, and is given an angular acceleration $\alpha = (0.3\theta)\ \mathrm{rad/s^2}$, where θ is in radians. Determine the magnitudes of the normal and tangential components of acceleration of a point P on the rim of the disk when $\theta = 1$ rev.



Probs. 16–12/13/14

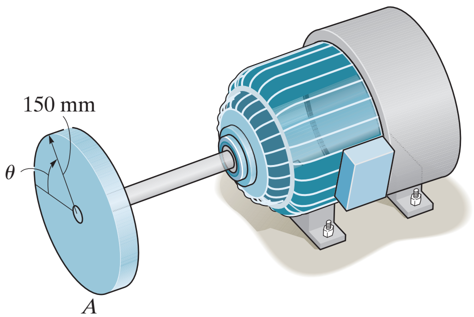
16–15. At the instant $\omega_A = 5$ rad/s, pulley A is given an angular acceleration $\alpha = (0.8\theta)\ \mathrm{rad/s^2}$, where θ is in radians. Determine the magnitude of acceleration of point B on pulley C when A rotates 3 revolutions. Pulley C has an inner hub which is fixed to its outer one and turns with it.

***16–16.** At the instant $\omega_A = 5$ rad/s, pulley A is given a constant angular acceleration $\alpha_A = 6\ \mathrm{rad/s^2}$. Determine the magnitude of acceleration of point B on pulley C when A rotates 2 revolutions. Pulley C has an inner hub which is fixed to its outer one and turns with it.



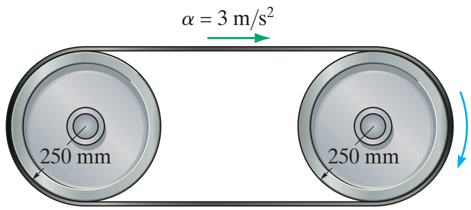
Probs. 16–15/16

16–17. The motor turns the disk with an angular velocity of $\omega = (5t^2 + 3t)$ rad/s, where t is in seconds. Determine the magnitudes of the velocity and the n and t components of acceleration of the point A on the disk when $t = 3$ s.



Prob. 16–17

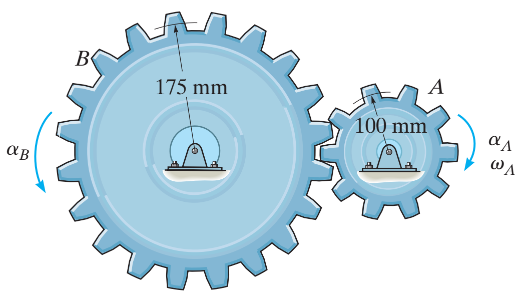
***16–20.** At the instant shown, the horizontal portion of the belt has an angular acceleration of 3 m/s^2, while points in contact with the outer edge of the pulleys have an acceleration magnitude of 5 m/s^2. If the belt does not slip on the pulleys, determine the belt's speed due to the motion.



Prob. 16–20

16–18. A motor gives gear A an angular acceleration of $\alpha_A = (2 + 0.006\,\theta^2)$ rad/s^2, where θ is in radians. If this gear is initially turning at $\omega_A = 15$ rad/s, determine the angular velocity of gear B after A undergoes an angular displacement of 10 rev.

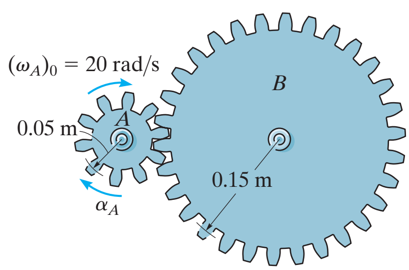
16–19. A motor gives gear A an angular acceleration of $\alpha_A = (2t^3)$ rad/s^2, where t is in seconds. If this gear is initially turning at $\omega_A = 15$ rad/s, determine the angular velocity of gear B when $t = 3$ s.



Probs. 16–18/19

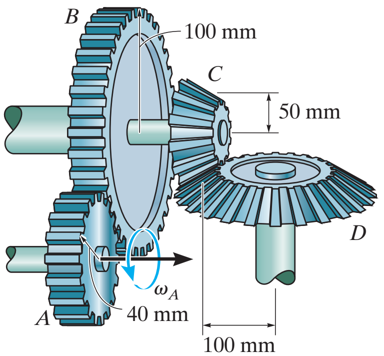
16–21. A motor gives gear A an angular acceleration of $\alpha_A = (4t^3)$ rad/s^2, where t is in seconds. If this gear is initially turning at $(\omega_A)_0 = 20$ rad/s, determine the angular velocity of gear B when $t = 2$ s.

16



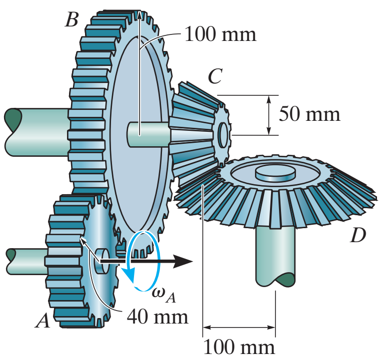
Prob. 16–21

16–22. If the motor turns gear A with an angular acceleration of $\alpha_A = 3\ \text{rad/s}^2$ when the angular velocity is $\omega_A = 60\ \text{rad/s}$, determine the angular acceleration and angular velocity of gear D.



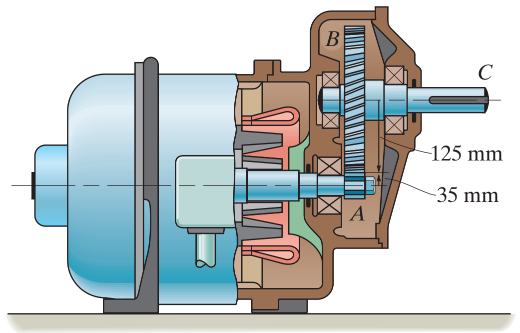
Prob. 16–22

16–23. If the motor turns gear A with an angular acceleration of $\alpha_A = 2\ \text{rad/s}^2$ when the angular velocity is $\omega_A = 20\ \text{rad/s}$, determine the angular acceleration and angular velocity of gear D.



Prob. 16–23

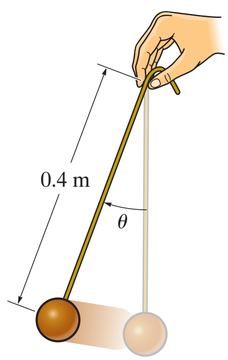
***16–24.** The pinion gear A on the motor shaft is given a constant angular acceleration $\alpha = 3\ \text{rad/s}^2$. If the gears A and B have the dimensions shown, determine the angular velocity and angular displacement of the output shaft C when $t = 2$ s, starting from rest. The shaft is fixed to B and turns with it.



Prob. 16–24

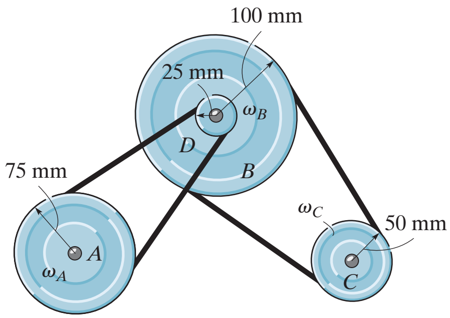
16–25. A pendulum has a swinging motion such that $\theta = [-0.101 \sin(4.95t) + 0.3 \cos(4.95t)]$ rad, where t is in seconds, and the arguments for the sine and cosine are in radians. Determine the magnitudes of the velocity and acceleration of the bob when $\theta = 0°$.

16–26. Determine the maximum angular displacement of the pendulum in Prob. 16–25.



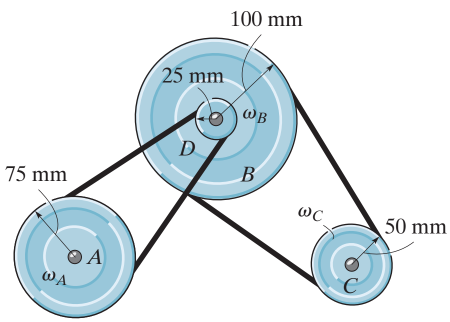
Probs. 16–25/26

16–27. The power of a bus engine is transmitted using the belt-and-pulley arrangement shown. If the engine turns pulley A at $\omega_A = (20t + 40)$ rad/s, where t is in seconds, determine the angular velocities of the generator pulley B and the air-conditioning pulley C when $t = 3$ s. The hub at D is rigidly *connected* to B and turns with it.



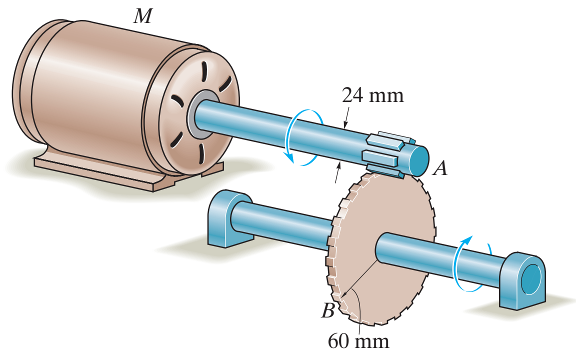
Prob. 16–27

*__16–28.__ The power of a bus engine is transmitted using the belt-and-pulley arrangement shown. If the engine turns pulley A at $\omega_A = 60$ rad/s, determine the angular velocities of the generator pulley B and the air-conditioning pulley C. The hub at D is rigidly *connected* to B and turns with it.



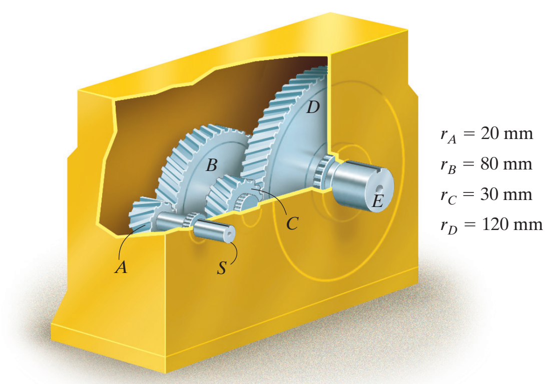
Prob. 16–28

16–29. Due to an increase in power, the motor M rotates the shaft A with an angular acceleration of $\alpha = (0.06\theta^2)$ rad/s^2, where θ is in radians. If the shaft is initially turning at $\omega_0 = 50$ rad/s, determine the angular velocity of gear B after the shaft undergoes an angular displacement $\Delta\theta = 10$ rev.



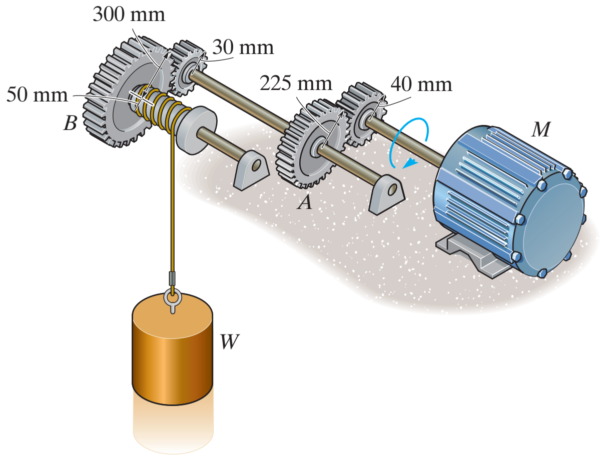
Prob. 16–29

16–30. Morse Industrial manufactures the speed reducer shown. If a motor drives the gear shaft S with an angular acceleration of $\alpha = (0.4e^t)$ rad/s^2, where t is in seconds, determine the angular velocity of shaft E when $t = 2$ s after starting from rest. The radius of each gear is listed in the figure. Note that gears B and C are fixed connected to the same shaft.



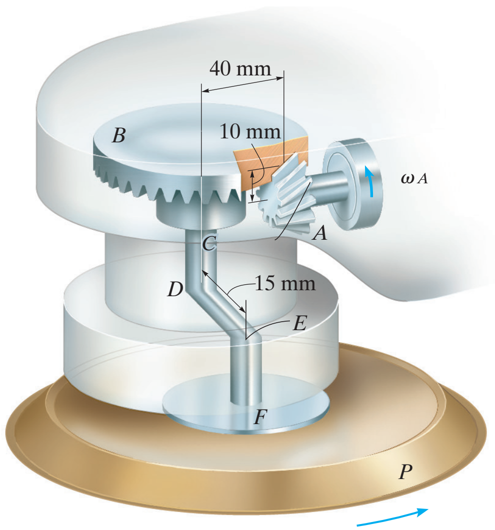
Prob. 16–30

16–31. Determine the distance the load W is lifted in $t = 5$ s using the hoist. The shaft of the motor M turns with an angular velocity $\omega = 100(4 + t)$ rad/s, where t is in seconds.



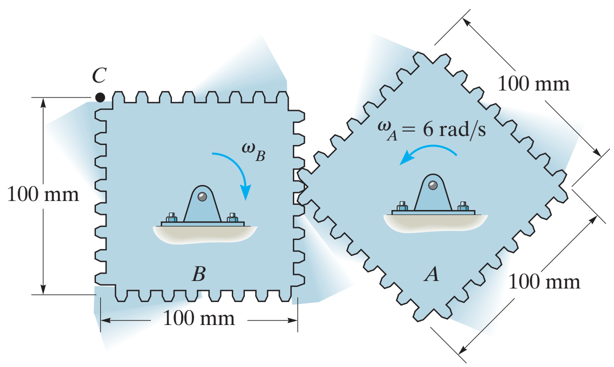
Prob. 16–31

16–33. For a short time a motor of the random-orbit sander drives the gear A with an angular velocity of $\omega_A = 40(t^3 + 6t)$ rad/s, where t is in seconds. This gear is connected to gear B, which is fixed connected to the shaft CD. The end of this shaft is connected to the eccentric spindle EF and pad P, which causes the pad to orbit around shaft CD at a radius of 15 mm. Determine the magnitudes of the velocity and the tangential and normal components of acceleration of the spindle EF when $t = 2$ s after starting from rest.



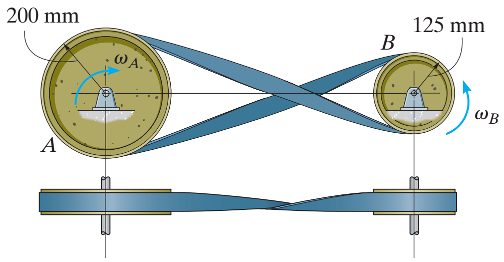
Prob. 16–33

***16–32.** At the instant shown, gear A is rotating with a constant angular velocity of $\omega_A = 6$ rad/s. Determine the largest angular velocity of gear B and the maximum speed of point C.



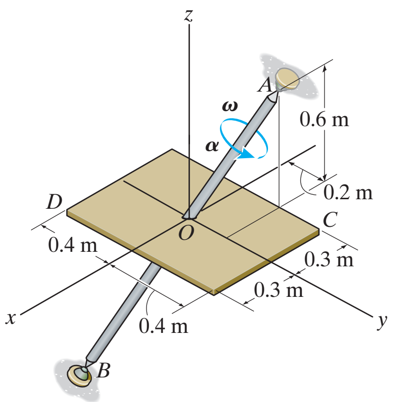
Prob. 16–32

16–34. The driving belt is twisted so that pulley B rotates in the opposite direction to that of drive wheel A. If the angular displacement of A is $\theta_A = (5t^3 + 10t^2)$ rad, where t is in seconds, determine the angular velocity and angular acceleration of B when $t = 3$ s.



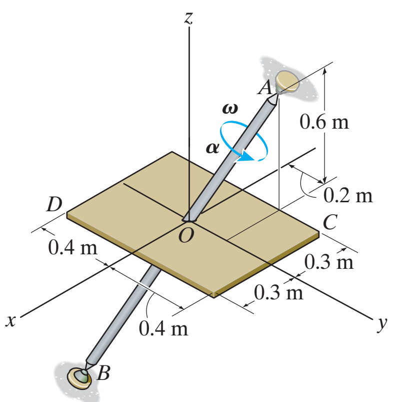
Prob. 16–34

16–35. At the instant shown, the shaft and plate rotates with an angular velocity of $\omega = 14$ rad/s and angular acceleration of $\alpha = 7$ rad/s^2. Determine the velocity and acceleration of point D located on the corner of the plate at this instant. Express the result in Cartesian vector form.



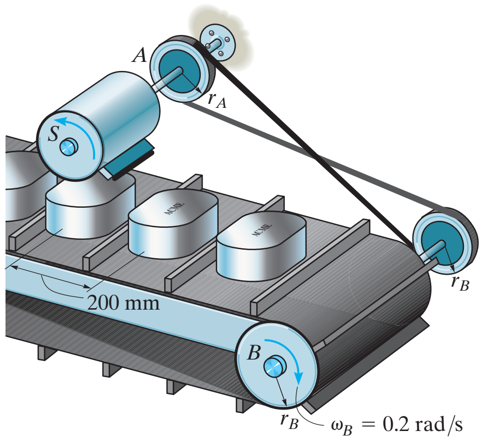
Prob. 16–35

***16–36.** If the shaft and plate rotates with a constant angular velocity of $\omega = 14$ rad/s, determine the velocity and acceleration of point C located on the corner of the plate at the instant shown. Express the result in Cartesian vector form.



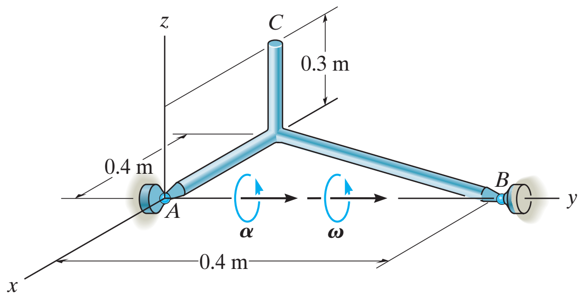
Prob. 16–36

16–37. A stamp S, located on the revolving drum, is used to label canisters. If the canisters are centered 200 mm apart on the conveyor, determine the radius r_A of the driving wheel A and the radius r_B of the conveyor belt drum so that for each revolution of the stamp it marks the top of a canister. How many canisters are marked per minute if the drum at B is rotating at $\omega_B = 0.2$ rad/s? Note that the driving belt is twisted as it passes between the wheels.



Prob. 16–37

16–38. The rod assembly is supported by ball-and-socket joints at A and B. At the instant shown it is rotating about the y axis with an angular velocity $\omega = 5$ rad/s and has an angular acceleration $\alpha = 8$ rad/s^2. Determine the magnitudes of the velocity and acceleration of point C at this instant. Solve the problem using Cartesian vectors and Eqs. 16–9 and 16–13.



Prob. 16–38

*16.4 ABSOLUTE MOTION ANALYSIS

The dumping bin on the truck rotates about a fixed axis passing through the pin at A. It is operated by the extension of the hydraulic cylinder BC. The angular position of the bin can be specified using the angular position coordinate θ, and the position of point C on the bin is specified using the rectilinear position coordinate s. Since a and b are fixed lengths, then the two coordinates can be related by the cosine law, $s = \sqrt{a^2 + b^2 - 2ab\cos\theta}$. The time derivative of this equation relates the speed at which the hydraulic cylinder extends to the angular velocity of the bin.

16

A body subjected to general plane motion undergoes a simultaneous translation and rotation. If the body is represented by a thin slab, the slab translates in the plane of the slab and rotates about an axis perpendicular to this plane. The motion can be completely specified by knowing *both* the angular rotation of a line fixed in the body and the motion of a point on the body. One way to relate these motions is to use a rectilinear position coordinate s to locate the point along its path and an angular position coordinate θ to specify the orientation of the line. The two coordinates are then related using the geometry of the problem. By direct application of the time-differential equations $v = ds/dt$, $a = dv/dt$, $\omega = d\theta/dt$, and $\alpha = d\omega/dt$, the motion of the point and the angular motion of the line can then be related. This procedure is similar to that used to solve dependent motion problems involving pulleys, Sec. 12.9. In some cases, this same procedure may be used to relate the motion of one body to the motion of a connected body.

PROCEDURE FOR ANALYSIS

The velocity and acceleration of a point P undergoing rectilinear motion can be related to the angular velocity and angular acceleration of a line contained within a body using the following procedure.

Position Coordinate Equation.

- Locate point P on the body using a position coordinate s, which is measured from a *fixed origin* and is *directed along the straight-line path of motion* of point P.
- Measure from a fixed reference line the angular position θ of a line lying in the body.
- From the dimensions of the body, relate s to θ, $s = f(\theta)$, using geometry and/or trigonometry.

Time Derivatives.

- Take the first time derivative of $s = f(\theta)$ to get a relationship between v and ω.
- Take the second time derivative to get a relationship between a and α.
- In each case the chain rule must be used when taking the time derivatives of the position coordinate equation. See Appendix C.

EXAMPLE 16.3

The end of rod R shown in Fig. 16–8 maintains contact with the cam by means of a spring. If the cam rotates about an axis passing through point O with an angular acceleration $\boldsymbol{\alpha}$ and angular velocity $\boldsymbol{\omega}$, determine the velocity and acceleration of the rod when the cam is in the arbitrary position θ.

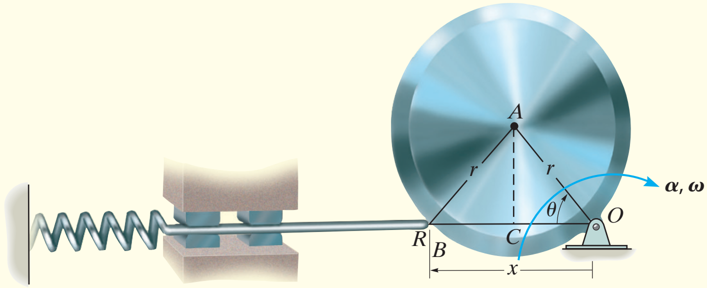


Fig. 16–8

SOLUTION

Position Coordinate Equation. Coordinates θ and x are chosen in order to relate the *rotational motion* of the line segment OA on the cam to the *rectilinear translation* of the rod. These coordinates are measured from the *fixed point O* and can be related to each other using trigonometry. Since $OC = CB = r\cos\theta$, Fig. 16–8, then

$$x = 2r\cos\theta$$

Time Derivatives. Using the chain rule, we have

$$\frac{dx}{dt} = -2r(\sin\theta)\frac{d\theta}{dt}$$

$$v = -2r\omega\sin\theta \qquad \textit{Ans.}$$

$$\frac{dv}{dt} = -2r\left(\frac{d\omega}{dt}\right)\sin\theta - 2r\omega(\cos\theta)\frac{d\theta}{dt}$$

$$a = -2r(\alpha\sin\theta + \omega^2\cos\theta) \qquad \textit{Ans.}$$

NOTE: The negative signs indicate that v and a are opposite to the direction of positive x. This seems reasonable if you visualize the motion.

16

EXAMPLE 16.4

At a given instant, the cylinder of radius r, shown in Fig. 16–9, has an angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$. Determine the velocity and acceleration of its center G if the cylinder rolls without slipping.

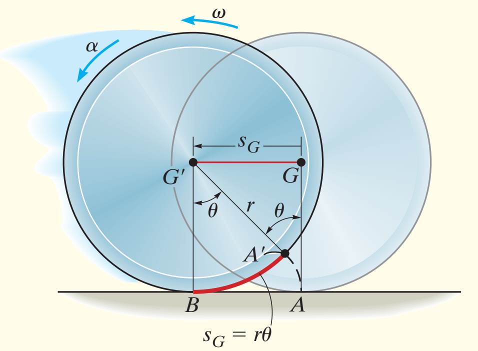


Fig. 16–9

SOLUTION

Position Coordinate Equation. The cylinder undergoes general plane motion since it simultaneously translates and rotates. By inspection, point G moves in a *straight line* to the left, from G to G', as the cylinder rolls, Fig. 16–9. Consequently its new position G' will be specified by the *horizontal* position coordinate s_G. Also, as the cylinder rolls (without slipping), the length of the arc $A'B$ on the rim, which was in contact with the ground from A to B, is equivalent to the length s_G. Consequently, the motion requires the radial line GA to rotate θ to the position $G'A'$. Since the arc $A'B = r\theta$, then G travels a distance

$$s_G = r\theta$$

Time Derivatives. Taking successive time derivatives of this equation, realizing that r is constant, $\omega = d\theta/dt$, and $\alpha = d\omega/dt$, gives the necessary relationships:

$$s_G = r\theta$$

$$v_G = r\omega \qquad \textit{Ans.}$$

$$a_G = r\alpha \qquad \textit{Ans.}$$

NOTE: Remember that these relationships are valid only if the cylinder (disk, wheel, ball, etc.) rolls *without* slipping.

16

EXAMPLE 16.5

The window in Fig. 16–10 is opened using a hydraulic cylinder AB. If the cylinder extends at a constant rate of 0.5 m/s, determine the angular velocity and angular acceleration of the window at the instant $\theta = 30°$.

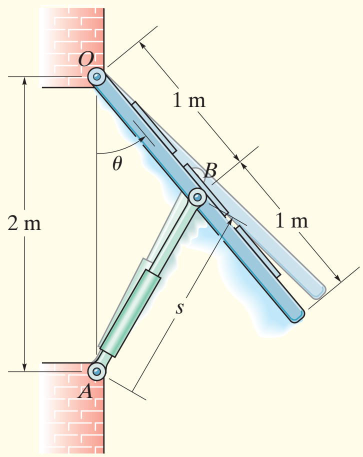


Fig. 16–10

SOLUTION

Position Coordinate Equation. The angular motion of the window can be obtained using the coordinate θ, whereas the extension or motion *along the hydraulic cylinder* is defined using the coordinate s. These coordinates can be related using the law of cosines, namely,

$$s^2 = (2\text{ m})^2 + (1\text{ m})^2 - 2(2\text{ m})(1\text{ m})\cos\theta$$

$$s^2 = 5 - 4\cos\theta \tag{1}$$

When $\theta = 30°$,

$$s = 1.239\text{ m}$$

Time Derivatives. Taking the time derivatives of Eq. 1, we have

$$2s\frac{ds}{dt} = 0 - 4(-\sin\theta)\frac{d\theta}{dt}$$

$$s(v_s) = 2(\sin\theta)\omega \tag{2}$$

Since $v_s = 0.5$ m/s, then at $\theta = 30°$,

$$(1.239\text{ m})(0.5\text{ m/s}) = 2\sin 30°\omega$$

$$\omega = 0.6197\text{ rad/s} = 0.620\text{ rad/s} \qquad \textit{Ans.}$$

Taking the time derivative of Eq. 2 yields

$$\frac{ds}{dt}v_s + s\frac{dv_s}{dt} = 2(\cos\theta)\frac{d\theta}{dt}\omega + 2(\sin\theta)\frac{d\omega}{dt}$$

$$v_s^2 + sa_s = 2(\cos\theta)\omega^2 + 2(\sin\theta)\alpha$$

Since $a_s = dv_s/dt = 0$, then

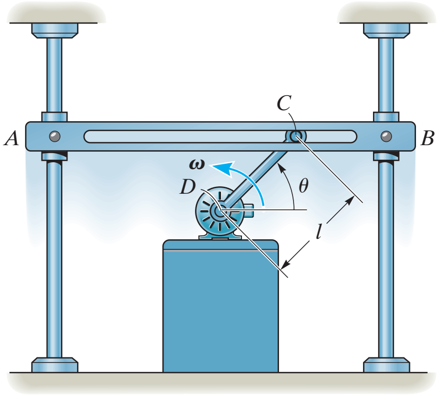
$$(0.5\text{ m/s})^2 + 0 = 2\cos 30°(0.6197\text{ rad/s})^2 + 2\sin 30°\alpha$$

$$\alpha = -0.415\text{ rad/s}^2 \qquad \textit{Ans.}$$

Because the result is negative, it indicates the window has an angular deceleration.

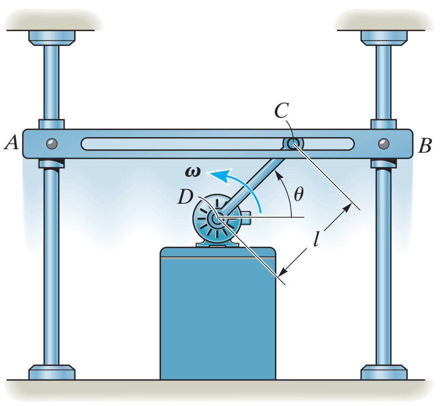
PROBLEMS

16–39. The bar DC rotates uniformly about the shaft at D with an angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$. Determine the velocity and acceleration of the bar AB, which is confined by the guides to move vertically.



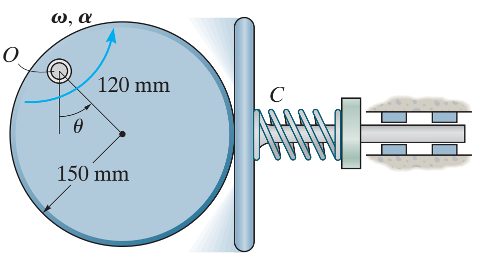
Prob. 16–39

***16–40.** The bar DC rotates uniformly about the shaft at D with a constant angular velocity $\boldsymbol{\omega}$. Determine the velocity and acceleration of the bar AB, which is confined by the guides to move vertically.



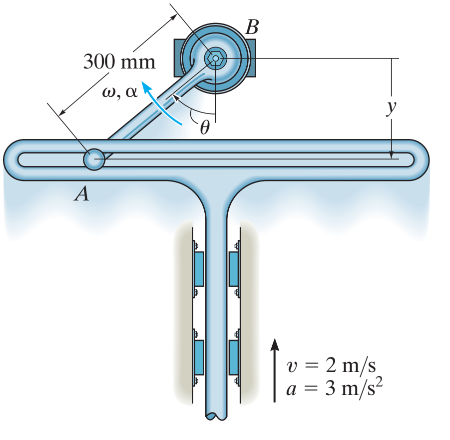
Prob. 16–40

16–41. Determine the velocity and acceleration of the plate at the instant $\theta = 30°$, if at this instant the circular cam is rotating about the fixed point O with an angular velocity $\omega = 4$ rad/s and an angular acceleration $\alpha = 2$ rad/s^2.



Prob. 16–41

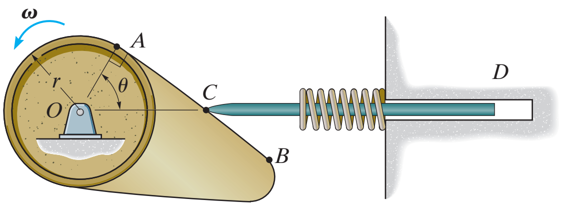
16–42. At the instant $\theta = 50°$, the slotted guide is moving upward with an acceleration of 3 m/s^2 and a velocity of 2 m/s. Determine the angular acceleration and angular velocity of link AB at this instant. *Note:* The upward motion of the guide is in the negative y direction.



Prob. 16–42

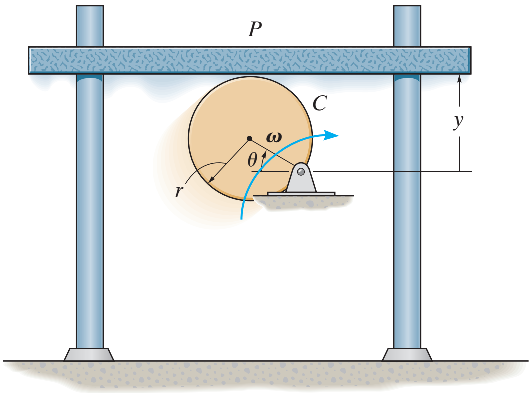
16

16–43. Determine the velocity and acceleration of the follower rod CD as a function of θ when the contact between the cam and follower is along the straight region AB on the face of the cam. The cam rotates with a constant counterclockwise angular velocity $\boldsymbol{\omega}$.



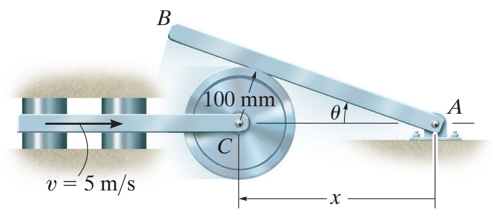
Prob. 16–43

***16–44.** Determine the velocity and acceleration of platform P as a function of the angle θ of cam C if the cam rotates with a constant angular velocity $\boldsymbol{\omega}$. The pin connection does not cause interference with the motion of P on C. The platform is constrained to move vertically by the smooth vertical guides.



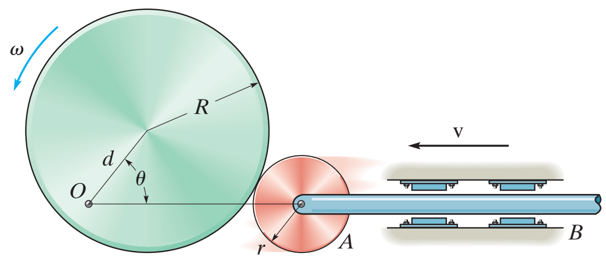
Prob. 16–44

16–45. Determine the angular velocity of rod AB when $\theta = 30°$. The shaft and the center of the roller C move forward at a constant rate of $v = 5$ m/s.



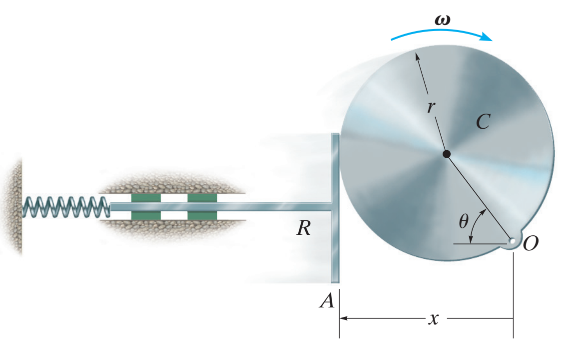
Prob. 16–45

16–46. The circular cam rotates about the fixed point O with a constant angular velocity $\boldsymbol{\omega}$. Determine the velocity v of the follower rod AB as a function of θ.



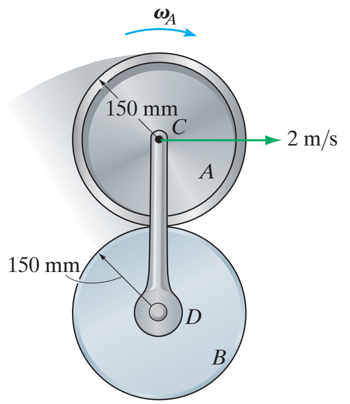
Prob. 16–46

16–47. Determine the velocity of the rod R for any angle θ of cam C as the cam rotates with a constant angular velocity $\boldsymbol{\omega}$. The pin connection at O does not cause an interference with the motion of plate A on C.



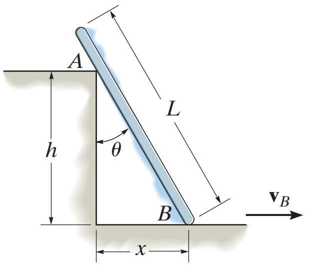
Prob. 16–47

***16–48.** Wheel A rolls without slipping over the surface of the *fixed* cylinder B. Determine the angular velocity of A if its center C has a speed of 2 m/s. How many revolutions will A make about its center after link DC completes one revolution?



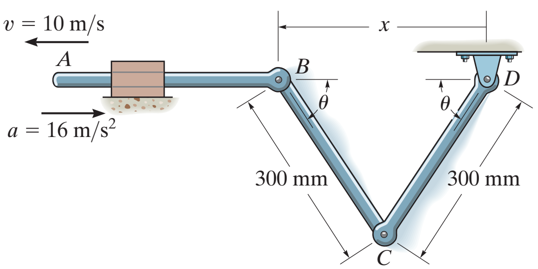
Prob. 16–48

16–50. The bar remains in contact with the floor and with point A. If point B moves to the right with a constant velocity $\mathbf{v}_B$, determine the angular velocity and angular acceleration of the bar as a function of x.



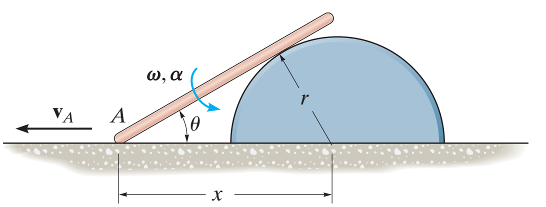
Prob. 16–50

16–49. At the instant shown, $\theta = 60°$, and rod AB is subjected to a deceleration of 16 m/s^2 when the velocity is 10 m/s. Determine the angular velocity and angular acceleration of link CD at this instant.



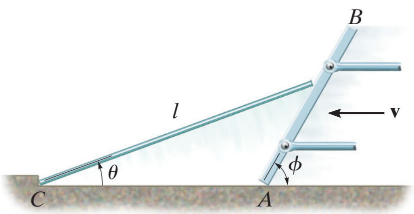
Prob. 16–49

16–51. End A of the bar moves to the left with a constant velocity $\mathbf{v}_A$. Determine the angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$ of the bar as a function of its position x.



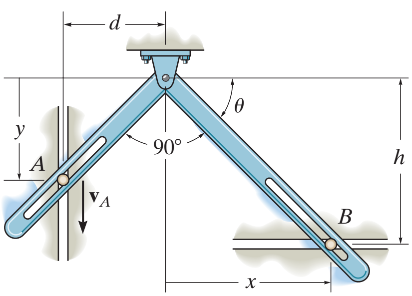
Prob. 16–51

***16–52.** The inclined plate moves to the left with a constant velocity **v**. Determine the angular velocity and angular acceleration of the slender rod of length l. The rod pivots about the step at C as its slides on the plate.



Prob. 16–52

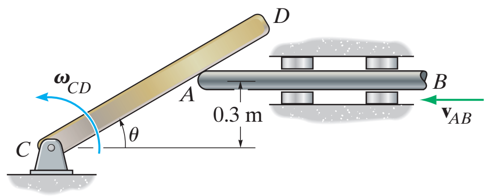
16–53. The pins at A and B are confined to move in the vertical and horizontal tracks. If the slotted arm is causing A to move downward at $\mathbf{v}_A$, determine the velocity of B at the instant shown.



Prob. 16–53

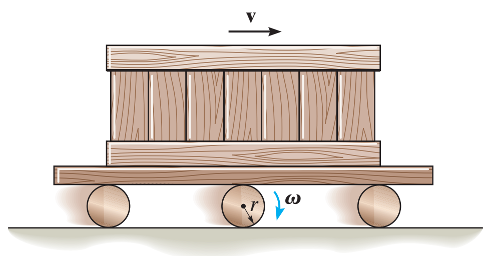
16–54. Determine the angular velocity of rod CD at the instant $\theta = 30°$. Rod AB moves to the left at a constant rate $v_{AB} = 5\,\text{m/s}$.

16–55. Determine the angular acceleration of rod CD at the instant $\theta = 30°$. Rod AB has zero velocity, i.e., $v_{AB} = 0$, and an acceleration of $a_{AB} = 2\ \text{m/s}^2$ to the right when $\theta = 30°$.

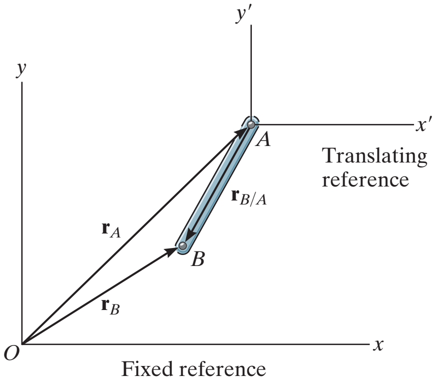


Probs. 16–54/55

***16–56.** The crate is transported on a platform which rests on rollers, each having a radius r. If the rollers do not slip, determine their angular velocity if the platform moves forward with a velocity **v**.



Prob. 16–56



(a)

16.5 RELATIVE-MOTION ANALYSIS: VELOCITY

The general plane motion of a rigid body can be described as a combination of translation and rotation. To view these "component" motions separately we will use a relative-motion analysis involving two sets of coordinate axes. The x, y coordinate system is fixed and measures the *absolute* position of two points A and B on the body, here represented as a bar, Fig. 16–11*a*. The origin of the x', y' coordinate system will be attached to the selected "base point" A, which generally has a known motion. The axes of this coordinate system *translate* with respect to the fixed frame but do not rotate with the bar.

Position. The position vector $\mathbf{r}_A$ in Fig. 16–11*a* specifies the location of the "base point" A, and the relative-position vector $\mathbf{r}_{B/A}$ locates point B with respect to point A. By vector addition, the position of B is then

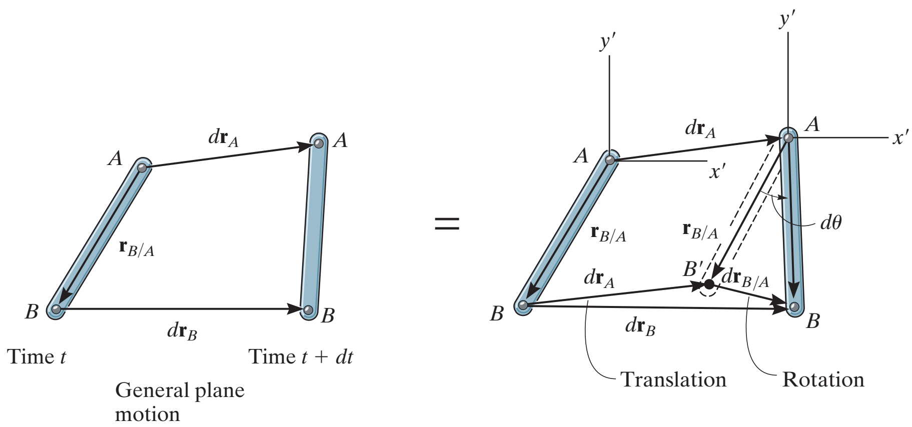
$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$$

Displacement. During an instant of time dt, points A and B undergo displacements $d\mathbf{r}_A$ and $d\mathbf{r}_B$ as shown in Fig. 16–11*b*. If we consider the general plane motion by its component parts then the entire bar first translates by an amount $d\mathbf{r}_A$ so that A, the base point, moves to its *final* position and point B moves to B', Fig. 16–11*c*. The bar is then rotated about A by an amount $d\theta$ so that B' undergoes the relative displacement $d\mathbf{r}_{B/A}$ and thus moves to its final position B. Due to the rotation about A, $dr_{B/A} = r_{B/A}\, d\theta$, and the displacement of B is

$$d\mathbf{r}_B = d\mathbf{r}_A + d\mathbf{r}_{B/A}$$

due to rotation about A (for $d\mathbf{r}_{B/A}$)
due to translation of A (for $d\mathbf{r}_A$)
due to translation and rotation (for $d\mathbf{r}_B$)

16



(b) (c)

Fig. 16–11

Velocity. To determine the relation between the velocities of points A and B, it is necessary to take the time derivative of the position equation, or simply divide the displacement equation by dt. This yields

$$\frac{d\mathbf{r}_B}{dt} = \frac{d\mathbf{r}_A}{dt} + \frac{d\mathbf{r}_{B/A}}{dt}$$

The terms $d\mathbf{r}_B/dt = \mathbf{v}_B$ and $d\mathbf{r}_A/dt = \mathbf{v}_A$ are measured with respect to the fixed x, y axes and represent the absolute velocities of points A and B, respectively. Since the relative displacement is caused by a rotation, the magnitude of the third term is $dr_{B/A}/dt = r_{B/A}\, d\theta/dt = r_{B/A}\dot{\theta} = r_{B/A}\omega$, where ω is the angular velocity of the body at the instant considered. We will denote this term as the relative velocity $\mathbf{v}_{B/A}$, since it represents the velocity of B with respect to A as measured by an observer fixed to the translating x', y' axes. In other words, *the bar appears to move as if it were rotating with an angular velocity* $\boldsymbol{\omega}$ *about the* z' *axis passing through* A. Consequently, $\mathbf{v}_{B/A}$ has a magnitude of $v_{B/A} = \omega r_{B/A}$ and a direction which is perpendicular to $\mathbf{r}_{B/A}$. We therefore have

$$\boxed{\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}} \qquad (16\text{–}15)$$

where

$\mathbf{v}_B$ = velocity of point B
$\mathbf{v}_A$ = velocity of the base point A
$\mathbf{v}_{B/A}$ = velocity of B with respect to A

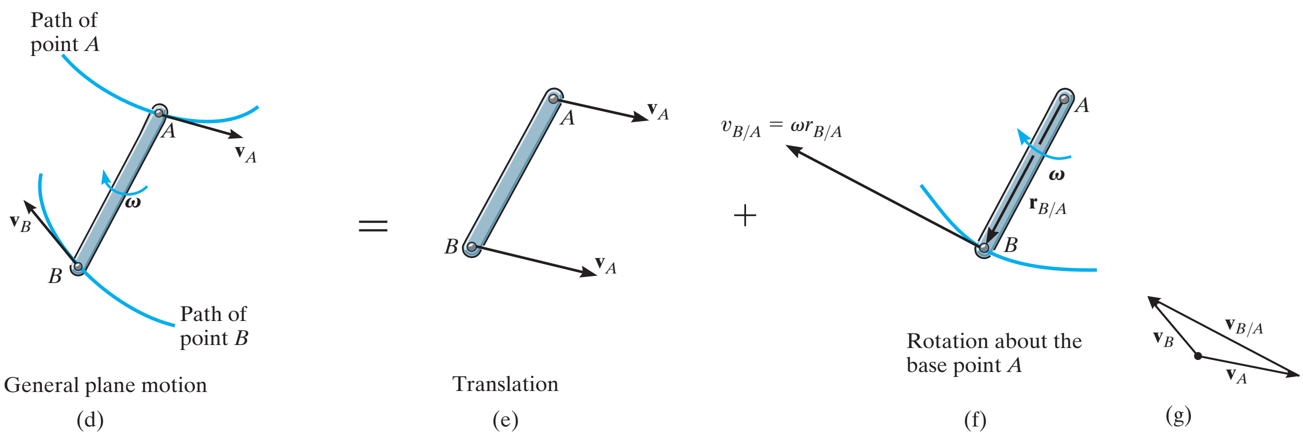


Fig. 16–11(cont.)

What the equation $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$ states is that the velocity of B, Fig. 16–11*d*, is determined by considering the entire bar to translate with a velocity of $\mathbf{v}_A$, Fig. 16–11*e*, and rotate about A with an angular velocity $\boldsymbol{\omega}$, Fig. 16–11*f*. Vector addition of these two effects, applied to B, yields $\mathbf{v}_B$, as shown in Fig. 16–11*g*.

Since the relative velocity $\mathbf{v}_{B/A}$ represents the effect of *circular motion* about A, this term can be expressed by the cross product $\mathbf{v}_{B/A} = \boldsymbol{\omega} \times \mathbf{r}_{B/A}$, Eq. 16–9. Hence, for application using Cartesian vector analysis, we can also write Eq. 16–15 as

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A} \tag{16–16}$$

where

$\mathbf{v}_B$ = velocity of B
$\mathbf{v}_A$ = velocity of the base point A
$\boldsymbol{\omega}$ = angular velocity of the body
$\mathbf{r}_{B/A}$ = position vector directed from A to B

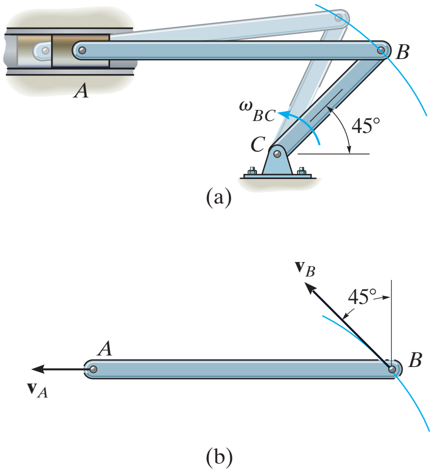


Fig. 16–12

When applying the velocity equation, points A and B should generally be selected as points on the body which are pin connected to other bodies, or as points in contact with adjacent bodies which have a *known motion*. For example, point A on link AB in Fig. 16–12*a* must move along a horizontal path, whereas point B moves on a circular path. The *directions* of $\mathbf{v}_A$ and $\mathbf{v}_B$ can therefore be established since they are always *tangent* to their paths of motion, Fig. 16–12*b*.

Rolling Without Slipping. Consider the case of a disk (wheel, ball, etc.) which rolls *without slipping,* Fig. 16–13*a*. If we select point A at the ground, then it (momentarily) has zero velocity since the ground does not move. Furthermore, the center of the disk G moves along a horizontal path so that $\mathbf{v}_G$ is horizontal. To obtain this velocity we will apply Eq. 16–16. From the kinematic diagram, Fig. 16–13*b*, we have

$$\mathbf{v}_G = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{G/A} = \mathbf{0} + (-\omega\mathbf{k}) \times (r\mathbf{j})$$

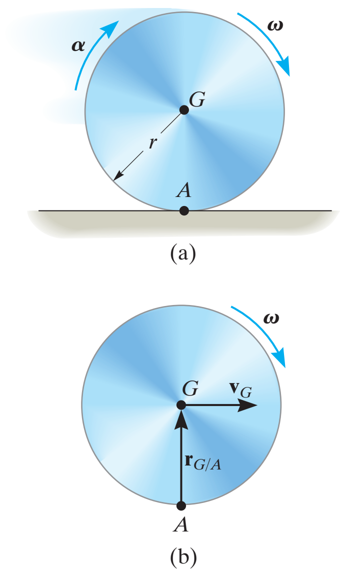


Fig. 16–13

So that

$$v_G = \omega r \tag{16–17}$$

This same result was obtained in Example 16–4.

PROCEDURE FOR ANALYSIS

The relative velocity equation can be applied either by using Cartesian vector analysis, or by writing the x and y scalar component equations directly. For application, it is suggested that the following procedure be used.

Vector Analysis

Kinematic Diagram.

- Establish the directions of the fixed x, y coordinates and draw a kinematic diagram of the body that shows the velocities $\mathbf{v}_A$, $\mathbf{v}_B$ of points A and B, the angular velocity $\boldsymbol{\omega}$, and the relative-position vector $\mathbf{r}_{B/A}$.
- If the magnitudes of $\mathbf{v}_A$, $\mathbf{v}_B$, or $\boldsymbol{\omega}$ are unknown, the sense of direction of these vectors can be assumed.

Velocity Equation.

- To apply $\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$, express the vectors in Cartesian vector form and substitute them into the equation. Evaluate the cross product and then equate the respective $\mathbf{i}$ and $\mathbf{j}$ components to obtain two scalar equations.
- If the solution yields a *negative* answer for an *unknown* magnitude, it indicates the sense of direction of the vector is opposite to that shown on the kinematic diagram.

Scalar Analysis

Kinematic Diagram.

- If the velocity equation is to be applied in scalar form, then the magnitude and direction of the relative velocity $\mathbf{v}_{B/A}$ must be established. Draw a kinematic diagram such as shown in Fig. 16–11*f*, which shows the relative motion. Since the body is considered to be "pinned" momentarily at the base point A, the magnitude of $\mathbf{v}_{B/A}$ is $v_{B/A} = \omega r_{B/A}$. The sense of direction of $\mathbf{v}_{B/A}$ is always perpendicular to $\mathbf{r}_{B/A}$ in accordance with the rotational motion $\boldsymbol{\omega}$ of the body.*

Velocity Equation.

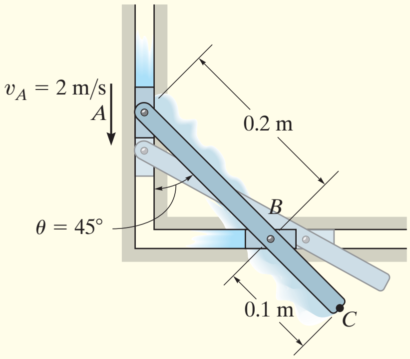
- Write Eq. 16–15 in symbolic form, $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$, and underneath each of the terms represent the vectors graphically by showing their magnitudes and directions. The scalar equations are determined from the x and y components of these vectors.

* The notation $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A(\text{pin})}$ may be helpful in recalling that A is "pinned."

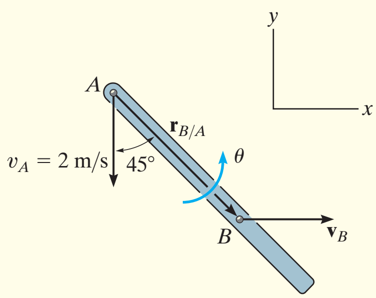
16

Refer to the companion website for Lecture Summary and Quiz videos.

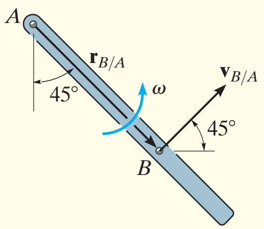
EXAMPLE 16.6



(a)



(b)



Relative motion

(c)

Fig. 16–14

The link shown in Fig. 16–14*a* is guided by two blocks at A and B, which move in the fixed slots. If the velocity of A is 2 m/s downward, determine the velocity of B at the instant $\theta = 45°$.

SOLUTION I (VECTOR ANALYSIS)

Kinematic Diagram. Since points A and B are restricted to move along the fixed slots and $\mathbf{v}_A$ is directed downward, then $\mathbf{v}_B$ must be directed horizontally to the right, Fig. 16–14*b*. This motion causes the link to rotate counterclockwise; that is, by the right-hand rule the angular velocity $\boldsymbol{\omega}$ is directed outward, perpendicular to the plane of motion.

Velocity Equation. Expressing each of the vectors in Fig. 16–14*b* in terms of their **i**, **j**, **k** components and applying Eq. 16–16 to A, the base point, and B, we have

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$
$$v_B\mathbf{i} = -2\mathbf{j} + [\omega\mathbf{k} \times (0.2 \sin 45°\mathbf{i} - 0.2 \cos 45°\mathbf{j})]$$
$$v_B\mathbf{i} = -2\mathbf{j} + 0.2\omega \sin 45°\mathbf{j} + 0.2\omega \cos 45°\mathbf{i}$$

Equating the **i** and **j** components gives

$$v_B = 0.2\omega \cos 45° \qquad (1)$$
$$0 = -2 + 0.2\omega \sin 45° \qquad (2)$$

Thus,

$$\omega = 14.1 \text{ rad/s} \circlearrowleft \qquad v_B = 2 \text{ m/s} \rightarrow \qquad \textit{Ans.}$$

SOLUTION II (SCALAR ANALYSIS)

The kinematic diagram of the relative "circular motion" which produces $\mathbf{v}_{B/A}$ is shown in Fig. 16–14*c*. Here $v_{B/A} = \omega(0.2 \text{ m})$.

Thus,

$$v_B = v_A + v_{B/A}$$

$$\begin{bmatrix} v_B \\ \rightarrow \end{bmatrix} = \begin{bmatrix} 2 \text{ m/s} \\ \downarrow \end{bmatrix} + \begin{bmatrix} \omega(0.2 \text{ m}) \\ \measuredangle 45° \end{bmatrix}$$

$(\overset{+}{\rightarrow})$ $$v_B = 0 + \omega(0.2) \cos 45°$$

$(+\uparrow)$ $$0 = -2 + \omega(0.2) \sin 45°$$

The solution produces the above Eqs. 1 and 2.

It should be emphasized that these results are *valid only* at the instant $\theta = 45°$. A recalculation for $\theta = 44°$ yields $v_B = 2.07$ m/s and $\omega = 14.4$ rad/s; whereas when $\theta = 46°$, $v_B = 1.93$ m/s and $\omega = 13.9$ rad/s, etc.

NOTE: Since v_A and ω are *known*, the velocity of any other point on the link can be determined. As an exercise, see if you can apply Eq. 16–15 or Eq. 16–16 to points A and C or to points B and C and show that when $\theta = 45°$, $v_C = 3.16$ m/s, directed at an angle of 18.4° up from the horizontal.

16

EXAMPLE 16.7

The cylinder shown in Fig. 16–15*a* rolls without slipping on the surface of a conveyor belt which is moving at 0.6 m/s. Determine the velocity of point A. The cylinder has a clockwise angular velocity $\omega = 15$ rad/s at the instant shown.

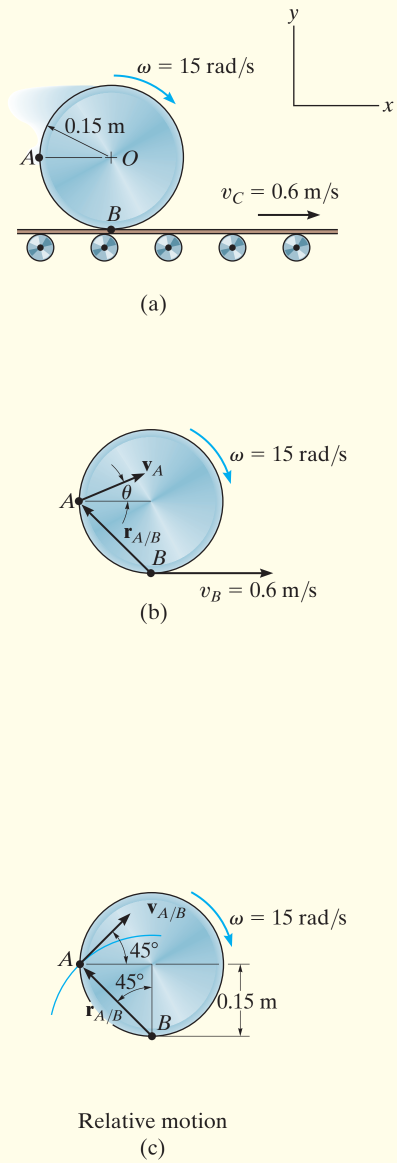


Fig. 16–15

SOLUTION I (VECTOR ANALYSIS)

Kinematic Diagram. Since no slipping occurs, point B on the cylinder has the same velocity as the conveyor, Fig. 16–15*b*. Also, the angular velocity of the cylinder is known, so we can apply the velocity equation to B, the base point, and A to determine $\mathbf{v}_A$.

Velocity Equation.

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{A/B}$$

$$(v_A)_x\mathbf{i} + (v_A)_y\mathbf{j} = 0.6\mathbf{i} + (-15\mathbf{k}) \times (-0.15\mathbf{i} + 0.15\mathbf{j})$$

$$(v_A)_x\mathbf{i} + (v_A)_y\mathbf{j} = 0.6\mathbf{i} + 2.25\mathbf{j} + 2.25\mathbf{i}$$

so that

$$(v_A)_x = 0.6 + 2.25 = 2.85 \text{ m/s} \quad (1)$$

$$(v_A)_y = 2.25 \text{ m/s} \quad (2)$$

Thus,

$$v_A = \sqrt{(2.85)^2 + (2.25)^2} = 3.63 \text{ m/s} \qquad \textit{Ans.}$$

$$\theta = \tan^{-1}\left(\frac{2.25}{2.85}\right) = 38.3^\circ \measuredangle \qquad \textit{Ans.}$$

SOLUTION II (SCALAR ANALYSIS)

As an alternative procedure, the scalar components of $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$ can be obtained directly. From the kinematic diagram showing the relative "circular" motion which produces $\mathbf{v}_{A/B}$, Fig. 16–15*c*, we have

$$v_{A/B} = \omega r_{A/B} = (15 \text{ rad/s})\left(\frac{0.15 \text{ m}}{\cos 45^\circ}\right) = 3.18 \text{ m/s}$$

Thus,

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

$$\begin{bmatrix}(v_A)_x \\ \rightarrow\end{bmatrix} + \begin{bmatrix}(v_A)_y \\ \uparrow\end{bmatrix} = \begin{bmatrix}0.6 \text{ m/s} \\ \rightarrow\end{bmatrix} + \begin{bmatrix}3.18 \text{ m/s} \\ \measuredangle 45^\circ\end{bmatrix}$$

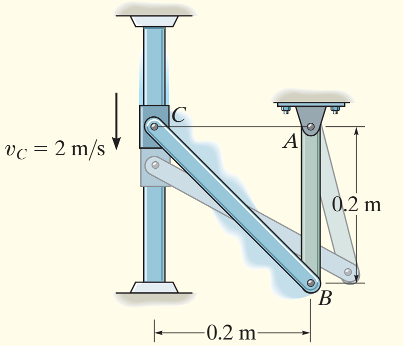
Equating the x and y components gives the same results as before, namely,

$$(\overset{+}{\rightarrow}) \qquad (v_A)_x = 0.6 + 3.18 \cos 45^\circ = 2.85 \text{ m/s}$$

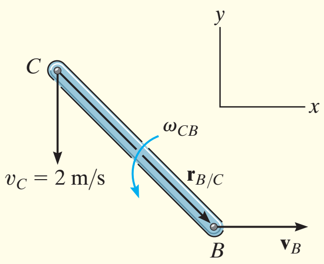
$$(+\uparrow) \qquad (v_A)_y = 0 \quad + 3.18 \sin 45^\circ = 2.25 \text{ m/s}$$

16

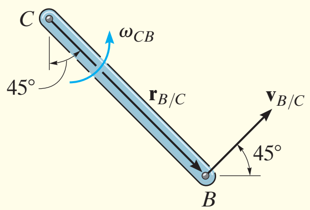
EXAMPLE 16.8



(a)

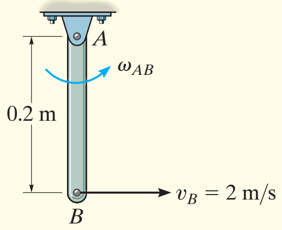


(b)



Relative motion

(c)



(d)

Fig. 16–16

Refer to the companion website for a self quiz of these Example problems.

The collar C in Fig. 16–16a is moving downward with a velocity of 2 m/s. Determine the angular velocity of CB at this instant.

SOLUTION I (VECTOR ANALYSIS)

Kinematic Diagram. The downward motion of C causes B to move to the right along a curved path. Also, CB and AB rotate counterclockwise.

Velocity Equation. Link CB (general plane motion): See Fig. 16–16b.

$$\mathbf{v}_B = \mathbf{v}_C + \boldsymbol{\omega}_{CB} \times \mathbf{r}_{B/C}$$

$$v_B\mathbf{i} = -2\mathbf{j} + \omega_{CB}\mathbf{k} \times (0.2\mathbf{i} - 0.2\mathbf{j})$$

$$v_B\mathbf{i} = -2\mathbf{j} + 0.2\omega_{CB}\mathbf{j} + 0.2\omega_{CB}\mathbf{i}$$

$$v_B = 0.2\omega_{CB} \qquad (1)$$

$$0 = -2 + 0.2\omega_{CB} \qquad (2)$$

$$\omega_{CB} = 10 \text{ rad/s} \circlearrowleft \qquad \textit{Ans.}$$

$$v_B = 2 \text{ m/s} \rightarrow$$

SOLUTION II (SCALAR ANALYSIS)

The scalar component equations of $\mathbf{v}_B = \mathbf{v}_C + \mathbf{v}_{B/C}$ can be obtained directly. The kinematic diagram in Fig. 16–16c shows the relative "circular" motion which produces $\mathbf{v}_{B/C}$. We have

$$\mathbf{v}_B = \mathbf{v}_C + \mathbf{v}_{B/C}$$

$$\begin{bmatrix} v_B \\ \rightarrow \end{bmatrix} = \begin{bmatrix} 2 \text{ m/s} \\ \downarrow \end{bmatrix} + \begin{bmatrix} \omega_{CB}(0.2\sqrt{2} \text{ m}) \\ \measuredangle 45° \end{bmatrix}$$

Resolving these vectors in the x and y directions yields

$$(\overset{+}{\rightarrow}) \qquad v_B = 0 + \omega_{CB}(0.2\sqrt{2}\cos 45°)$$

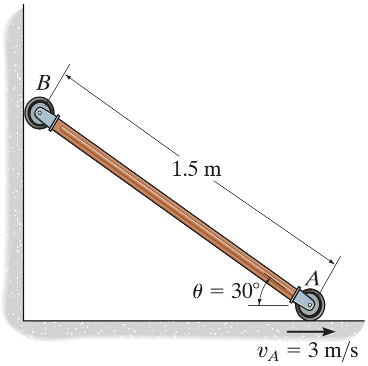
$$(+\uparrow) \qquad 0 = -2 + \omega_{CB}(0.2\sqrt{2}\sin 45°)$$

which is the same as Eqs. 1 and 2.

NOTE: Since link AB rotates about a fixed axis and v_B is known, Fig. 16–16d, its angular velocity is found from $v_B = \omega_{AB} r_{AB}$ or $2 \text{ m/s} = \omega_{AB}(0.2 \text{ m})$, $\omega_{AB} = 10$ rad/s.

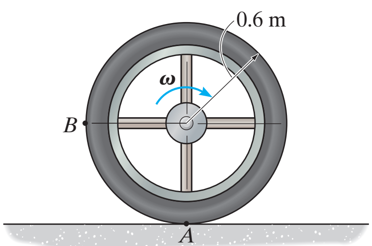
FUNDAMENTAL PROBLEMS

F16–7. If roller A moves to the right with a constant velocity of $v_A = 3\text{ m/s}$, determine the angular velocity of the link and the velocity of roller B at the instant $\theta = 30°$.



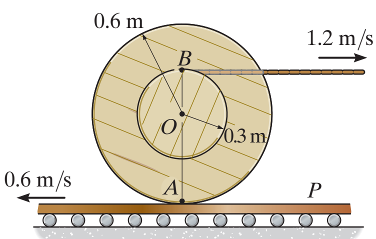
Prob. F16–7

F16–8. The wheel rolls without slipping with an angular velocity of $\omega = 10\text{ rad/s}$. Determine the magnitude of the velocity of point B at the instant shown.



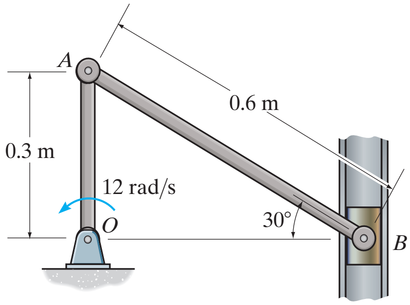
Prob. F16–8

F16–9. Determine the angular velocity of the spool. The cable wraps around the inner core, and the spool does not slip on the platform P.



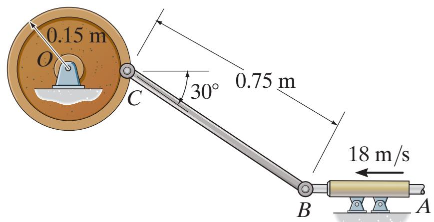
Prob. F16–9

F16–10. If crank OA rotates with an angular velocity of $\omega = 12\text{ rad/s}$, determine the velocity of piston B and the angular velocity of rod AB at the instant shown.



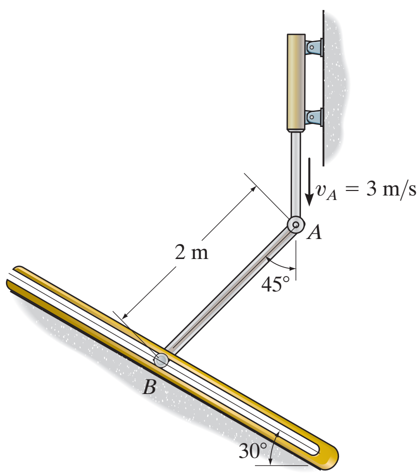
Prob. F16–10

F16–11. If rod AB slides along the horizontal collar with a velocity of 18 m/s, determine the angular velocity of link BC and the velocity of point C at the instant shown.



Prob. F16–11

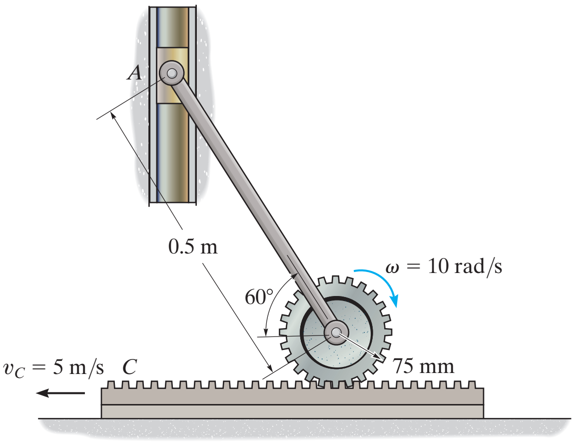
F16–12. End A of the link has a velocity of $v_A = 3\text{ m/s}$. Determine the velocity of the peg at B at this instant. The peg is constrained to move along the slot.



Prob. F16–12

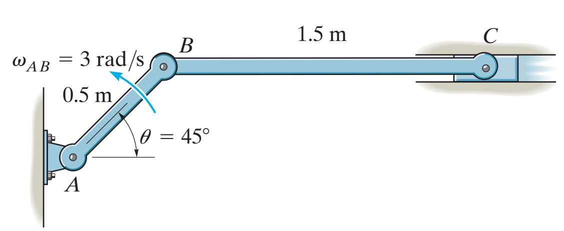
PROBLEMS

16–57. If the gear rotates with an angular velocity of $\omega = 10$ rad/s and the gear rack moves at $v_C = 5$ m/s, determine the velocity of the slider block A at the instant shown.



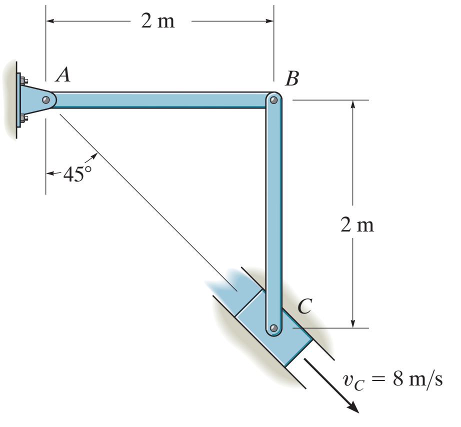
Prob. 16–57

16–58. The link AB has an angular velocity of 3 rad/s. Determine the velocity of block C and the angular velocity of link BC at the instant $\theta = 45°$. Also, sketch the position of link BC when $\theta = 60°$, 45°, and 30° to show its general plane motion.



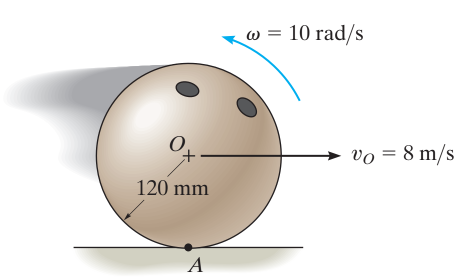
Prob. 16–58

16–59. The slider block C moves at 8 m/s down the inclined groove. Determine the angular velocities of links AB and BC, at the instant shown.



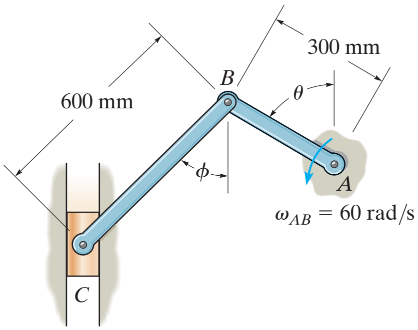
Prob. 16–59

***16–60.** A bowling ball is cast on the "alley" with a backspin of $\omega = 10$ rad/s while its center O has a forward velocity of $v_O = 8$ m/s. Determine the velocity of the contact point A in contact with the alley.



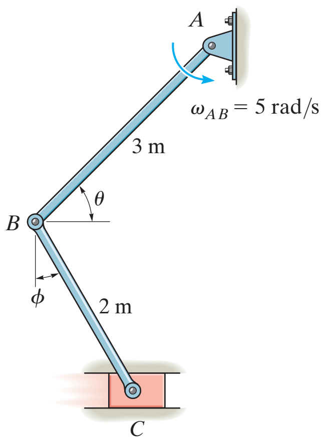
Prob. 16–60

16–61. Rod AB is rotating with an angular velocity of $\omega_{AB} = 60$ rad/s. Determine the velocity of the slider C at the instant $\theta = 60°$ and $\phi = 45°$. Also, sketch the position of bar BC when $\theta = 30°$, $60°$ and $90°$ to show its general plane motion.



Prob. 16–61

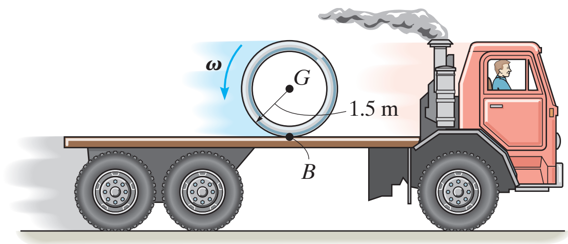
16–62. The angular velocity of link AB is $\omega_{AB} = 5$ rad/s. Determine the velocity of block C and the angular velocity of link BC at the instant $\theta = 45°$ and $\phi = 30°$. Also, sketch the position of link CB when $\theta = 45°$, $60°$, and $75°$ to show its general plane motion.



Prob. 16–62

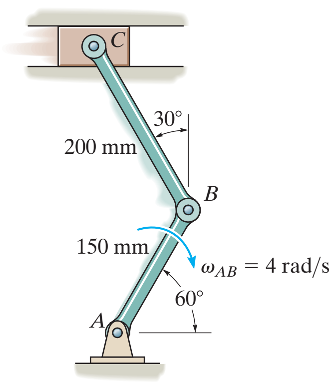
16–63. At the instant shown, the truck is traveling to the right at 3 m/s, while the pipe is rolling counterclockwise at angular $\omega = 8$ rad/s without slipping at B. Determine the velocity of the pipe's center G.

***16–64.** At the instant shown, the truck is traveling to the right at 8 m/s. If the spool does not slip at B, determine its angular velocity so that its mass center G appears to an observer on the ground to remain stationary.



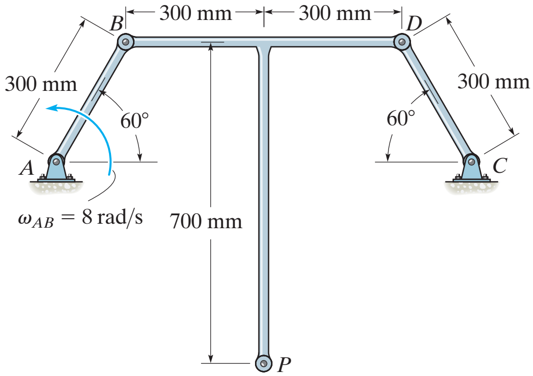
Probs. 16–63/64

16–65. If bar AB has an angular velocity $\omega_{AB} = 4$ rad/s, determine the velocity of the slider block C at the instant shown.



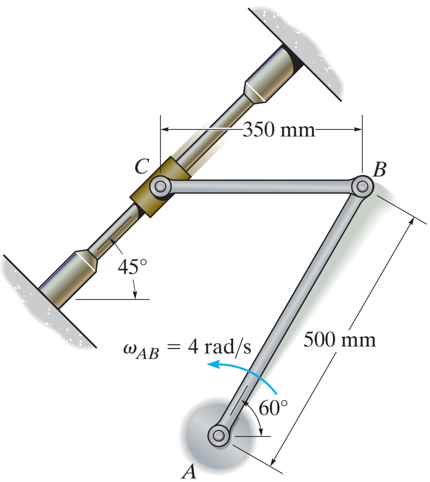
Prob. 16–65

16–66. The similar links AB and CD rotate about the fixed pins at A and C. If AB has an angular velocity $\omega_{AB} = 8$ rad/s, determine the angular velocity of BDP and the velocity of point P.



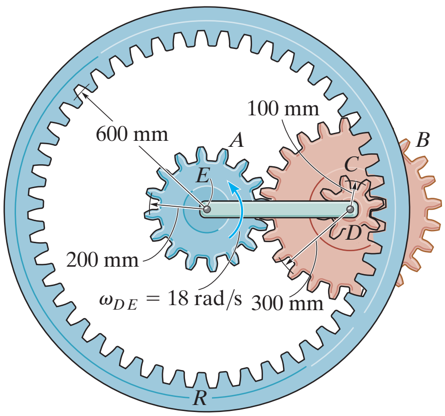
Prob. 16–66

16–67. Knowing that the angular velocity of link AB is $\omega_{AB} = 4$ rad/s, determine the velocity of the collar at C and the angular velocity of link CB at the instant shown. Link CB is horizontal at this instant.



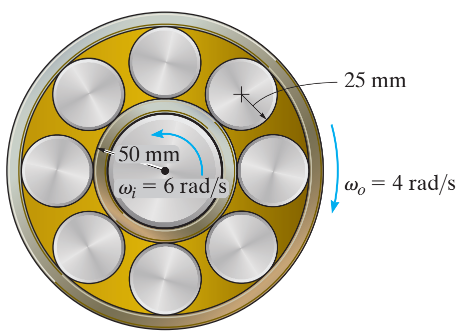
Prob. 16–67

***16–68.** The epicyclic gear train consists of the sun gear A which is in mesh with the planet gear B. This gear has an inner hub C which is fixed to B and in mesh with the fixed ring gear R. If the connecting link DE pinned to B and C is rotating at $\omega_{DE} = 18$ rad/s about the pin at E, determine the angular velocities of the planet and sun gears.



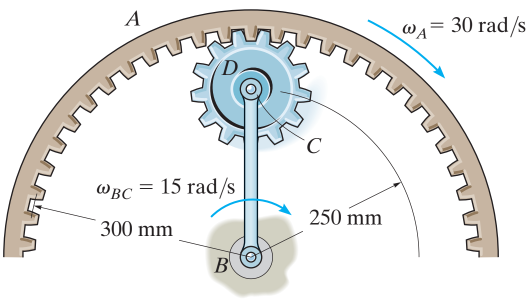
Prob. 16–68

16–69. The inner hub of the roller bearing is rotating with an angular velocity of $\omega_i = 6$ rad/s, while the outer hub is rotating in the opposite direction at $\omega_o = 4$ rad/s. Determine the angular velocity of each of the rollers if they roll on the hubs without slipping.



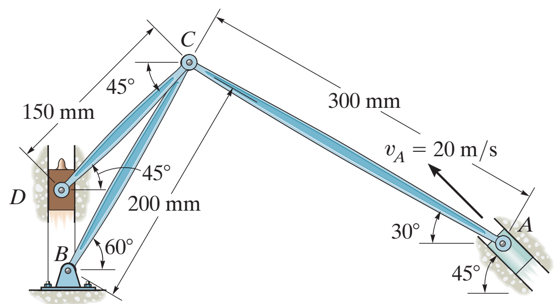
Prob. 16–69

16–70. If the ring gear A rotates clockwise with an angular velocity of $\omega_A = 30\ \text{rad/s}$, while link BC rotates clockwise with an angular velocity of $\omega_{BC} = 15\ \text{rad/s}$, determine the angular velocity of gear D.



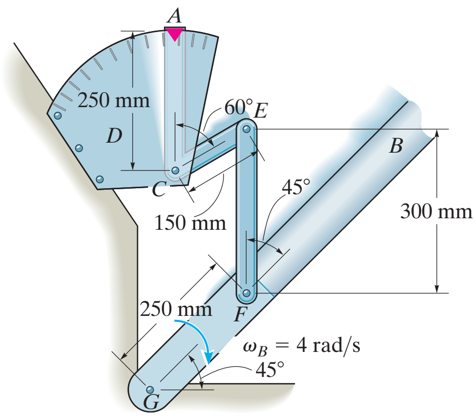
Prob. 16–70

16–71. The mechanism shown is used in a riveting machine. It consists of a driving piston A, three links, and a riveter which is attached to the slider block D. Determine the velocity of D at the instant shown, when the piston at A is traveling at $v_A = 20\ \text{m/s}$.



Prob. 16–71

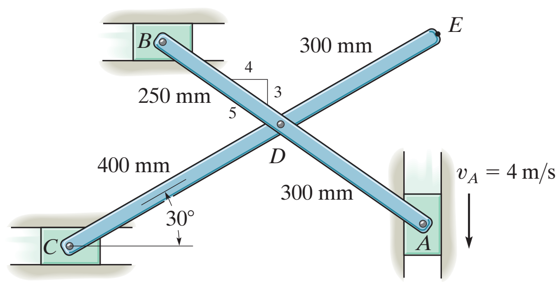
***16–72.** The gage is used to indicate the safe load acting at the end of the boom, B, when it is in any angular position. It consists of a fixed dial plate D and an indicator arm ACE, which is pinned to the plate at C and to a short link EF. If the boom is pin connected to the trunk frame at G and is rotating downward at $\omega_B = 4\ \text{rad/s}$, determine the velocity of the dial pointer A at the instant shown, i.e., when EF and AC are in the vertical position.



Prob. 16–72

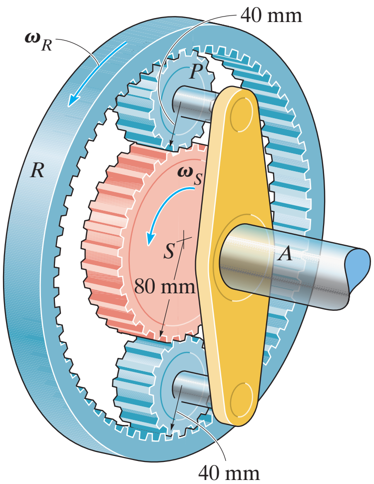
16–73. If the slider block A is moving downward at $v_A = 4\ \text{m/s}$, determine the velocities of blocks B and C at the instant shown.

16–74. If the slider block A is moving downward at $v_A = 4\ \text{m/s}$, determine the velocity of point E at the instant shown.



Probs. 16–73/74

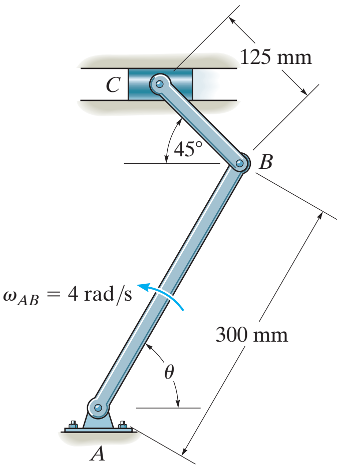
16–75. The planetary gear system is used in an automatic transmission for an automobile. By locking or releasing certain gears, it has the advantage of operating the car at different speeds. Consider the case where the ring gear R is held fixed, $\omega_R = 0$, and the sun gear S is rotating at $\omega_S = 5$ rad/s. Determine the angular velocity of each of the planet gears P and shaft A.



Prob. 16–75

***16–76.** The shaper mechanism is designed to give a slow cutting stroke and a quick return to a blade attached to the slider at C. Determine the velocity of the slider block C at the instant $\theta = 60°$, if link AB is rotating at 4 rad/s.

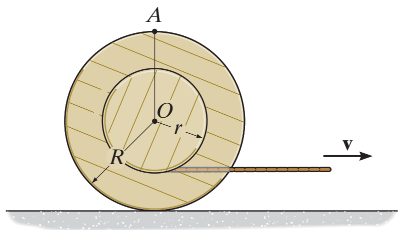
16–77. Determine the velocity of the slider block at C at the instant $\theta = 45°$, if link AB is rotating at 4 rad/s.



Probs. 16–76/77

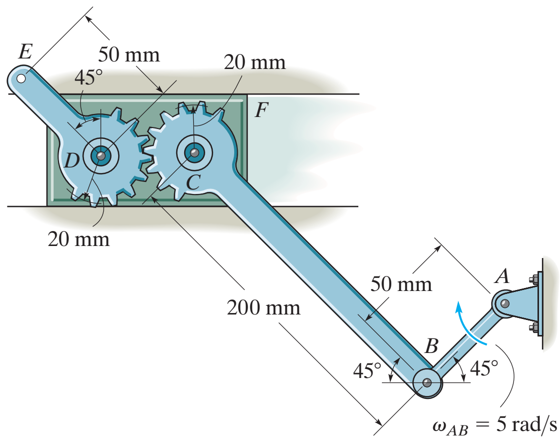
16–78. Determine the velocity of the center O of the spool when the cable is pulled to the right with a velocity of **v**. The spool rolls without slipping.

16–79. Determine the velocity of point A on the outer rim of the spool at the instant shown when the cable is pulled to the right with a velocity of **v**. The spool rolls without slipping.



Probs. 16–78/79

***16–80.** The mechanism is used on a machine for the manufacturing of a wire product. Because of the rotational motion of link AB and the sliding of block F, the segmental gear lever DE undergoes general plane motion. If AB is rotating at $\omega_{AB} = 5$ rad/s, determine the velocity of point E at the instant shown.



Prob. 16–80

16.6 INSTANTANEOUS CENTER OF ZERO VELOCITY

The velocity of any point B located on a rigid body can be obtained in a very direct way by choosing the base point A to be a point that has zero velocity at the instant considered. In this case, $\mathbf{v}_A = \mathbf{0}$, and therefore the velocity equation, $\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$, becomes $\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/A}$. For a body having general plane motion, point A so chosen is called the ***instantaneous center of zero velocity (IC)***, and it lies on the ***instantaneous axis of zero velocity***. This axis is always perpendicular to the plane of motion, and the intersection of the axis with this plane defines the location of the *IC*. Since point A coincides with the *IC*, then

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/IC}$$

and so point B moves momentarily about the *IC* in a *circular path*. The *magnitude* of $\mathbf{v}_B$ is simply $v_B = \omega r_{B/IC}$, where ω is the angular velocity of the body. Due to the circular motion, the *direction* of $\mathbf{v}_B$ must always be *perpendicular* to $\mathbf{r}_{B/IC}$.

For example, the *IC* for the bicycle wheel in Fig. 16–17 is at the contact point with the ground since the wheel does not slip. If one imagines that the wheel is momentarily *pinned* at the *IC*, the velocities of various points on the wheel can be found using $v = \omega r$, where the radial distances shown in the photo must be determined from the geometry of the wheel.

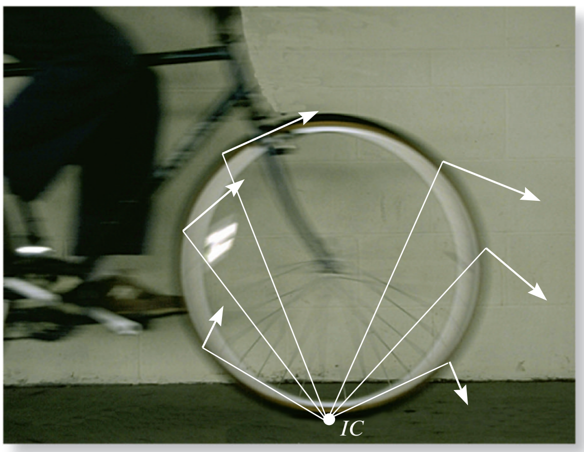
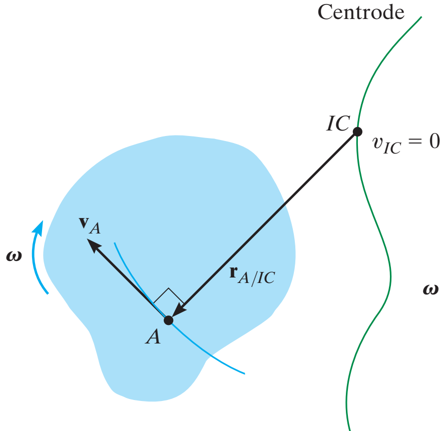
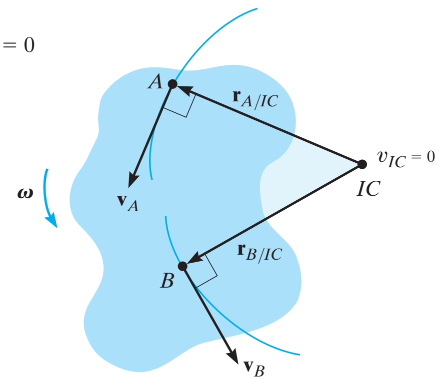


Fig. 16–17



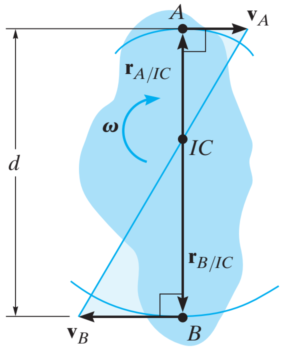
Location of IC
knowing $\mathbf{v}_A$ and $\boldsymbol{\omega}$

(a)

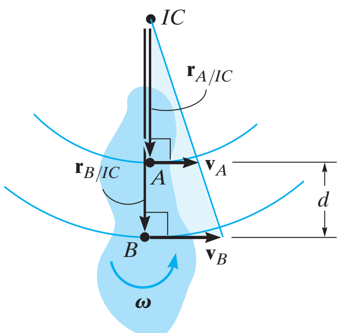


Location of IC
knowing the directions
of $\mathbf{v}_A$ and $\mathbf{v}_B$

(b)



(c)



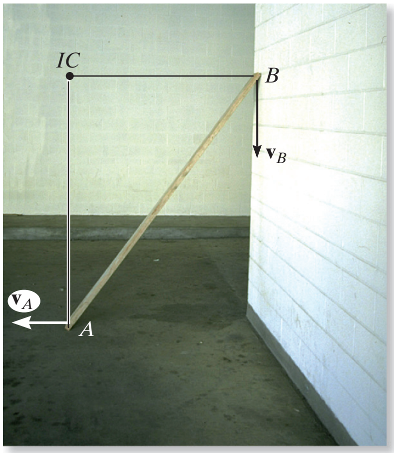
Location of IC
knowing $\mathbf{v}_A$ and $\mathbf{v}_B$

(d)

Fig. 16–18

Location of the *IC*. To locate the IC we can use the fact that the *velocity* of a point on the body is *always perpendicular* to the *relative-position vector* directed from the IC to the point. Several possibilities exist:

- *The velocity $\mathbf{v}_A$ of a point A on the body and the angular velocity $\boldsymbol{\omega}$ of the body are known*, Fig. 16–18*a*. In this case, the IC is located along the line drawn perpendicular to $\mathbf{v}_A$ at A, such that the distance from A to the IC is $r_{A/IC} = v_A/\omega$. Note that the IC lies up and to the right of A since $\mathbf{v}_A$ must cause a clockwise angular velocity $\boldsymbol{\omega}$ about the IC.
- *The lines of action of two nonparallel velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ are known*, Fig. 16–18*b*. Construct at points A and B line segments that are perpendicular to $\mathbf{v}_A$ and $\mathbf{v}_B$. Extending these perpendiculars to their *point of intersection*, as shown, locates the IC at the instant considered.
- *The magnitude and direction of two parallel velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ are known*. Here the location of the IC is determined by proportional triangles. Examples are shown in Figs. 16–18*c* and *d*. In both cases $r_{A/IC} = v_A/\omega$ and $r_{B/IC} = v_B/\omega$. If d is a known, distance between points A and B, then in Fig. 16–18*c*, $r_{A/IC} + r_{B/IC} = d$, and in Fig. 16–18*d*, $r_{B/IC} - r_{A/IC} = d$.



As the board slides downward to the left it is subjected to general plane motion. Since the directions of the velocities of its ends A and B are known, the IC is located as shown. At this instant the board will momentarily rotate about this point.

Realize that the point chosen as the instantaneous center of zero velocity for the body *can only be used at the instant considered* since the body changes its position from one instant to the next. The path described by points which define the location of the IC during the body's motion is called a ***centrode***, Fig. 16–18*a*, and so each point on the centrode acts as the IC for the body only for an instant.

Although the IC may be conveniently used to determine the velocity of any point in a body, it generally *does not have zero acceleration* and therefore it *should not* be used for finding the accelerations of points in a body.

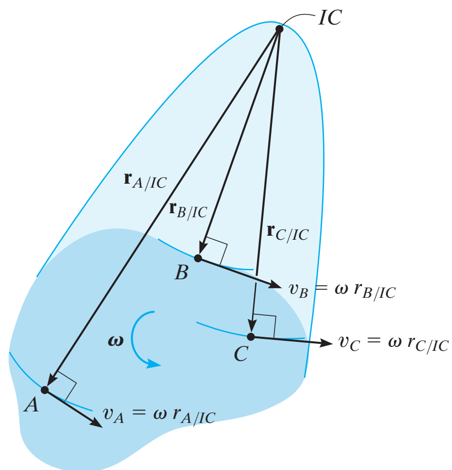


Fig. 16–19

16

PROCEDURE FOR ANALYSIS

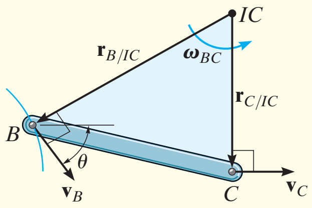
The velocity of a point on a body which is subjected to general plane motion can be determined with reference to its instantaneous center of zero velocity provided the location of the IC is first established using one of the three methods described in Fig. 16–18.

- As shown on the kinematic diagram in Fig. 16–19, the body is imagined as "extended and pinned" at the IC so that, at the instant considered, it rotates about this pin with its angular velocity $\boldsymbol{\omega}$.
- The *magnitude* of velocity for each of the arbitrary points A, B, and C on the body can be determined by using the equation $v = \omega r$, where r is the radial distance from the IC to each point.
- The line of action of each velocity vector $\mathbf{v}$ is *perpendicular* to its associated radial line $\mathbf{r}$, and the velocity has a *sense of direction* which tends to move the point in a manner consistent with the angular rotation $\boldsymbol{\omega}$ of the radial line, Fig. 16–19.

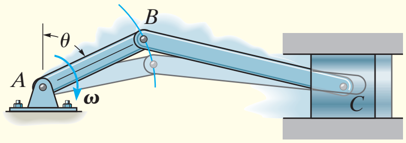
Refer to the companion website for Lecture Summary and Quiz videos.

EXAMPLE 16.9

Show how to determine the location of the instantaneous center of zero velocity for (a) member BC shown in Fig. 16–20*a*; and (b) the link CB shown in Fig. 16–20*c*.



(b)

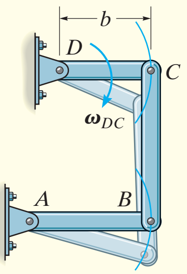


(a)

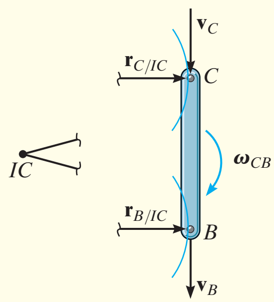
SOLUTION

Part (a). As shown in Fig. 16–20*a*, point B moves in a circular path such that $\mathbf{v}_B$ is perpendicular to AB. Therefore, it acts at an angle θ from the horizontal as shown in Fig. 16–20*b*. The motion of point B causes the piston to move forward *horizontally* with a velocity $\mathbf{v}_C$. When lines are drawn perpendicular to $\mathbf{v}_B$ and $\mathbf{v}_C$, Fig. 16–20*b*, they intersect at the IC.

Part (b). Points B and C follow circular paths of motion since links AB and DC are each subjected to rotation about a fixed axis, Fig. 16–20*c*. Since the velocity is always tangent to the path, at the instant considered, $\mathbf{v}_C$ on rod DC and $\mathbf{v}_B$ on rod AB are both directed vertically downward, along the axis of link CB, Fig. 16–20*d*. Radial lines drawn perpendicular to these two velocities form parallel lines which intersect at "infinity;" i.e., $r_{C/IC} \rightarrow \infty$ and $r_{B/IC} \rightarrow \infty$. Thus, $\omega_{CB} = (v_C/r_{C/IC}) \rightarrow 0$. As a result, link CB momentarily *translates*. An instant later, however, CB will move to a tilted position, causing the IC to move to some finite location.



(c)



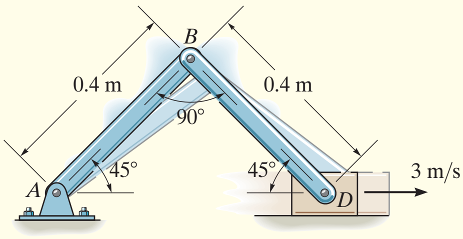
(d)

Fig. 16–20

16

EXAMPLE 16.10

Block D shown in Fig. 16–21a moves with a speed of 3 m/s. Determine the angular velocities of links BD and AB at the instant shown.



(a)

SOLUTION

As D moves to the right, it causes AB to rotate clockwise about point A. Hence, $\mathbf{v}_B$ is directed perpendicular to AB, Fig. 16–21b. The instantaneous center of zero velocity for BD is located at the intersection of the line segments drawn perpendicular to $\mathbf{v}_B$ and $\mathbf{v}_D$. From the geometry,

$$r_{B/IC} = 0.4 \tan 45^\circ \text{ m} = 0.4 \text{ m}$$

$$r_{D/IC} = \frac{0.4 \text{ m}}{\cos 45^\circ} = 0.5657 \text{ m}$$

Since the magnitude of $\mathbf{v}_D$ is known, the angular velocity of link BD is

$$\omega_{BD} = \frac{v_D}{r_{D/IC}} = \frac{3 \text{ m/s}}{0.5657 \text{ m}} = 5.30 \text{ rad/s} \circlearrowleft \qquad \textit{Ans.}$$

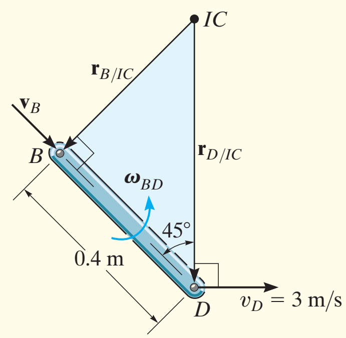
The velocity of B is therefore

$$v_B = \omega_{BD}(r_{B/IC}) = 5.30 \text{ rad/s } (0.4 \text{ m}) = 2.12 \text{ m/s} \measuredangle 45^\circ$$

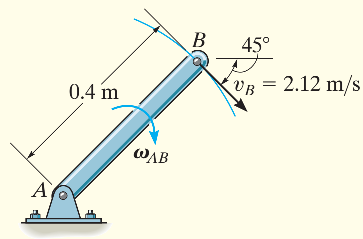
From Fig. 16–21c, the angular velocity of AB is

$$\omega_{AB} = \frac{v_B}{r_{B/A}} = \frac{2.12 \text{ m/s}}{0.4 \text{ m}} = 5.30 \text{ rad/s} \circlearrowright \qquad \textit{Ans.}$$

NOTE: Try to solve this problem by applying $\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{D/B}$ to member BD.



(b)



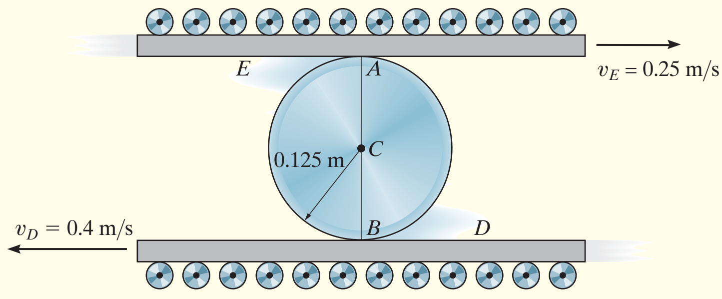
(c)

Fig. 16–21

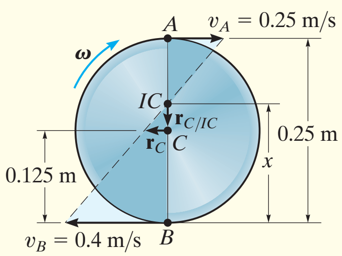
16

EXAMPLE 16.11

The cylinder shown in Fig. 16–22*a* rolls without slipping between the two moving plates *E* and *D*. Determine the angular velocity of the cylinder and the velocity of its center *C*.



(a)



(b)

Fig. 16–22

16

SOLUTION

Since no slipping occurs, the contact points *A* and *B* on the cylinder have the same velocities as the plates *E* and *D*, respectively. The radial lines drawn from these points are collinear, so that by the proportionality of right triangles the *IC* is located at a point on line *AB*, Fig. 16–22*b*. Assuming this point to be a distance *x* from *B*, we have

$$v_B = \omega x; \qquad 0.4 \text{ m/s} = \omega x$$

$$v_A = \omega(0.25 \text{ m} - x); \qquad 0.25 \text{ m/s} = \omega(0.25 \text{ m} - x)$$

Dividing one equation into the other eliminates ω and yields

$$0.4(0.25 - x) = 0.25x$$

$$x = \frac{0.1}{0.65} = 0.1538 \text{ m}$$

Hence, the angular velocity of the cylinder is

$$\omega = \frac{v_B}{x} = \frac{0.4 \text{ m/s}}{0.1538 \text{ m}} = 2.60 \text{ rad/s} \circlearrowright \qquad \textit{Ans.}$$

The velocity of point *C* is therefore

$$v_C = \omega r_{C/IC} = 2.60 \text{ rad/s } (0.1538 \text{ m} - 0.125 \text{ m})$$

$$= 0.0750 \text{ m/s} \leftarrow \qquad \textit{Ans.}$$

EXAMPLE 16.12

The crankshaft AB turns with a clockwise angular velocity of 10 rad/s, Fig. 16–23a. Determine the velocity of the piston at the instant shown.

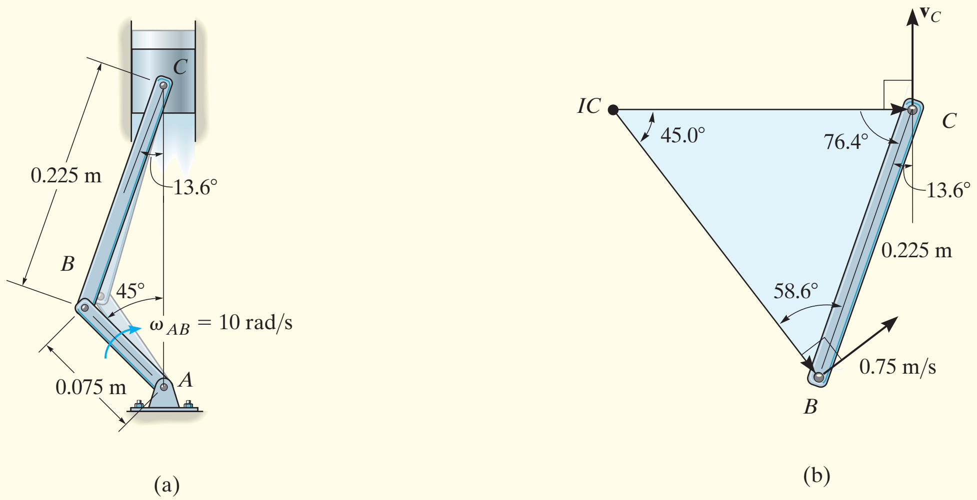


Fig. 16–23

SOLUTION

The crankshaft rotates about a fixed axis, and so the velocity of point B is

$$v_B = 10 \text{ rad/s } (0.075 \text{ m}) = 0.75 \text{ m/s} \measuredangle 45^\circ$$

Since the directions of the velocities of B and C are known, then the location of the IC for rod BC is at the intersection of the lines extended from these points, perpendicular to $\mathbf{v}_B$ and $\mathbf{v}_C$, Fig. 16–23b. The interior angles to the shaded triangle are determined, and the magnitudes of $\mathbf{r}_{B/IC}$ and $\mathbf{r}_{C/IC}$ can be obtained from the law of sines, i.e.,

$$\frac{0.225 \text{ m}}{\sin 45^\circ} = \frac{r_{B/IC}}{\sin 76.4^\circ}$$

$$r_{B/IC} = 0.3092 \text{ m}$$

$$\frac{0.225 \text{ m}}{\sin 45^\circ} = \frac{r_{C/IC}}{\sin 58.6^\circ}$$

$$r_{C/IC} = 0.2717 \text{ m}$$

The rotational sense of $\boldsymbol{\omega}_{BC}$ must be the same as the rotation caused by $\mathbf{v}_B$ about the IC, which is counterclockwise. Therefore,

$$\omega_{BC} = \frac{v_B}{r_{B/IC}} = \frac{0.75 \text{ m/s}}{0.3092 \text{ m}} = 2.425 \text{ rad/s}$$

Using this result, the velocity of the piston is

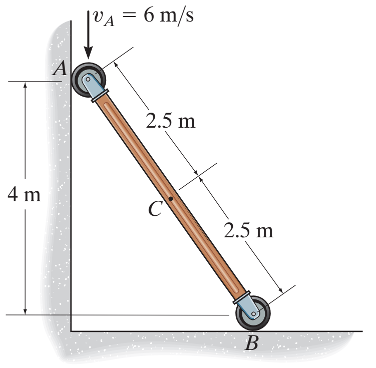
$$v_C = \omega_{BC} r_{C/IC} = (2.425 \text{ rad/s})(0.2717 \text{ m}) = 0.659 \text{ m/s} \quad \textit{Ans.}$$

Refer to the companion website for a self quiz of these Example problems.

16

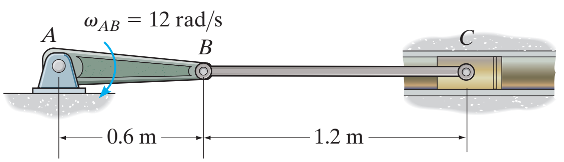
FUNDAMENTAL PROBLEMS

F16–13. Determine the angular velocity of the rod and the velocity of point C at the instant shown.



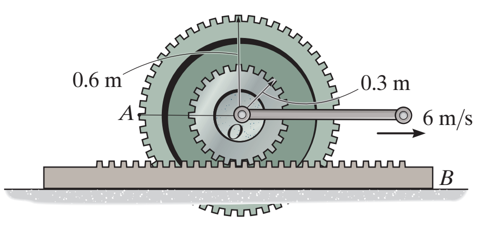
Prob. F16–13

F16–14. Determine the angular velocity of link BC and velocity of the piston C at the instant shown.



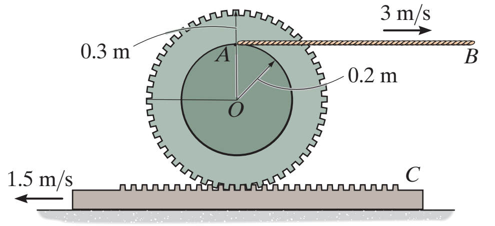
Prob. F16–14

F16–15. If the center O of the gear is moving with a speed of $v_O = 6$ m/s, determine the velocity of point A on the gear. The gear rack B is fixed.



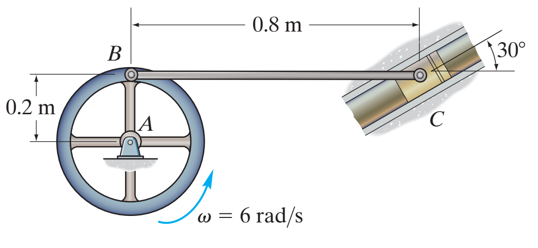
Prob. F16–15

F16–16. If cable AB is unwound with a speed of 3 m/s, and the gear rack C has a speed of 1.5 m/s, determine the angular velocity of the gear and the velocity of its center O.



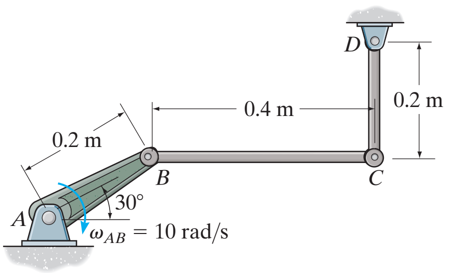
Prob. F16–16

F16–17. Determine the angular velocity of link BC and the velocity of the piston C at the instant shown.



Prob. F16–17

F16–18. Determine the angular velocity of links BC and CD at the instant shown.

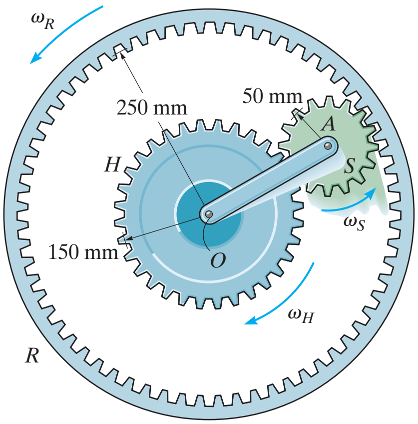


Prob. F16–18

PROBLEMS

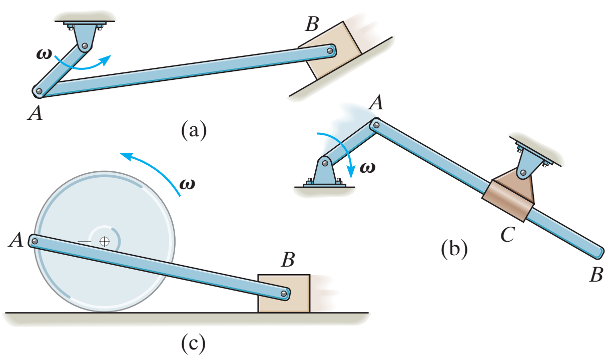
16–81. If the hub gear H and ring gear R have angular velocities $\omega_H = 5$ rad/s and $\omega_R = 20$ rad/s, respectively, determine the angular velocity ω_S of the spur gear S and the angular velocity of its attached arm OA.

16–82. If the hub gear H has an angular velocity $\omega_H = 5$ rad/s, determine the angular velocity of the ring gear R so that the arm OA attached to the spur gear S remains stationary ($\omega_{OA} = 0$). What is the angular velocity of the spur gear?



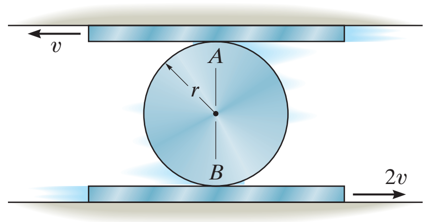
Probs. 16–81/82

16–83. In each case show graphically how to locate the instantaneous center of zero velocity of link AB. Assume the geometry is known. The disk rolls without slipping.



Prob. 16–83

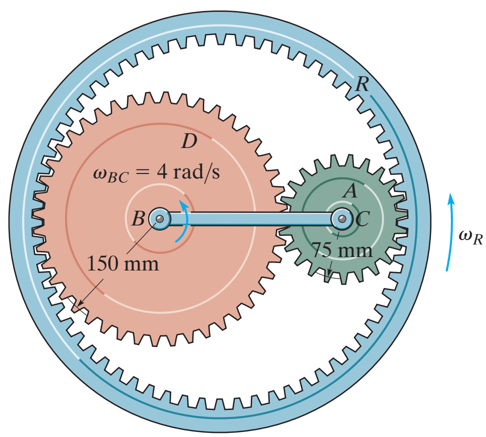
***16–84.** The disk of radius r is confined to roll without slipping at A and B. If the plates have the velocities shown, determine the angular velocity of the disk.



Prob. 16–84

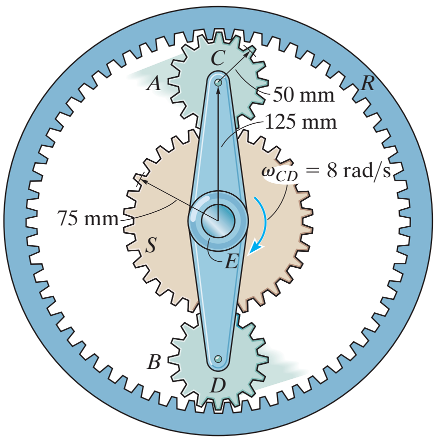
16–85. The planet gear A is pin connected to the end of the link BC. If the link rotates about the fixed point B at 4 rad/s, determine the angular velocity of the ring gear R. The sun gear D is fixed from rotating.

16–86. Solve Prob. 16–85 if the sun gear D is rotating clockwise at $\omega_D = 5$ rad/s while link BC rotates counterclockwise at $\omega_{BC} = 4$ rad/s.



Probs. 16–85/86

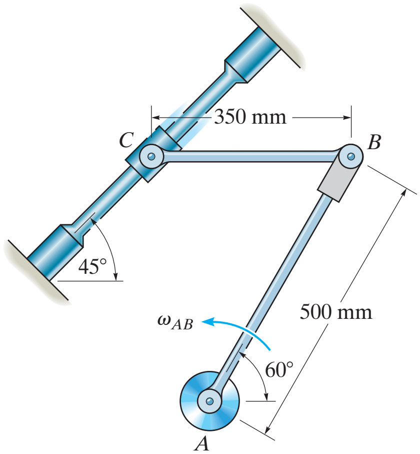
16–87. In an automobile transmission the planet pinions A and B rotate on shafts that are mounted on the planet-pinion carrier CD. As shown, CD is attached to a shaft at E which is aligned wih the center of the *fixed* sun gear S. This shaft is not attached to the sun gear. If CD is rotating at $\omega_{CD} = 8$ rad/s, determine the angular velocity of the ring gear R.



Prob. 16–87

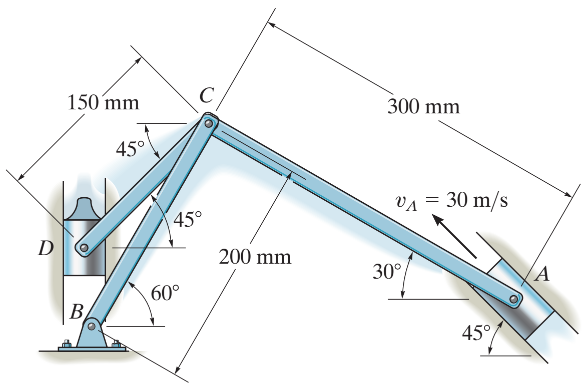
16–89. Knowing that the angular velocity of link AB is $\omega_{AB} = 4$ rad/s, determine the velocity of the collar at C and the angular velocity of link CB at the instant shown. Link CB is horizontal at this instant.

16–90. If the collar at C is moving downward to the left at $v_C = 8$ m/s, determine the angular velocity of link AB at the instant shown.



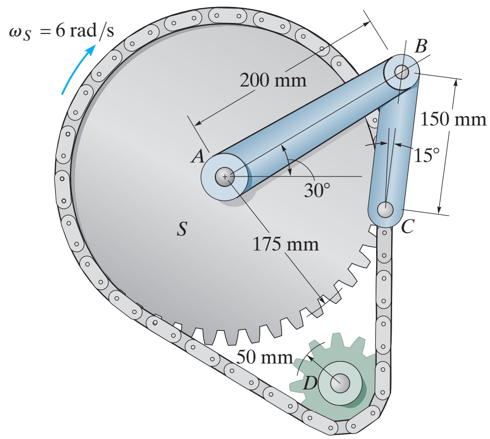
Probs. 16–89/90

***16–88.** The mechanism shown is used in a riveting machine. It consists of a driving piston A, three members, and a riveter which is attached to the slider block D. Determine the velocity of D at the instant shown, when the piston at A is traveling at $v_A = 30$ m/s.



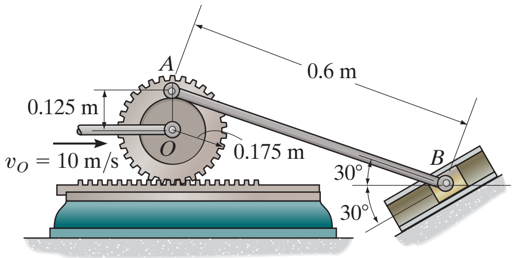
Prob. 16–88

16–91. The mechanism produces intermittent motion of link AB. If the sprocket S is turning with an angular velocity of $\omega_S = 6$ rad/s, determine the angular velocity of link AB at this instant. The sprocket S is mounted on a shaft which is separate from a collinear shaft attached to AB at A. The pin at C is attached to the vertical chain links.



Prob. 16–91

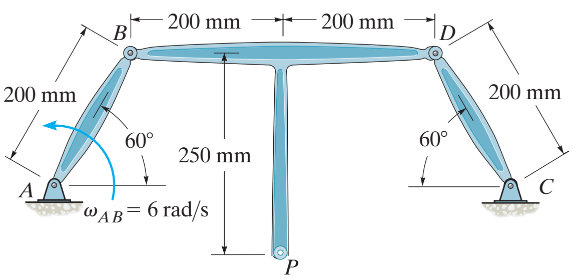
***16–92.** If the center O of the gear is given a velocity of $v_O = 10$ m/s, determine the velocity of the slider block B at the instant shown.



Prob. 16–92

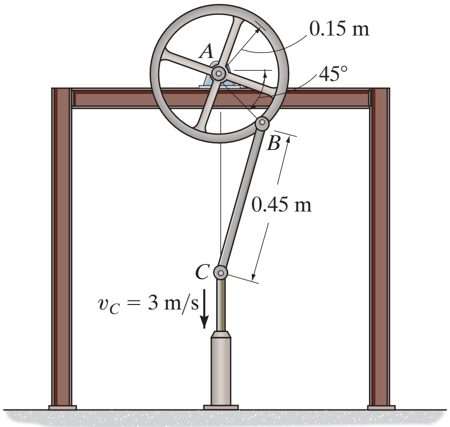
16–93. Member AB is rotating at $\omega_{AB} = 6$ rad/s. Determine the velocity of point D and the angular velocity of members BPD and CD.

16–94. Member AB is rotating at $\omega_{AB} = 6$ rad/s. Determine the velocity of point P, and the angular velocity of member BPD.



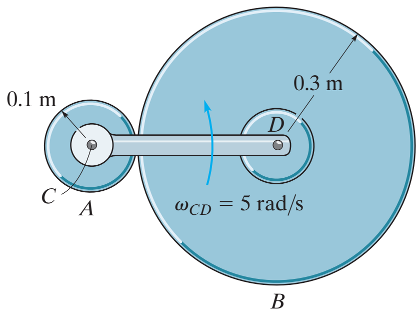
Probs. 16–93/94

16–95. If C has a velocity of $v_C = 3$ m/s, determine the angular velocity of the wheel at the instant shown.



Prob. 16–95

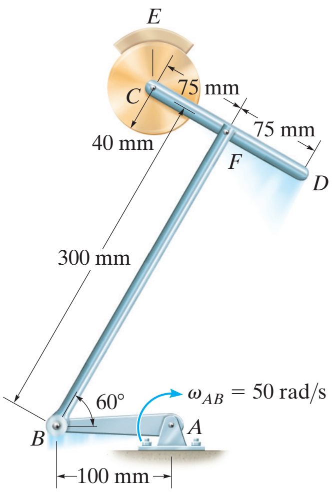
***16–96.** The cylinder B rolls on the fixed cylinder A without slipping. If connected bar CD is rotating with an angular velocity $\omega_{CD} = 5$ rad/s, determine the angular velocity of cylinder B. Point C is a fixed point.



Prob. 16–96

16

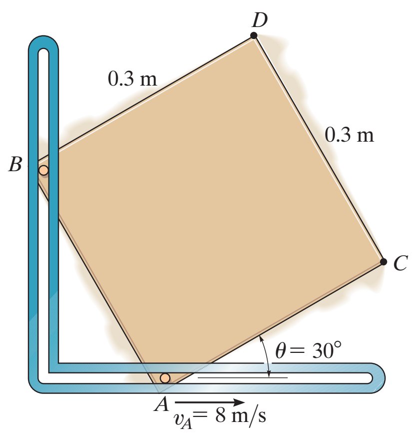
16–97. The crankshaft AB rotates at $\omega_{AB} = 50$ rad/s about the fixed axis through point A, and the disk at C is held fixed in its support at E. Determine the angular velocity of rod CD at the instant shown.



Prob. 16–97

16–98. The square plate is confined within the slots at A and B. When $\theta = 30°$, point A is moving at $v_A = 8$ m/s. Determine the velocity of point C at this instant.

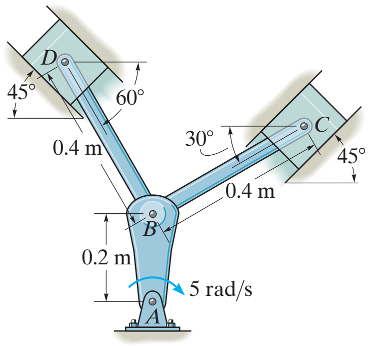
16–99. The square plate is confined within the slots at A and B. When $\theta = 30°$, point A is moving at $v_A = 8$ m/s. Determine the velocity of point D at this instant.



Probs. 16–98/99

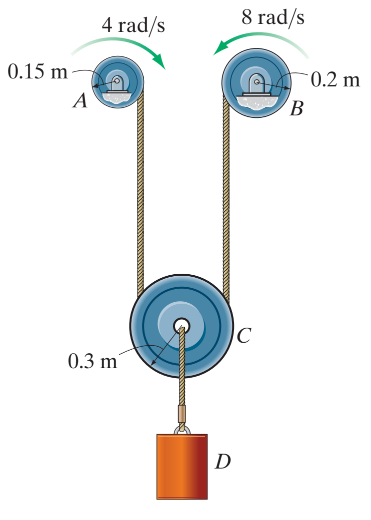
***16–100.** The mechanism used in a marine engine consists of a single crank AB and two connecting rods BC and BD. Determine the velocity of the piston at C the instant the crank is in the position shown and has an angular velocity of 5 rad/s.

16–101. The mechanism used in a marine engine consists of a single crank AB and two connecting rods BC and BD. Determine the velocity of the piston at D the instant the crank is in the position shown and has an angular velocity of 5 rad/s.



Probs. 16–100/101

16–102. The drums have the angular velocities at the instant shown. Determine the angular velocity of the pulley C and the velocity of the load D.



Prob. 16–102

16.7 RELATIVE-MOTION ANALYSIS: ACCELERATION

An equation that relates the accelerations of two points A and B on the bar (rigid body) shown in Fig. 16–24a is determined by differentiating $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$ with respect to time. This yields

$$\frac{d\mathbf{v}_B}{dt} = \frac{d\mathbf{v}_A}{dt} + \frac{d\mathbf{v}_{B/A}}{dt}$$

The terms $d\mathbf{v}_B/dt = \mathbf{a}_B$ and $d\mathbf{v}_A/dt = \mathbf{a}_A$ are measured with respect to a set of *fixed x, y axes* and represent the *absolute accelerations* of points B and A. The last term represents the acceleration of B with respect to A as measured by an observer fixed to translating x', y' axes which have their origin at the base point A. In Sec. 16.5 it was shown that to this observer point B appears to move along a *circular arc* that has a radius of curvature $r_{B/A}$. Consequently, $\mathbf{a}_{B/A}$ can be expressed in terms of its tangential and normal components; i.e., $\mathbf{a}_{B/A} = (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$, where $(a_{B/A})_t = \alpha r_{B/A}$ and $(a_{B/A})_n = \omega^2 r_{B/A}$. Hence, the relative-acceleration equation can be written in the form

$$\boxed{\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n} \qquad (16\text{–}18)$$

where

$\mathbf{a}_B$ = acceleration of point B

$\mathbf{a}_A$ = acceleration of point A

$(\mathbf{a}_{B/A})_t$ = tangential acceleration component of B with respect to A. The *magnitude* is $(a_{B/A})_t = \alpha r_{B/A}$, and the *direction* is perpendicular to $\mathbf{r}_{B/A}$.

$(\mathbf{a}_{B/A})_n$ = normal acceleration component of B with respect to A. The *magnitude* is $(a_{B/A})_n = \omega^2 r_{B/A}$, and the *direction* is always from B toward A.

These terms are represented graphically in Fig. 16–24. Here the acceleration of B, Fig. 16–24a, is determined by considering the bar to *translate* with an acceleration $\mathbf{a}_A$, Fig. 16–24b, and simultaneously *rotate* about the base point A with an instantaneous angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$, Fig. 16–24c. Vector addition of these two effects, applied to B, yields $\mathbf{a}_B$, as shown in Fig. 16–24d.

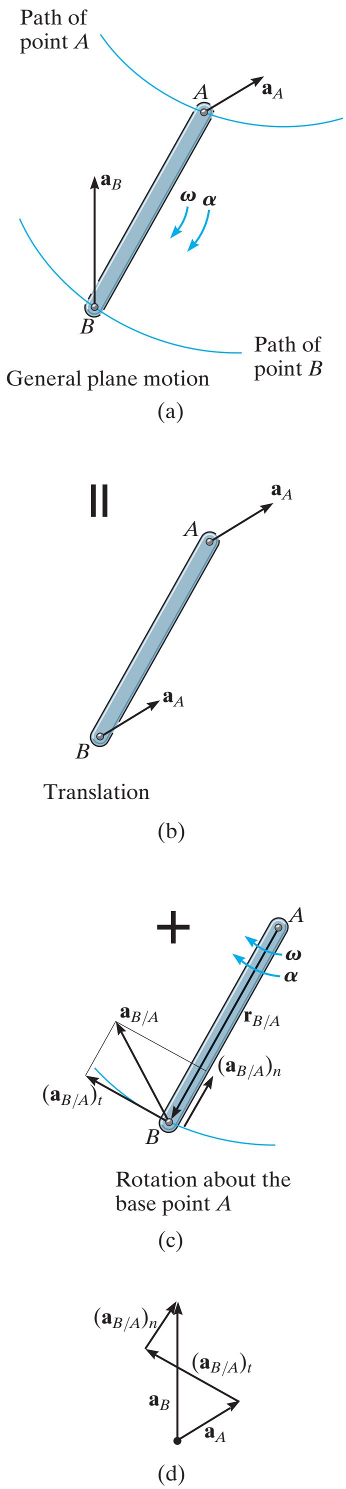


Fig. 16–24

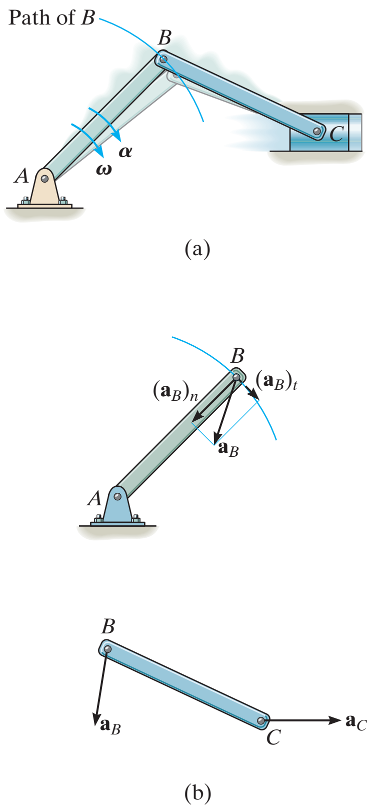


Fig. 16–25

For a vector analysis, the relative-acceleration components can be expressed as $(\mathbf{a}_{B/A})_t = \boldsymbol{\alpha} \times \mathbf{r}_{B/A}$ and $(\mathbf{a}_{B/A})_n = -\omega^2 \mathbf{r}_{B/A}$, Eq. 16–14, and so, Eq. 16–18 then becomes

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A} \tag{16–19}$$

where

$\mathbf{a}_B$ = acceleration of point B
$\mathbf{a}_A$ = acceleration of the base point A
$\boldsymbol{\alpha}$ = angular acceleration of the body
$\boldsymbol{\omega}$ = angular velocity of the body
$\mathbf{r}_{B/A}$ = position vector directed from A to B

When Eq. 16–18 or Eq. 16–19 is applied to study the accelerated motion of a rigid body which is *pin connected* to two other bodies, it should be realized that points which are *coincident at the pin* move with the *same acceleration*, since the path of motion over which they travel is the *same*. For example, point B lying on either rod BA or BC of the crank mechanism shown in Fig. 16–25*a* has the same acceleration, since the rods are pin connected at B. Here the motion of B is along a *circular path*, so that $\mathbf{a}_B$ can be expressed in terms of its tangential and normal components. At the other end of rod BC point C moves along a *straight-lined path*, which is defined by the piston. Hence, $\mathbf{a}_C$ is horizontal, Fig. 16–25*b*.

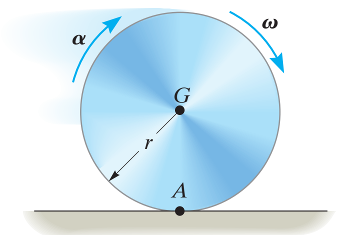


Fig. 16–26

Rolling without slipping. Consider a wheel that rolls without slipping, Fig. 16–26. We have shown that the velocity of its center can be found using Eq. 16–17, namely

$$v_G = \omega r$$

Since G moves along a *straight line*, its acceleration can be determined from the time derivative of its velocity.

$$\frac{dv_G}{dt} = \frac{d\omega}{dt} r$$
$$a_G = \alpha r \tag{16–20}$$

This result was also obtained in Example 16–4. It applies as well to any circular object, such as a ball, gear, disk, etc., that *rolls without slipping*.

16

PROCEDURE FOR ANALYSIS

The relative-acceleration equation can be applied between any two points A and B on a body either by using a Cartesian vector analysis, or by writing the x and y scalar component equations directly.

Velocity Analysis.

- Determine the angular velocity $\boldsymbol{\omega}$ of the body by using a velocity analysis as discussed in Sec. 16.5 or 16.6. Also, determine the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ of points A and B *if these points move along curved paths*.

Vector Analysis

Kinematic Diagram.

- Establish the directions of the fixed x, y coordinates and draw a kinematic diagram of the body that shows $\mathbf{a}_A$, $\mathbf{a}_B$, $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$, and $\mathbf{r}_{B/A}$.
- If points A and B move along *curved paths*, then their accelerations should be indicated in terms of their tangential and normal components, i.e., $\mathbf{a}_A = (\mathbf{a}_A)_t + (\mathbf{a}_A)_n$ and $\mathbf{a}_B = (\mathbf{a}_B)_t + (\mathbf{a}_B)_n$.

Acceleration Equation.

- To apply $\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2\mathbf{r}_{B/A}$, express the vectors in Cartesian vector form and substitute them into the equation. Evaluate the cross product and then equate the respective $\mathbf{i}$ and $\mathbf{j}$ components to obtain two scalar equations.
- If the solution yields a *negative* answer for an *unknown* magnitude, it indicates that the sense of direction of the vector is opposite to that shown on the kinematic diagram.

Scalar Analysis

Kinematic Diagram.

- If the acceleration equation is applied in scalar form, then the magnitudes and directions of the relative-acceleration components $(\mathbf{a}_{B/A})_t$ and $(\mathbf{a}_{B/A})_n$ must be established. To do this draw a kinematic diagram such as shown in Fig. 16–27*c*. Since the body is considered to be momentarily "pinned" at the base point A, the *magnitudes* of these components are $(a_{B/A})_t = \alpha r_{B/A}$ and $(a_{B/A})_n = \omega^2 r_{B/A}$. Their *sense of direction* is established from the diagram such that $(\mathbf{a}_{B/A})_t$ acts perpendicular to $\mathbf{r}_{B/A}$, in accordance with the rotational motion $\boldsymbol{\alpha}$ of the body, and $(\mathbf{a}_{B/A})_n$ is directed from B toward A.*

Acceleration Equation.

- Represent the vectors $\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$ graphically by showing their magnitudes and directions underneath each term. The scalar equations are determined from the x and y components of these vectors.

* The notation $\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A(\text{pin})})_t + (\mathbf{a}_{B/A(\text{pin})})_n$ may be helpful in recalling that A is assumed to be pinned.

The mechanism for a window is shown. Here CA rotates about a fixed axis through C, and AB undergoes general plane motion. Since point A moves along a curved path it has two components of acceleration, whereas point B moves along a straight track and so the direction of its acceleration is specified.

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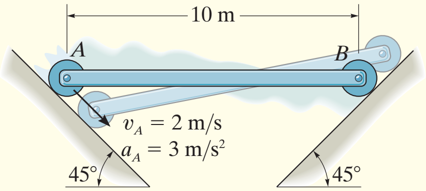
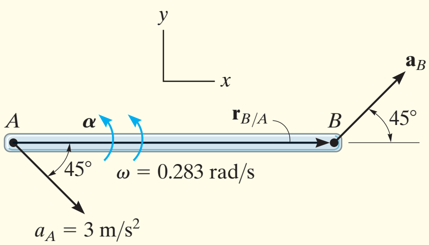
Refer to the companion website for Lecture Summary and Quiz videos.

EXAMPLE 16.13



(a)



(b)

The rod AB shown in Fig. 16–27a is confined to move along the inclined planes. If point A has an acceleration of 3 m/s^2 and a velocity of 2 m/s, both directed down the plane at the instant the rod is horizontal, determine the angular acceleration of the rod at this instant.

SOLUTION I (VECTOR ANALYSIS)

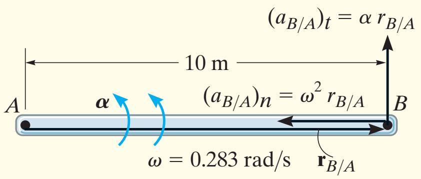
We will apply the acceleration equation to points A and B on the rod. To do so it is first necessary to determine the angular velocity of the rod. Show that it is $\omega = 0.283$ rad/s $\circlearrowleft$ using either the velocity equation or the method of instantaneous centers.

Kinematic Diagram. Since points A and B both move along straight-line paths, they have *no* components of acceleration normal to the paths. There are two unknowns in Fig. 16–27b, namely, a_B and α.

Acceleration Equation.

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$$

$$a_B \cos 45°\mathbf{i} + a_B \sin 45°\mathbf{j} = 3\cos 45°\mathbf{i} - 3\sin 45°\mathbf{j} + (\alpha\mathbf{k}) \times (10\mathbf{i}) - (0.283)^2(10\mathbf{i})$$



(c)

Fig. 16–27

Carrying out the cross product and equating the $\mathbf{i}$ and $\mathbf{j}$ components yields

$$a_B \cos 45° = 3\cos 45° - (0.283)^2(10) \quad (1)$$

$$a_B \sin 45° = -3\sin 45° + \alpha(10) \quad (2)$$

Solving, we have

$$a_B = 1.87 \text{ m/s}^2 \ \measuredangle 45°$$

$$\alpha = 0.344 \text{ rad/s}^2 \circlearrowleft \qquad \textit{Ans.}$$

SOLUTION II (SCALAR ANALYSIS)

From the kinematic diagram, showing the relative-acceleration components $(\mathbf{a}_{B/A})_t$ and $(\mathbf{a}_{B/A})_n$, Fig. 16–27c, we have

$$\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$$

$$\begin{bmatrix} a_B \\ \measuredangle 45° \end{bmatrix} = \begin{bmatrix} 3 \text{ m/s}^2 \\ \searrow 45° \end{bmatrix} + \begin{bmatrix} \alpha(10 \text{ m}) \\ \uparrow \end{bmatrix} + \begin{bmatrix} (0.283 \text{ rad/s})^2(10 \text{ m}) \\ \leftarrow \end{bmatrix}$$

Equating the x and y components of these four terms yields Eq. 1 and 2, and the solution proceeds as before.

EXAMPLE 16.14

The disk rolls without slipping and has the angular motion shown in Fig. 16–28*a*. Determine the acceleration of point *A* at this instant.

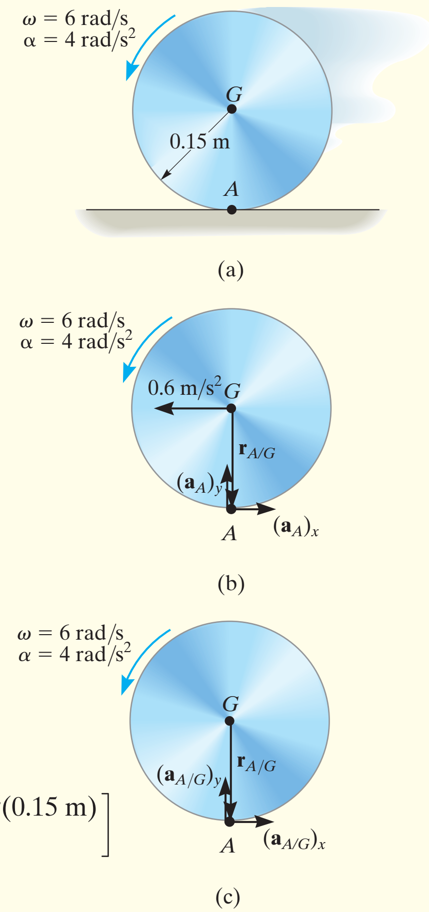


Fig. 16–28

SOLUTION I (VECTOR ANALYSIS)

Kinematic Diagram. Since no slipping occurs, applying Eq. 16–20,

$$a_G = \alpha r = (4\ \text{rad/s}^2)(0.15\ \text{m}) = 0.6\ \text{m/s}^2$$

Acceleration Equation.
We will apply the acceleration equation to points *G* and *A*, Fig. 16–28*b*.

$$\mathbf{a}_A = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{A/G} - \omega^2 \mathbf{r}_{A/G}$$

$$\mathbf{a}_A = -0.6\mathbf{i} + (4\mathbf{k}) \times (-0.15\mathbf{j}) - (6)^2(-0.15\mathbf{j})$$

$$= \{5.4\mathbf{j}\}\ \text{m/s}^2 \qquad \textit{Ans.}$$

SOLUTION II (SCALAR ANALYSIS)

Using the result for $a_G = 0.6\ \text{m/s}^2$ determined above, and from the kinematic diagram, showing the relative motion $\mathbf{a}_{A/G}$, Fig. 16–28*c*, we have

$$\mathbf{a}_A = \mathbf{a}_G + (\mathbf{a}_{A/G})_x + (\mathbf{a}_{A/G})_y$$

$$\begin{bmatrix} (a_A)_x \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (a_A)_y \\ \uparrow \end{bmatrix} = \begin{bmatrix} 0.6\ \text{m/s}^2 \\ \leftarrow \end{bmatrix} + \begin{bmatrix} (4\ \text{rad/s}^2)(0.15\ \text{m}) \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (6\ \text{rad/s})^2(0.15\ \text{m}) \\ \uparrow \end{bmatrix}$$

$\overset{+}{\rightarrow}$ $\qquad (a_A)_x = -0.6 + 0.6 = 0$

$+\uparrow$ $\qquad (a_A)_y = 5.4\ \text{m/s}^2$

Therefore,

$$a_A = \sqrt{(0)^2 + (5.4\ \text{m/s}^2)^2} = 5.4\ \text{m/s}^2$$

$$\mathbf{a}_A = \{5.4\mathbf{j}\}\ \text{m/s}^2 \qquad \textit{Ans.}$$

NOTE: The fact that $a_A = 5.4\ \text{m/s}^2$ indicates that the instantaneous center of zero velocity, point *A*, is *not* a point of zero acceleration.

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EXAMPLE 16.15

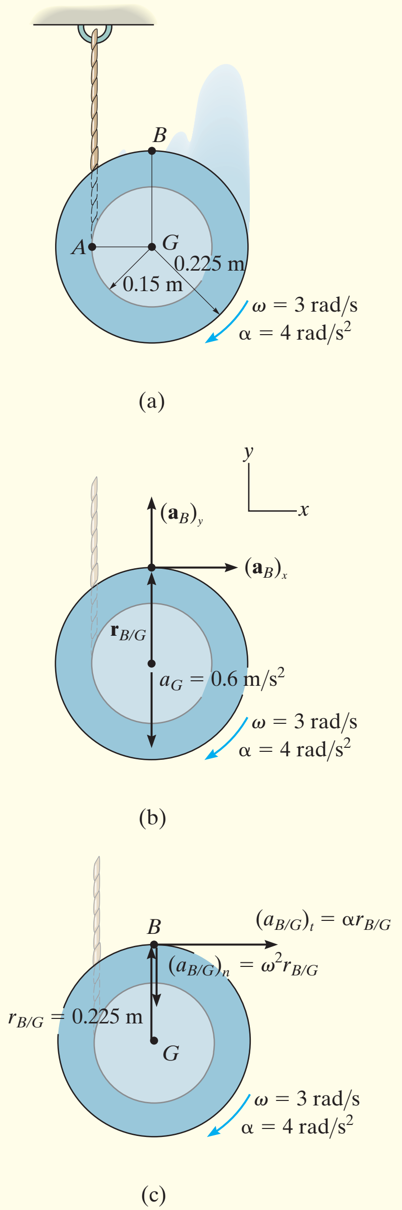

Fig. 16–29

The spool in Fig. 16–29*a* unravels from the cord, such that at the instant shown it has an angular velocity of 3 rad/s and an angular acceleration of 4 rad/s^2. Determine the magnitude of acceleration of point B.

SOLUTION I (VECTOR ANALYSIS)

The spool "appears" to be rolling downward without slipping at point A. Therefore, we can use the results of Eq. 16-20 to determine the acceleration of point G, i.e.,

$$a_G = \alpha r = (4 \text{ rad/s}^2)(0.15 \text{ m}) = 0.6 \text{ m/s}^2$$

We will apply the acceleration equation to points G and B.

Kinematic Diagram. Point B moves along a *curved path* having an *unknown* radius of curvature.* Its acceleration will be represented by its unknown x and y components as shown in Fig. 16–29*b*.

Acceleration Equation.

$$\mathbf{a}_B = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{B/G} - \omega^2 \mathbf{r}_{B/G}$$

$$(a_B)_x\mathbf{i} + (a_B)_y\mathbf{j} = -0.6\mathbf{j} + (-4\mathbf{k}) \times (0.225\mathbf{j}) - (3)^2(0.225\mathbf{j})$$

Equating the $\mathbf{i}$ and $\mathbf{j}$ terms, the component equations are

$$(a_B)_x = 4(0.225) = 0.9 \text{ m/s}^2 \rightarrow \qquad (1)$$

$$(a_B)_y = -0.6 - 2.025 = -2.625 \text{ m/s}^2 = 2.625 \text{ m/s}^2 \downarrow \qquad (2)$$

The magnitude and direction of $\mathbf{a}_B$ are therefore

$$a_B = \sqrt{(0.9)^2 + (2.625)^2} = 2.78 \text{ m/s}^2 \qquad \textit{Ans.}$$

$$\theta = \tan^{-1}\left(\frac{2.625}{0.9}\right) = 71.1°$$

SOLUTION II (SCALAR ANALYSIS)

This problem may be solved by writing the scalar component equations directly. The kinematic diagram in Fig. 16–29*c* shows the relative-acceleration components $(\mathbf{a}_{B/G})_t$ and $(\mathbf{a}_{B/G})_n$. Thus,

$$\mathbf{a}_B = \mathbf{a}_G + (\mathbf{a}_{B/G})_t + (\mathbf{a}_{B/G})_n$$

$$\begin{bmatrix} (a_B)_x \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (a_B)_y \\ \uparrow \end{bmatrix} = \begin{bmatrix} 0.6 \text{ m/s}^2 \\ \downarrow \end{bmatrix} + \begin{bmatrix} 4 \text{ rad/s}^2 \, (0.225 \text{ m}) \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (3 \text{ rad/s})^2(0.225 \text{ m}) \\ \downarrow \end{bmatrix}$$

The x and y components yield Eqs. 1 and 2 above.

Refer to the companion website for a self quiz of these Example problems.

* Realize that the path's radius of curvature ρ is not equal to the radius of the spool since the spool is not rotating about point G. Furthermore, ρ is not defined as the distance from A (IC) to B, since the location of the IC depends only on the velocity of a point and not the geometry of its path.

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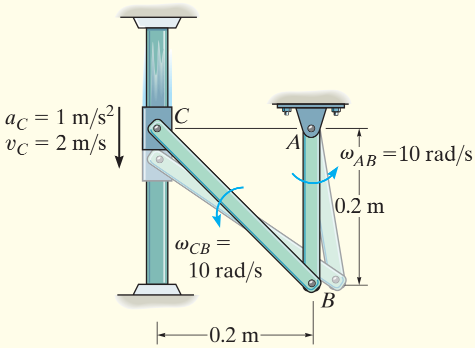
EXAMPLE 16.16

The collar C in Fig. 16–30a moves downward with an acceleration of 1 m/s^2. At the instant shown, it has a speed of 2 m/s which gives links CB and AB an angular velocity $\omega_{AB} = \omega_{CB} = 10$ rad/s. (See Example 16.8.) Determine the angular accelerations of CB and AB at this instant.



(a)

SOLUTION (VECTOR ANALYSIS)

Kinematic Diagram. The kinematic diagrams of *both* links AB and CB are shown in Fig. 16–30b. To solve, we will apply the appropriate kinematic equation to each link.

Acceleration Equation.

Link AB (rotation about a fixed axis):

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_B - \omega_{AB}^2 \mathbf{r}_B$$
$$\mathbf{a}_B = (\alpha_{AB}\mathbf{k}) \times (-0.2\mathbf{j}) - (10)^2(-0.2\mathbf{j})$$
$$\mathbf{a}_B = 0.2\alpha_{AB}\mathbf{i} + 20\mathbf{j}$$

Note that $\mathbf{a}_B$ has n and t components since it moves along a *circular path*.

Link BC (general plane motion): Using the result for $\mathbf{a}_B$ and applying Eq. 16–18, we have

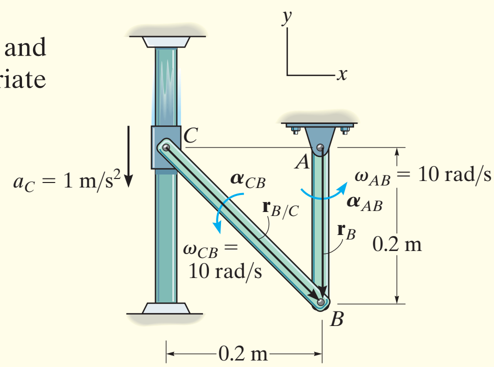
$$\mathbf{a}_B = \mathbf{a}_C + \boldsymbol{\alpha}_{CB} \times \mathbf{r}_{B/C} - \omega_{CB}^2 \mathbf{r}_{B/C}$$
$$0.2\alpha_{AB}\mathbf{i} + 20\mathbf{j} = -1\mathbf{j} + (\alpha_{CB}\mathbf{k}) \times (0.2\mathbf{i} - 0.2\mathbf{j}) - (10)^2(0.2\mathbf{i} - 0.2\mathbf{j})$$
$$0.2\alpha_{AB}\mathbf{i} + 20\mathbf{j} = -1\mathbf{j} + 0.2\alpha_{CB}\mathbf{j} + 0.2\alpha_{CB}\mathbf{i} - 20\mathbf{i} + 20\mathbf{j}$$

Thus,

$$0.2\alpha_{AB} = 0.2\alpha_{CB} - 20$$
$$20 = -1 + 0.2\alpha_{CB} + 20$$

Solving,

$$\alpha_{CB} = 5 \text{ rad/s}^2 \circlearrowleft \qquad \textit{Ans.}$$
$$\alpha_{AB} = -95 \text{ rad/s}^2 = 95 \text{ rad/s}^2 \circlearrowright \qquad \textit{Ans.}$$



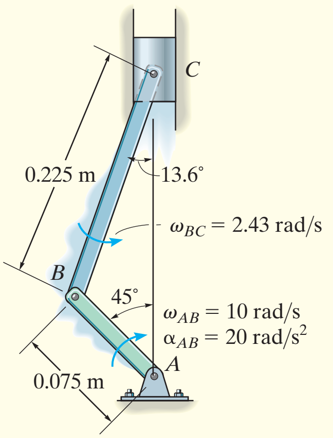
(b)

Fig. 16–30

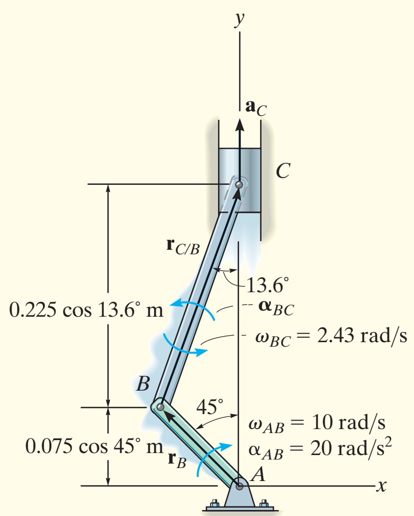
16

Refer to the companion website for a self quiz of these Example problems.

EXAMPLE 16.17



(a)



(b)

Fig. 16–31

The crankshaft AB turns with a clockwise angular acceleration of 20 rad/s^2, Fig. 16–31a. Determine the acceleration of the piston at the instant AB is in the position shown. At this instant $\omega_{AB} = 10$ rad/s and $\omega_{BC} = 2.43$ rad/s. (See Example 16.12.)

SOLUTION (VECTOR ANALYSIS)

Kinematic Diagram. The kinematic diagrams for both AB and BC are shown in Fig. 16–31b. Here $\mathbf{a}_C$ is vertical since C moves along a straight-line path.

Acceleration Equation. Expressing each of the position vectors in Cartesian vector form,

$$\mathbf{r}_B = \{-0.075 \sin 45°\mathbf{i} + 0.075 \cos 45°\mathbf{j}\} \text{ m} = \{-0.0530\mathbf{i} + 0.0530\mathbf{j}\} \text{ m}$$

$$\mathbf{r}_{C/B} = \{0.225 \sin 13.6°\mathbf{i} + 0.225 \cos 13.6°\mathbf{j}\} \text{ m} = \{0.0530\mathbf{i} + 0.219\mathbf{j}\} \text{ m}$$

Crankshaft AB (rotation about a fixed axis):

$$\begin{aligned}\mathbf{a}_B &= \boldsymbol{\alpha}_{AB} \times \mathbf{r}_B - \omega_{AB}^2\mathbf{r}_B \\ &= (-20\mathbf{k}) \times (-0.0530\mathbf{i} + 0.0530\mathbf{j}) - (10)^2(-0.0530\mathbf{i} + 0.0530\mathbf{j}) \\ &= \{6.364\mathbf{i} - 4.243\mathbf{j}\} \text{ m/s}^2\end{aligned}$$

Connecting Rod BC (general plane motion): Using the result for $\mathbf{a}_B$ and noting that $\mathbf{a}_C$ is in the vertical direction, we have

$$\begin{aligned}\mathbf{a}_C &= \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2\mathbf{r}_{C/B} \\ a_C\mathbf{j} &= 6.364\mathbf{i} - 4.243\mathbf{j} + (\alpha_{BC}\mathbf{k}) \times (0.0530\mathbf{i} + 0.219\mathbf{j}) \\ &\quad - (2.43)^2(0.0530\mathbf{i} + 0.219\mathbf{j}) \\ a_C\mathbf{j} &= 6.364\mathbf{i} - 4.243\mathbf{j} + 0.0530\alpha_{BC}\mathbf{j} - 0.219\alpha_{BC}\mathbf{i} - 0.312\mathbf{i} - 1.286\mathbf{j} \\ 0 &= 6.052 - 0.219\alpha_{BC} \\ a_C &= 0.0530\alpha_{BC} - 5.529\end{aligned}$$

Solving yields

$$\alpha_{BC} = 27.7 \text{ rad/s}^2 ↺$$

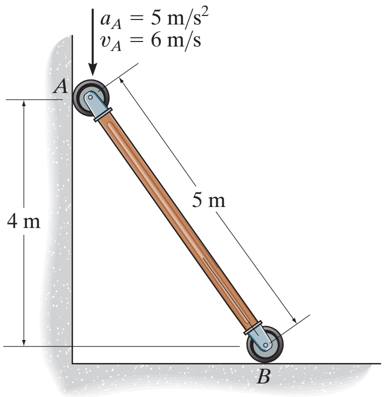
$$a_C = -4.06 \text{ m/s}^2$$

$$\mathbf{a}_C = \{-4.06\mathbf{j}\} \text{ m/s}^2 \qquad \textit{Ans.}$$

NOTE: Since the piston is moving upward, the negative sign for a_C indicates that the piston is decelerating, i.e., $\mathbf{a}_C = \{-4.06\mathbf{j}\} \text{ m/s}^2$. This causes the speed of the piston to decrease until AB becomes vertical, at which time the piston is momentarily at rest.

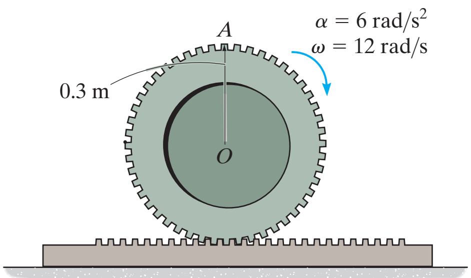
FUNDAMENTAL PROBLEMS

F16–19. At the instant shown, end A of the rod has the velocity and acceleration shown. Determine the angular acceleration of the rod and acceleration of end B of the rod.



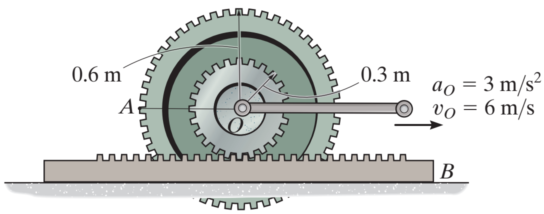
Prob. F16–19

F16–20. The gear rolls on the fixed rack with an angular velocity of $\omega = 12\ \text{rad/s}$ and angular acceleration of $\alpha = 6\ \text{rad/s}^2$. Determine the horizontal and vertical components of acceleration of point A.



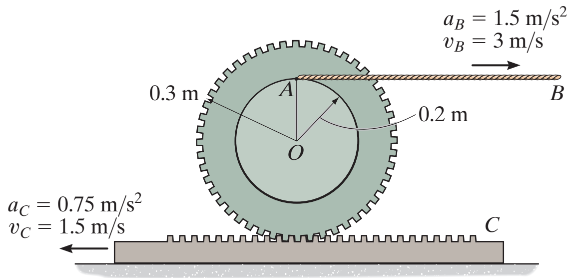
Prob. F16–20

F16–21. The gear rolls on the fixed rack B. At the instant shown, the center O of the gear moves with a velocity of $v_O = 6\ \text{m/s}$ and acceleration of $a_O = 3\ \text{m/s}^2$. Determine the angular acceleration of the gear and the horizontal and vertical components of acceleration of point A at this instant.



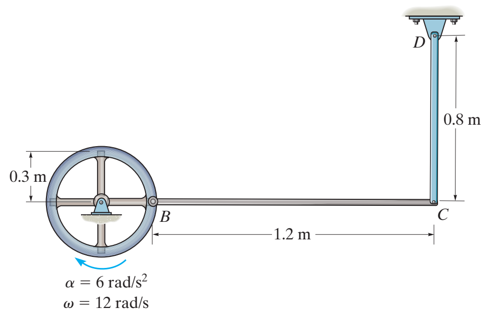
Prob. F16–21

F16–22. At the instant shown, cable AB has a velocity of $3\ \text{m/s}$ and acceleration of $1.5\ \text{m/s}^2$, while the gear rack has a velocity of $1.5\ \text{m/s}$ and acceleration of $0.75\ \text{m/s}^2$. Determine the angular acceleration of the gear at this instant.



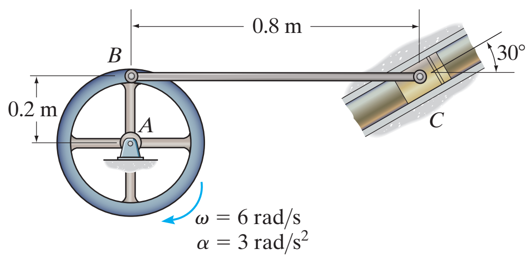
Prob. F16–22

F16-23 At the instant shown, the wheel rotates with an angular velocity of $\omega = 12\ \text{rad/s}$ and an angular acceleration of $\alpha = 6\ \text{rad/s}^2$. Determine the angular acceleration of link BC at the instant shown.



Prob. F16–23

F16–24. At the instant shown, wheel A rotates with an angular velocity of $\omega = 6\ \text{rad/s}$ and an angular acceleration of $\alpha = 3\ \text{rad/s}^2$. Determine the angular acceleration of link BC and the acceleration of piston C at this instant.

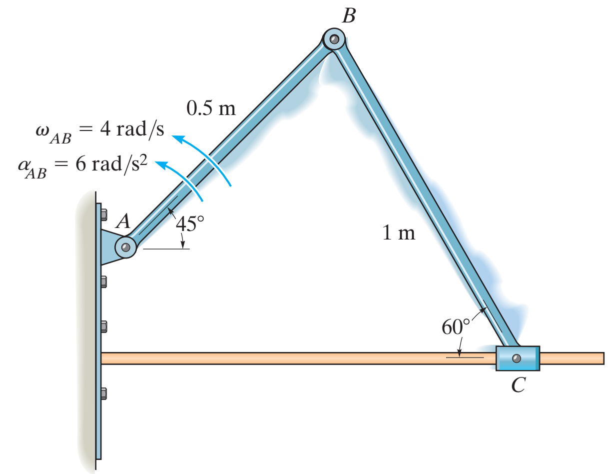


Prob. F16–24

16

PROBLEMS

16–103. Bar AB has the angular motions shown. Determine the velocity and acceleration of the slider block C at this instant.

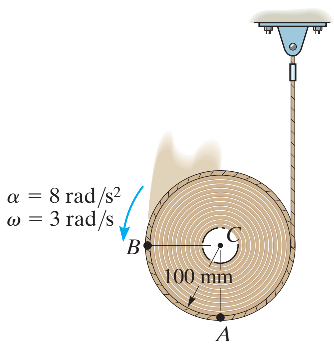


Prob. 16–103

16

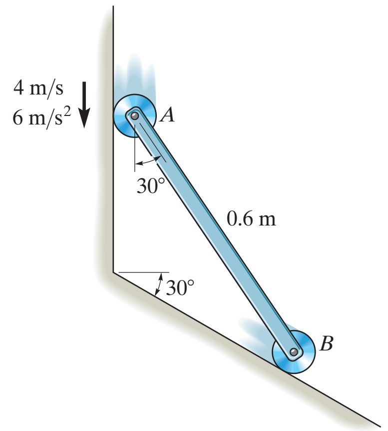
***16–104.** The reel of rope has the angular motion shown. Determine the velocity and acceleration of point A at the instant shown.

16–105. The reel of rope has the angular motion shown. Determine the velocity and acceleration of point B at the instant shown.



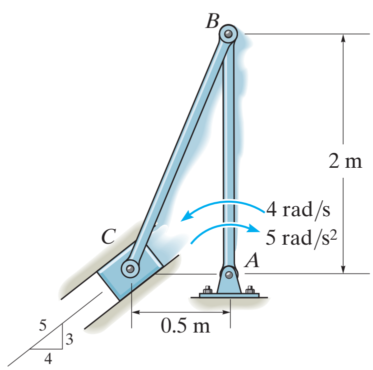
Probs. 16–104/105

16–106. At a given instant the roller A on the bar has the velocity and acceleration shown. Determine the velocity and acceleration of the roller B, and the bar's angular velocity and angular acceleration at this instant.



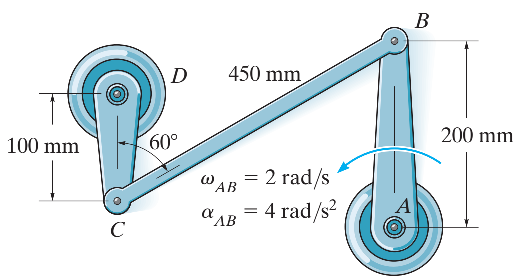
Prob. 16–106

16–107. Member AB has the angular motions shown. Determine the velocity and acceleration of the slider block C at this instant.



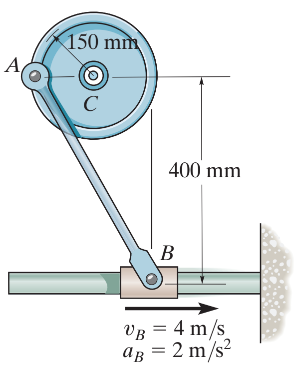
Prob. 16–107

*16–108. Member AB has the angular motions shown. Determine the angular velocity and angular acceleration of members CB and DC.



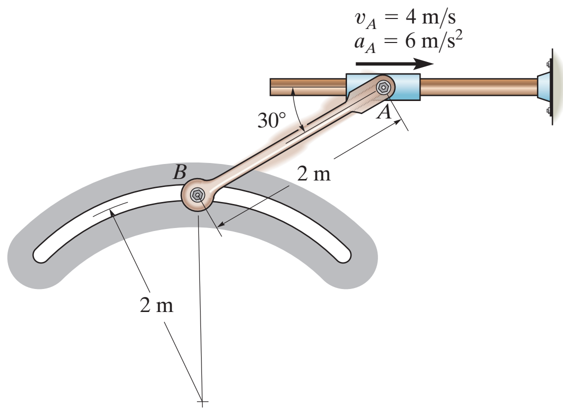
Prob. 16–108

16–109. The slider block has the motion shown. Determine the angular velocity and angular acceleration of the wheel at this instant.



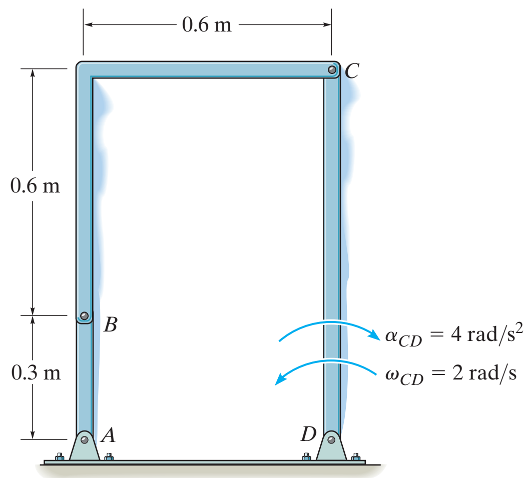
Prob. 16–109

16–110. At a given instant the slider block A is moving to the right with the motion shown. Determine the angular acceleration of link AB and the acceleration of point B at this instant.



Prob. 16–110

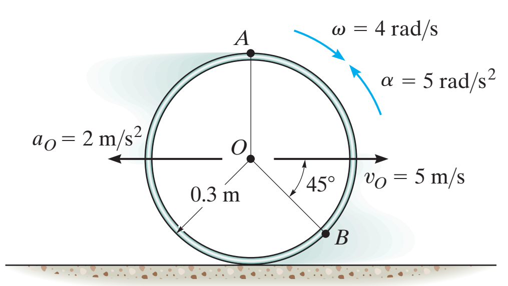
16–111. Determine the angular acceleration of link AB if link CD has the angular velocity and angular deceleration shown.



Prob. 16–111

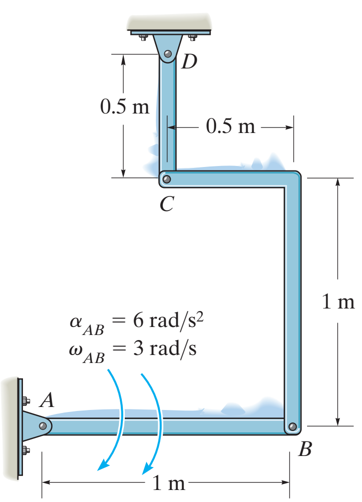
***16–112.** The hoop is cast on the rough surface such that it has an angular velocity $\omega = 4$ rad/s and an angular acceleration $\alpha = 5$ rad/s^2. Also, its center has a velocity $v_O = 5$ m/s and a deceleration $a_O = 2$ m/s^2. Determine the acceleration of point A at this instant.

16–113. The hoop is cast on the rough surface such that it has an angular velocity $\omega = 4$ rad/s and an angular acceleration $\alpha = 5$ rad/s^2. Also, its center has a velocity of $v_O = 5$ m/s and a deceleration $a_O = 2$ m/s^2. Determine the acceleration of point B at this instant.



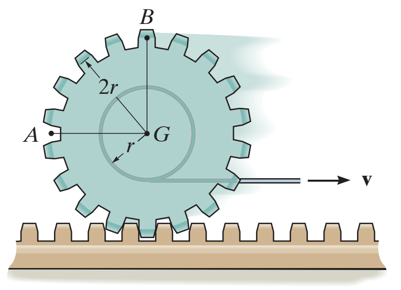
Probs. 16–112/113

16–114. Determine the angular acceleration of link CD if link AB has the angular velocity and angular acceleration shown.



Prob. 16–114

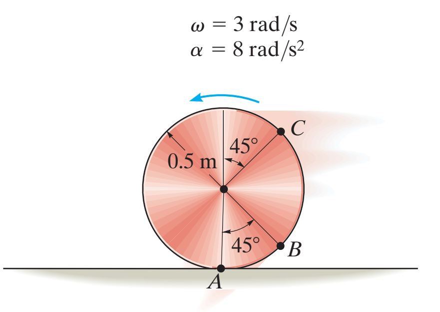
16–115. A cord is wrapped around the inner spool of the gear. If it is pulled with a constant velocity **v**, determine the velocities and accelerations of points A and B. The gear rolls on the fixed gear rack.



Prob. 16–115

***16–116.** The disk has an angular acceleration $\alpha = 8$ rad/s^2 and angular velocity $\omega = 3$ rad/s at the instant shown. If it does not slip at A, determine the acceleration of point B.

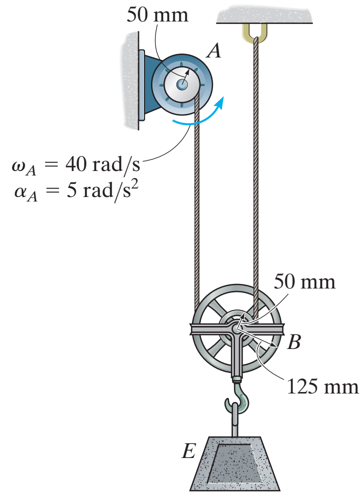
16–117. The disk has an angular acceleration $\alpha = 8$ rad/s^2 and angular velocity $\omega = 3$ rad/s at the instant shown. If it does not slip at A, determine the acceleration of point C.



Probs. 16–116/117

16–118. Pulley A rotates with the angular velocity and angular acceleration shown. Determine the angular acceleration of pulley B at the instant shown.

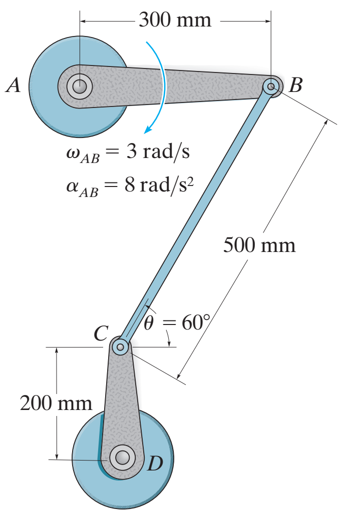
16–119. Pulley A rotates with the angular velocity and angular acceleration shown. Determine the acceleration of block E at the instant shown.



Probs. 16–118/119

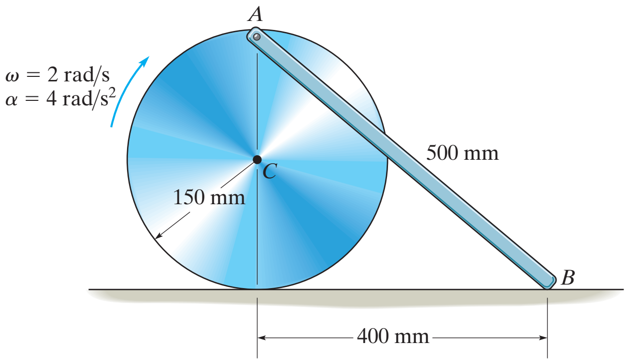
***16–120.** If member AB has the angular motion shown, determine the angular velocity and angular acceleration of member CD at the instant shown.

16–121. If member AB has the angular motion shown, determine the velocity and acceleration of point C at the instant shown.



Probs. 16–120/121

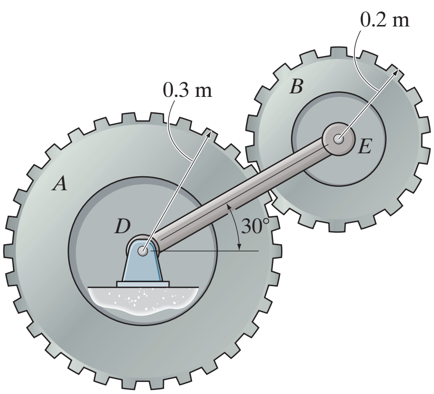
16–122. The disk rolls without slipping such that it has an angular acceleration of $\alpha = 4\ \text{rad/s}^2$ and angular velocity of $\omega = 2\ \text{rad/s}$ at the instant shown. Determine the acceleration of points A and B on the link and the link's angular acceleration at this instant. Assume point A lies on the periphery of the disk, 150 mm from C.



Prob. 16–122

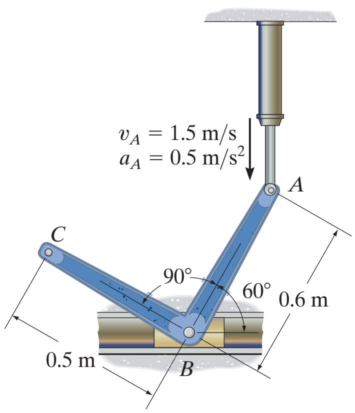
16–123. Gear A is held fixed, and arm DE rotates clockwise with an angular velocity of $\omega_{DE} = 6\ \text{rad/s}$ and an angular acceleration of $\alpha_{DE} = 3\ \text{rad/s}^2$. Determine the angular acceleration of gear B at the instant shown.

***16–124.** Gear A rotates counterclockwise with a constant angular velocity of $\omega_A = 10\ \text{rad/s}$, while arm DE rotates clockwise with an angular velocity of $\omega_{DE} = 6\ \text{rad/s}$ and an angular acceleration of $\alpha_{DE} = 3\ \text{rad/s}^2$. Determine the angular acceleration of gear B at the instant shown.



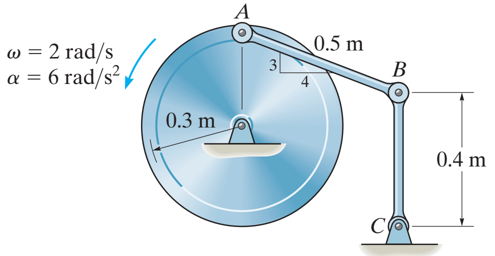
Probs. 16–123/124

16–125. The hydraulic cylinder extends with a velocity of $v_A = 1.5$ m/s and an acceleration of $\alpha_A = 0.5$ m/s^2. Determine the angular acceleration of link ABC and the acceleration of end C at the instant shown. Point B is pin connected to the slider block.



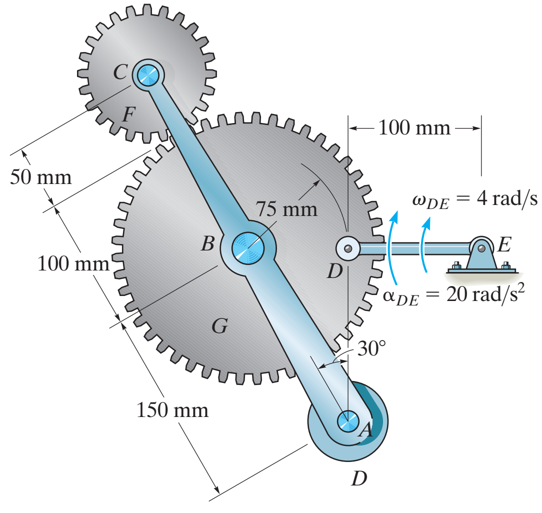
Prob. 16–125

16–126. The flywheel rotates with an angular velocity $\omega = 2$ rad/s and an angular acceleration $\alpha = 6$ rad/s^2. Determine the angular acceleration of links AB and BC at this instant.



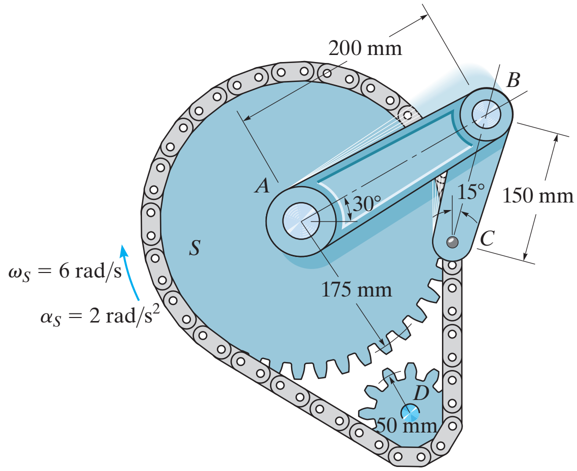
Prob. 16–126

16–127. The tied crank and gear mechanism gives rocking motion to crank AC, necessary for the operation of a printing press. If link DE has the angular motion shown, determine the respective angular velocities of gear F and crank AC at this instant, and the angular acceleration of crank AC.



Prob. 16–127

***16–128.** The mechanism produces intermittent motion of link AB. If the sprocket S is turning with an angular acceleration $\alpha_S = 2$ rad/s^2 and has an angular velocity $\omega_S = 6$ rad/s at the instant shown, determine the angular velocity and angular acceleration of link AB at this instant. The sprocket S is mounted on a shaft which is *separate* from a collinear shaft attached to AB at A. The pin at C is attached to one of the chain links such that it moves vertically downward.



Prob. 16–128

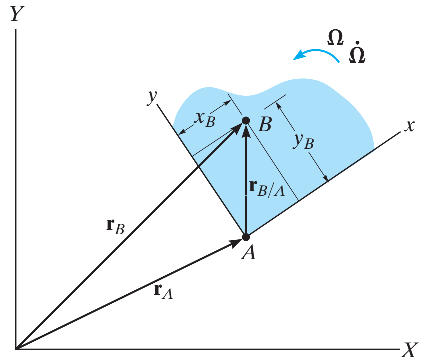
16

*16.8 RELATIVE-MOTION ANALYSIS USING ROTATING AXES

In the previous sections the relative-motion analysis for velocity and acceleration was described using a translating coordinate system. This type of analysis is useful for determining the motion of points on the *same* rigid body, or the motion of points located on several pin-connected bodies. In some problems, however, rigid bodies (mechanisms) are constructed such that *sliding* will occur at their connections. The kinematic analysis for such cases is best performed if the motion is analyzed using a coordinate system which both *translates* and *rotates*. Furthermore, this frame of reference is useful for analyzing the motions of two points on a mechanism which are *not* located in the *same* body and for specifying the kinematics of particle motion when the particle moves along a rotating path.

In the following analysis two equations will be developed which relate the velocity and acceleration of two points, one of which is the origin of a moving frame of reference subjected to both a translation and a rotation in the plane.*

Position. Consider the two points A and B shown in Fig. 16–32*a*. Their location is specified by the position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$, which are measured with respect to the fixed X, Y, Z coordinate system. As shown in the figure, the "base point" A represents the origin of the x, y, z coordinate system, which is assumed to be both translating and rotating with respect to the X, Y, Z system. The position of B with respect to A is specified by the relative-position vector $\mathbf{r}_{B/A}$. The components of this vector may be expressed either in terms of unit vectors along the X, Y axes, i.e., $\mathbf{I}$ and $\mathbf{J}$, or by unit vectors along the x, y axes, i.e., $\mathbf{i}$ and $\mathbf{j}$. For the development which follows, $\mathbf{r}_{B/A}$ will be measured with respect to the moving x, y frame of reference. Thus, if B has coordinates (x_B, y_B), Fig. 16–32*a*, then



(a)

Fig. 16–32

$$\mathbf{r}_{B/A} = x_B\mathbf{i} + y_B\mathbf{j}$$

The three position vectors in Fig. 16–32*a* are related by vector addition.

$$\boxed{\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}} \qquad (16\text{–}21)$$

At the instant considered, point A has a velocity $\mathbf{v}_A$ and an acceleration $\mathbf{a}_A$, while the angular velocity and angular acceleration of the x, y axes are $\mathbf{\Omega}$ (omega) and $\dot{\mathbf{\Omega}} = d\mathbf{\Omega}/dt$, respectively.

* The more general, three-dimensional motion of the points is developed in Sec. 20.4.

16

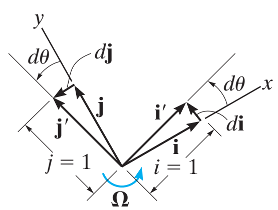
Velocity. The velocity of point B is determined by taking the time derivative of Eq. 16–21, which yields

$$\mathbf{v}_B = \mathbf{v}_A + \frac{d\mathbf{r}_{B/A}}{dt} \tag{16–22}$$

The last term in this equation is evaluated as follows:

$$\begin{aligned}\frac{d\mathbf{r}_{B/A}}{dt} &= \frac{d}{dt}(x_B\mathbf{i} + y_B\mathbf{j}) \\ &= \frac{dx_B}{dt}\mathbf{i} + x_B\frac{d\mathbf{i}}{dt} + \frac{dy_B}{dt}\mathbf{j} + y_B\frac{d\mathbf{j}}{dt} \\ &= \left(\frac{dx_B}{dt}\mathbf{i} + \frac{dy_B}{dt}\mathbf{j}\right) + \left(x_B\frac{d\mathbf{i}}{dt} + y_B\frac{d\mathbf{j}}{dt}\right)\end{aligned} \tag{16–23}$$

The two terms in the first set of parentheses represent the components of velocity of point B as measured by an observer attached to the moving x, y, z coordinate system, $(\mathbf{v}_{B/A})_{xyz}$. In the second set of parentheses the time rate of change of the unit vectors $\mathbf{i}$ and $\mathbf{j}$ are measured by an observer located in the fixed X, Y, Z coordinate system. These changes, $d\mathbf{i}$ and $d\mathbf{j}$, are due *only* to the *rotation* $d\theta$ of the x, y, z axes, causing $\mathbf{i}$ to become $\mathbf{i}' = \mathbf{i} + d\mathbf{i}$ and $\mathbf{j}$ to become $\mathbf{j}' = \mathbf{j} + d\mathbf{j}$, Fig. 16–32*b*. As shown, the *magnitudes* of both $d\mathbf{i}$ and $d\mathbf{j}$ equal 1 $d\theta$, since $i = i' = j = j' = 1$. The *direction* of $d\mathbf{i}$ is defined by $+\mathbf{j}$, since $d\mathbf{i}$ is tangent to the path described by the arrowhead of $\mathbf{i}$ in the limit as $\Delta t \to dt$. Likewise, $d\mathbf{j}$ acts in the $-\mathbf{i}$ direction, Fig. 16–32*b*. Hence,

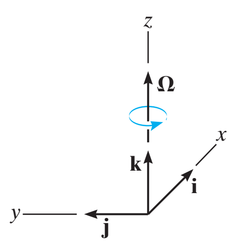


(b)

$$\frac{d\mathbf{i}}{dt} = \frac{d\theta}{dt}(\mathbf{j}) = \Omega\mathbf{j} \qquad \frac{d\mathbf{j}}{dt} = \frac{d\theta}{dt}(-\mathbf{i}) = -\Omega\mathbf{i}$$

Viewing the axes in three dimensions, Fig. 16–32*c*, and noting that $\boldsymbol{\Omega} = \Omega\mathbf{k}$, we can express the above derivatives in terms of the cross product as

$$\frac{d\mathbf{i}}{dt} = \boldsymbol{\Omega} \times \mathbf{i} \qquad \frac{d\mathbf{j}}{dt} = \boldsymbol{\Omega} \times \mathbf{j} \tag{16–24}$$



(c)

Fig. 16–32 (cont.)

Substituting these results into Eq. 16–23 and using the distributive property of the vector cross product, we obtain

$$\begin{aligned}\frac{d\mathbf{r}_{B/A}}{dt} &= (\mathbf{v}_{B/A})_{xyz} + \boldsymbol{\Omega} \times (x_B\mathbf{i} + y_B\mathbf{j}) \\ &= (\mathbf{v}_{B/A})_{xyz} + \boldsymbol{\Omega} \times \mathbf{r}_{B/A}\end{aligned} \tag{16–25}$$

Hence, Eq. 16–22 becomes

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz} \qquad (16\text{–}26)$$

where

$\mathbf{v}_B$ = velocity of B, measured from the X, Y, Z reference
$\mathbf{v}_A$ = velocity of the origin A of the x, y, z reference, measured from the X, Y, Z reference
$(\mathbf{v}_{B/A})_{xyz}$ = velocity of "B with respect to A," as measured by an observer attached to the rotating x, y, z reference
$\mathbf{\Omega}$ = angular velocity of the x, y, z reference, measured from the X, Y, Z reference
$\mathbf{r}_{B/A}$ = position of B with respect to A

Comparing Eq. 16–26 with Eq. 16–16 ($\mathbf{v}_B = \mathbf{v}_A + \mathbf{\Omega} \times \mathbf{r}_{B/A}$), which is valid for a translating frame of reference, it can be seen that the only difference between these two equations is represented by the term $(\mathbf{v}_{B/A})_{xyz}$.

When applying Eq. 16–26 it is often useful to understand what each of the terms represents. In order of appearance, they are as follows:

$\mathbf{v}_B$	absolute velocity of B	motion of B observed from the X, Y, Z frame
	(equals)	
$\mathbf{v}_A$	absolute velocity of the origin of x, y, z frame	motion of x, y, z frame observed from the X, Y, Z frame
	(plus)	
$\mathbf{\Omega} \times \mathbf{r}_{B/A}$	angular velocity effect caused by the rotation of x, y, z frame	
	(plus)	
$(\mathbf{v}_{B/A})_{xyz}$	velocity of B with respect to A	motion of B observed from the x, y, z frame

16

Acceleration. The acceleration of B, observed from the X, Y, Z coordinate system, may be expressed in terms of its motion measured with respect to the rotating system of coordinates by taking the time derivative of Eq. 16–26.

$$\frac{d\mathbf{v}_B}{dt} = \frac{d\mathbf{v}_A}{dt} + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times \frac{d\mathbf{r}_{B/A}}{dt} + \frac{d(\mathbf{v}_{B/A})_{xyz}}{dt}$$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times \frac{d\mathbf{r}_{B/A}}{dt} + \frac{d(\mathbf{v}_{B/A})_{xyz}}{dt} \qquad (16\text{–}27)$$

Here $\dot{\boldsymbol{\Omega}} = d\boldsymbol{\Omega}/dt$ is the angular acceleration of the x, y, z coordinate system. Since $\boldsymbol{\Omega}$ is always perpendicular to the plane of motion, then $\dot{\boldsymbol{\Omega}}$ measures *only the change in magnitude* of $\boldsymbol{\Omega}$. The derivative $d\mathbf{r}_{B/A}/dt$ is defined by Eq. 16–25, so that

$$\boldsymbol{\Omega} \times \frac{d\mathbf{r}_{B/A}}{dt} = \boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A}) \qquad (16\text{–}28)$$

Finding the time derivative of $(\mathbf{v}_{B/A})_{xyz} = (v_{B/A})_x\mathbf{i} + (v_{B/A})_y\mathbf{j}$,

$$\frac{d(\mathbf{v}_{B/A})_{xyz}}{dt} = \left[\frac{d(v_{B/A})_x}{dt}\mathbf{i} + \frac{d(v_{B/A})_y}{dt}\mathbf{j}\right] + \left[(v_{B/A})_x\frac{d\mathbf{i}}{dt} + (v_{B/A})_y\frac{d\mathbf{j}}{dt}\right]$$

The two terms in the first set of brackets represent the components of acceleration of point B as measured by an observer attached to the rotating coordinate system. These terms will be denoted by $(\mathbf{a}_{B/A})_{xyz}$. The terms in the second set of brackets can be simplified using Eqs. 16–24.

$$\frac{d(\mathbf{v}_{B/A})_{xyz}}{dt} = (\mathbf{a}_{B/A})_{xyz} + \boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz}$$

Substituting this and Eq. 16–28 into Eq. 16–27 and rearranging terms,

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz} \qquad (16\text{–}29)$$

where

$\mathbf{a}_B$ = acceleration of B, measured from the X, Y, Z reference

$\mathbf{a}_A$ = acceleration of the origin A of the x, y, z reference, measured from the X, Y, Z reference

$(\mathbf{a}_{B/A})_{xyz}$, $(\mathbf{v}_{B/A})_{xyz}$ = acceleration and velocity of B with respect to A, as measured by an observer attached to the rotating x, y, z reference

$\dot{\boldsymbol{\Omega}}$, $\boldsymbol{\Omega}$ = angular acceleration and angular velocity of the x, y, z reference, measured from the X, Y, Z reference

$\mathbf{r}_{B/A}$ = position of B with respect to A

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If Eq. 16–29 is compared with Eq. 16–18, $\mathbf{a}_B = \mathbf{a}_A + \dot{\mathbf{\Omega}} \times \mathbf{r}_{B/A} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{B/A})$, which is valid for a translating frame of reference, it can be seen that the difference between these two equations is represented by the terms $2\mathbf{\Omega} \times (\mathbf{v}_{B/A})_{xyz}$ and $(\mathbf{a}_{B/A})_{xyz}$. In particular, $2\mathbf{\Omega} \times (\mathbf{v}_{B/A})_{xyz}$ is called the ***Coriolis acceleration***, named after the French engineer G. C. Coriolis, who was the first to determine it. This term represents the difference in the acceleration of B as measured from nonrotating and rotating x, y, z axes. As indicated by the vector cross product, the Coriolis acceleration will *always* be perpendicular to both $\mathbf{\Omega}$ and $(\mathbf{v}_{B/A})_{xyz}$. It is an important component of acceleration which must be considered whenever rotating reference frames are used. This often occurs, for example, when studying the accelerations and forces which act on rockets, long-range projectiles, or other bodies having motions whose measurements are significantly affected by the rotation of the earth.

The following interpretation of the terms in Eq. 16–29 may be useful when applying this equation to the solution of problems.

$\mathbf{a}_B$	absolute acceleration of B	motion of B observed from the X, Y, Z frame
	(equals)	
$\mathbf{a}_A$	absolute acceleration of the origin of x, y, z frame	motion of x, y, z frame observed from the X, Y, Z frame
	(plus)	
$\dot{\mathbf{\Omega}} \times \mathbf{r}_{B/A}$	angular acceleration effect caused by rotation of x, y, z frame	
	(plus)	
$\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{B/A})$	angular velocity effect caused by rotation of x, y, z frame	
	(plus)	
$2\mathbf{\Omega} \times (\mathbf{v}_{B/A})_{xyz}$	combined effect of B moving relative to x, y, z coordinates and rotation of x, y, z frame	interacting motion
	(plus)	
$(\mathbf{a}_{B/A})_{xyz}$	acceleration of B with respect to A	motion of B observed from the x, y, z frame

PROCEDURE FOR ANALYSIS

Equations 16–26 and 16–29 can be applied to the solution of problems involving the planar motion of particles or rigid bodies using the following procedure.

Coordinate Axes.

- Choose an appropriate location for the origin and proper orientation of the axes for both fixed X, Y, Z and moving x, y, z reference frames.
- Most often solutions are easily obtained if at the instant considered:
  1. the origins are coincident
  2. the corresponding axes are collinear
  3. the corresponding axes are parallel
- The moving frame should be selected fixed to the body or device along which the relative motion occurs.

Kinematic Equations.

- After defining the origin A of the moving reference and specifying the moving point B, Eqs. 16–26 and 16–29 should be written in symbolic form.

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz}$$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}$$

- The Cartesian components of all these vectors may be expressed along either the X, Y, Z axes or the x, y, z axes. The choice is arbitrary provided a consistent set of unit vectors is used.
- Motion of the moving reference is expressed by $\mathbf{v}_A$, $\mathbf{a}_A$, $\boldsymbol{\Omega}$, and $\dot{\boldsymbol{\Omega}}$; and motion of B with respect to the moving reference is expressed by $\mathbf{r}_{B/A}$, $(\mathbf{v}_{B/A})_{xyz}$, and $(\mathbf{a}_{B/A})_{xyz}$.

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The rotation of the dumping bin of the truck about point C is operated by the extension of the hydraulic cylinder AB. To determine the rotation of the bin due to this extension, we can use the equations of relative motion and fix the rotating x, y, z axes to the cylinder so that the relative motion of the cylinder's extension occurs along the y axis.

EXAMPLE 16.18

At the instant $\theta = 60°$, the rod in Fig. 16–33 has an angular velocity of 3 rad/s and an angular acceleration of 2 rad/s^2. At this same instant, collar C travels outward along the rod such that when $x = 0.2$ m its velocity is 2 m/s and its acceleration is 3 m/s^2, both measured relative to the rod. Determine the Coriolis acceleration and the velocity and acceleration of the collar at this instant.

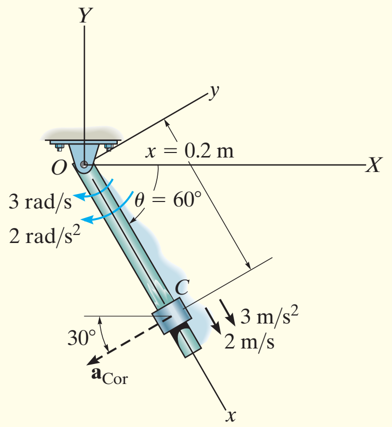

Fig. 16–33

SOLUTION

Coordinate Axes. The origin of both coordinate systems is located at point O, Fig. 16–33. Since motion of the collar is reported relative to the rod, the moving x, y, z frame of reference is attached to the rod.

Kinematic Equations.

$$\mathbf{v}_C = \mathbf{v}_O + \boldsymbol{\Omega} \times \mathbf{r}_{C/O} + (\mathbf{v}_{C/O})_{xyz} \qquad (1)$$

$$\mathbf{a}_C = \mathbf{a}_O + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/O} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/O}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/O})_{xyz} + (\mathbf{a}_{C/O})_{xyz} \qquad (2)$$

It will be simpler to express all the vectors in terms of their **i**, **j**, **k** components rather than **I**, **J**, **K** components. Hence,

Motion of moving reference	Motion of C with respect to moving reference
$\mathbf{v}_O = \mathbf{0}$	$\mathbf{r}_{C/O} = \{0.2\mathbf{i}\}$ m
$\mathbf{a}_O = \mathbf{0}$	$(\mathbf{v}_{C/O})_{xyz} = \{2\mathbf{i}\}$ m/s
$\boldsymbol{\Omega} = \{-3\mathbf{k}\}$ rad/s	$(\mathbf{a}_{C/O})_{xyz} = \{3\mathbf{i}\}$ m/s^2
$\dot{\boldsymbol{\Omega}} = \{-2\mathbf{k}\}$ rad/s^2	

The Coriolis acceleration is defined as

$$\mathbf{a}_{Cor} = 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/O})_{xyz} = 2(-3\mathbf{k}) \times (2\mathbf{i}) = \{-12\mathbf{j}\} \text{ m/s}^2 \qquad \textit{Ans.}$$

This vector is shown dashed in Fig. 16–33. If desired, it may be resolved into **I**, **J** components acting along the X and Y axes, respectively.

The velocity and acceleration of the collar are determined by substituting the data into Eqs. 1 and 2 and evaluating the cross products, which yields

$$\begin{aligned}\mathbf{v}_C &= \mathbf{v}_O + \boldsymbol{\Omega} \times \mathbf{r}_{C/O} + (\mathbf{v}_{C/O})_{xyz} \\ &= \mathbf{0} + (-3\mathbf{k}) \times (0.2\mathbf{i}) + 2\mathbf{i} \\ &= \{2\mathbf{i} - 0.6\mathbf{j}\} \text{ m/s} \qquad \textit{Ans.}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_C &= \mathbf{a}_O + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/O} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/O}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/O})_{xyz} + (\mathbf{a}_{C/O})_{xyz} \\ &= \mathbf{0} + (-2\mathbf{k}) \times (0.2\mathbf{i}) + (-3\mathbf{k}) \times [(-3\mathbf{k}) \times (0.2\mathbf{i})] + 2(-3\mathbf{k}) \times (2\mathbf{i}) + 3\mathbf{i} \\ &= \mathbf{0} - 0.4\mathbf{j} - 1.80\mathbf{i} - 12\mathbf{j} + 3\mathbf{i} \\ &= \{1.20\mathbf{i} - 12.4\mathbf{j}\} \text{ m/s}^2 \qquad \textit{Ans.}\end{aligned}$$

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EXAMPLE 16.19

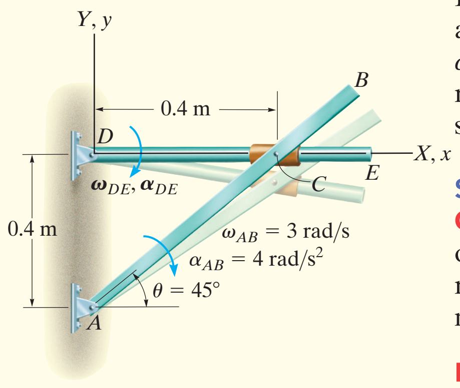

Fig. 16–34

Rod AB, shown in Fig. 16–34, rotates clockwise such that it has an angular velocity $\omega_{AB} = 3$ rad/s and angular acceleration $\alpha_{AB} = 4$ rad/s^2 when $\theta = 45°$. Determine the angular motion of rod DE at this instant. The collar at C is pin connected to AB and slides over rod DE.

SOLUTION

Coordinate Axes. The origin of both the fixed and moving frames of reference is located at D, Fig. 16–34. Furthermore, the x, y, z reference is attached to and rotates with rod DE so that the relative motion of the collar is easy to follow.

Kinematic Equations.

$$\mathbf{v}_C = \mathbf{v}_D + \boldsymbol{\Omega} \times \mathbf{r}_{C/D} + (\mathbf{v}_{C/D})_{xyz} \tag{1}$$

$$\mathbf{a}_C = \mathbf{a}_D + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/D} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/D}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/D})_{xyz} + (\mathbf{a}_{C/D})_{xyz} \tag{2}$$

All vectors will be expressed in terms of **i**, **j**, **k** components.

Motion of moving reference	Motion of C with respect to moving reference
$\mathbf{v}_D = \mathbf{0}$	$\mathbf{r}_{C/D} = \{0.4\mathbf{i}\}$ m
$\mathbf{a}_D = \mathbf{0}$	$(\mathbf{v}_{C/D})_{xyz} = (v_{C/D})_{xyz}\mathbf{i}$
$\boldsymbol{\Omega} = -\omega_{DE}\mathbf{k}$	$(\mathbf{a}_{C/D})_{xyz} = (a_{C/D})_{xyz}\mathbf{i}$
$\dot{\boldsymbol{\Omega}} = -\alpha_{DE}\mathbf{k}$	

Motion of C: Since the collar moves along a *circular path* of radius AC, its velocity and acceleration can be determined using Eqs. 16–9 and 16–14.

$$\mathbf{v}_C = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{C/A} = (-3\mathbf{k}) \times (0.4\mathbf{i} + 0.4\mathbf{j}) = \{1.2\mathbf{i} - 1.2\mathbf{j}\} \text{ m/s}$$

$$\mathbf{a}_C = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{C/A} - \omega_{AB}^2\mathbf{r}_{C/A}$$
$$= (-4\mathbf{k}) \times (0.4\mathbf{i} + 0.4\mathbf{j}) - (3)^2(0.4\mathbf{i} + 0.4\mathbf{j}) = \{-2\mathbf{i} - 5.2\mathbf{j}\} \text{ m/s}^2$$

Substituting the data into Eqs. 1 and 2, we have

$$\mathbf{v}_C = \mathbf{v}_D + \boldsymbol{\Omega} \times \mathbf{r}_{C/D} + (\mathbf{v}_{C/D})_{xyz}$$
$$1.2\mathbf{i} - 1.2\mathbf{j} = \mathbf{0} + (-\omega_{DE}\mathbf{k}) \times (0.4\mathbf{i}) + (v_{C/D})_{xyz}\mathbf{i}$$
$$1.2\mathbf{i} - 1.2\mathbf{j} = \mathbf{0} - 0.4\omega_{DE}\mathbf{j} + (v_{C/D})_{xyz}\mathbf{i}$$
$$(v_{C/D})_{xyz} = 1.2 \text{ m/s}$$
$$\omega_{DE} = 3 \text{ rad/s} \circlearrowright \qquad \textit{Ans.}$$

$$\mathbf{a}_C = \mathbf{a}_D + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/D} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/D}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/D})_{xyz} + (\mathbf{a}_{C/D})_{xyz}$$
$$-2\mathbf{i} - 5.2\mathbf{j} = \mathbf{0} + (-\alpha_{DE}\mathbf{k}) \times (0.4\mathbf{i}) + (-3\mathbf{k}) \times [(-3\mathbf{k}) \times (0.4\mathbf{i})] + 2(-3\mathbf{k}) \times (1.2\mathbf{i}) + (a_{C/D})_{xyz}\mathbf{i}$$
$$-2\mathbf{i} - 5.2\mathbf{j} = -0.4\alpha_{DE}\mathbf{j} - 3.6\mathbf{i} - 7.2\mathbf{j} + (a_{C/D})_{xyz}\mathbf{i}$$
$$(a_{C/D})_{xyz} = 1.6 \text{ m/s}^2$$
$$\alpha_{DE} = -5 \text{ rad/s}^2 = 5 \text{ rad/s}^2 \circlearrowleft \qquad \textit{Ans.}$$

EXAMPLE 16.20

Planes A and B fly at the same elevation and have the motions shown in Fig. 16–35. Determine the velocity and acceleration of A as measured by the pilot of B.

SOLUTION

Coordinate Axes. Since the relative motion of A with respect to the pilot in B is being sought, the x, y, z axes are attached to plane B, Fig. 16–35. At the *instant* considered, the origin B coincides with the origin of the fixed X, Y, Z frame.

Kinematic Equations.

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\Omega} \times \mathbf{r}_{A/B} + (\mathbf{v}_{A/B})_{xyz} \quad (1)$$

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{xyz} + (\mathbf{a}_{A/B})_{xyz} \quad (2)$$

Motion of Moving Reference: Using $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components,

$$\mathbf{v}_B = \{600\mathbf{j}\} \text{ km/h}$$

$$(a_B)_n = \frac{v_B^2}{\rho} = \frac{(600)^2}{400} = 900 \text{ km/h}^2$$

$$\mathbf{a}_B = (\mathbf{a}_B)_n + (\mathbf{a}_B)_t = \{900\mathbf{i} - 100\mathbf{j}\} \text{ km/h}^2$$

$$\Omega = \frac{v_B}{\rho} = \frac{600 \text{ km/h}}{400 \text{ km}} = 1.5 \text{ rad/h} \circlearrowright \qquad \boldsymbol{\Omega} = \{-1.5\mathbf{k}\} \text{ rad/h}$$

$$\dot{\Omega} = \frac{(a_B)_t}{\rho} = \frac{100 \text{ km/h}^2}{400 \text{ km}} = 0.25 \text{ rad/h}^2 \circlearrowleft \qquad \dot{\boldsymbol{\Omega}} = \{0.25\mathbf{k}\} \text{ rad/h}^2$$

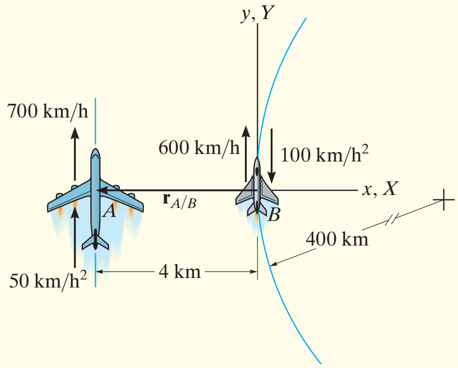

Fig. 16–35

Motion of A with Respect to Moving Reference:

$$\mathbf{r}_{A/B} = \{-4\mathbf{i}\} \text{ km} \quad (\mathbf{v}_{A/B})_{xyz} = ? \quad (\mathbf{a}_{A/B})_{xyz} = ?$$

Substituting the data into Eqs. 1 and 2, realizing that $\mathbf{v}_A = \{700\mathbf{j}\}$ km/h and $\mathbf{a}_A = \{50\mathbf{j}\}$ km/h^2, we have

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\Omega} \times \mathbf{r}_{A/B} + (\mathbf{v}_{A/B})_{xyz}$$

$$700\mathbf{j} = 600\mathbf{j} + (-1.5\mathbf{k}) \times (-4\mathbf{i}) + (\mathbf{v}_{A/B})_{xyz}$$

$$(\mathbf{v}_{A/B})_{xyz} = \{94\mathbf{j}\} \text{ km/h} \qquad \textit{Ans.}$$

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{xyz} + (\mathbf{a}_{A/B})_{xyz}$$

$$50\mathbf{j} = (900\mathbf{i} - 100\mathbf{j}) + (0.25\mathbf{k}) \times (-4\mathbf{i})$$

$$+ (-1.5\mathbf{k}) \times [(-1.5\mathbf{k}) \times (-4\mathbf{i})] + 2(-1.5\mathbf{k}) \times (94\mathbf{j}) + (\mathbf{a}_{A/B})_{xyz}$$

$$(\mathbf{a}_{A/B})_{xyz} = \{-1191\mathbf{i} + 151\mathbf{j}\} \text{ km/h}^2 \qquad \textit{Ans.}$$

NOTE: The solution of this problem should be compared with that of Example 12.26, where it is seen that $(v_{B/A})_{xyz} \neq (v_{A/B})_{xyz}$ and $(a_{B/A})_{xyz} \neq (a_{A/B})_{xyz}$.

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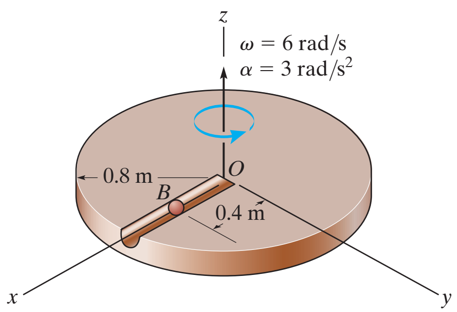
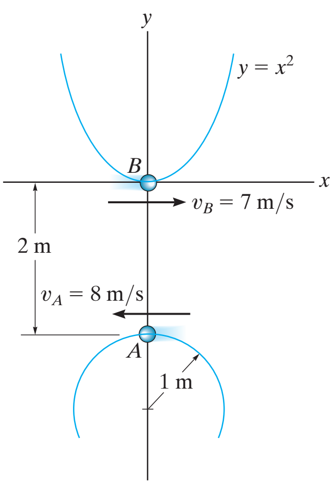
PROBLEMS

16–129. At the instant shown, ball B is rolling along the slot in the disk with a velocity of 600 mm/s and an acceleration of 150 mm/s^2, both measured relative to the disk and directed away from O. If at the same instant the disk has the angular velocity and angular acceleration shown, determine the velocity and acceleration of the ball at this instant.



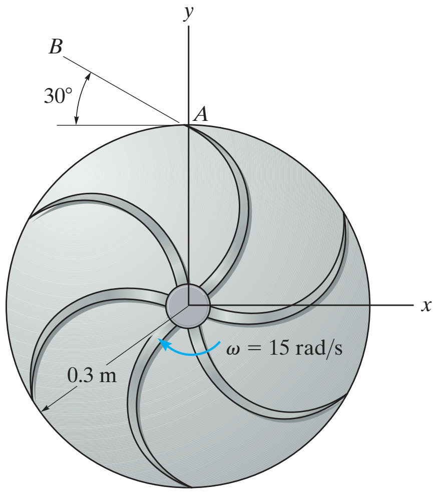
Prob. 16–129

16–130. Particles B and A move along the parabolic and circular paths, respectively. If B has a velocity of 7 m/s in the direction shown and its speed is increasing at 4 m/s^2, while A has a velocity of 8 m/s in the direction shown and its speed is decreasing at 6 m/s^2, determine the relative velocity and relative acceleration of B with respect to A.



Prob. 16–130

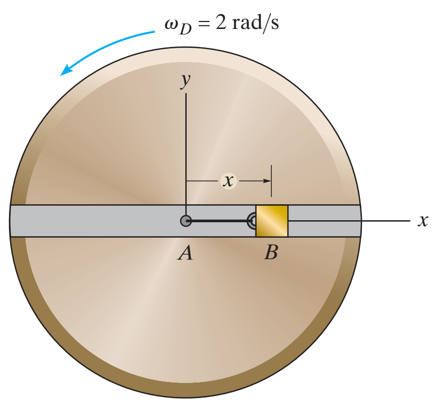
16–131. Water leaves the impeller of the centrifugal pump with a velocity of 25 m/s and acceleration of 30 m/s^2, both measured relative to the impeller along the blade line AB. Determine the velocity and acceleration of a water particle at A as it leaves the impeller at the instant shown. The impeller rotates with a constant angular velocity of $\omega = 15$ rad/s.



Prob. 16–131

***16–132.** The slider block B, which is attached to a cord, moves along the slot of the horizontal circular disk. If the cord is pulled down through the central hole A in the disk at a constant rate of $\dot{x} = -3$ m/s, measured relative to the disk, determine the acceleration of the block at the instant $x = 0.1$ m. The disk has a constant angular velocity of $\omega_D = 2$ rad/s.

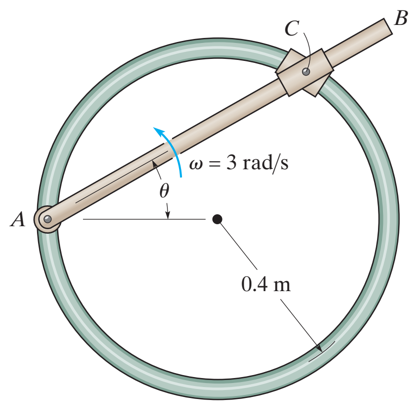
16–133. Solve Prob. 16–132 assuming that at the instant $x = 0.1$ m, $\dot{x} = -3$ m/s, $\ddot{x} = 1.25$ m/s^2, $\omega_D = 2$ rad/s, and the disk has an angular deceleration $\alpha_D = -4$ rad/s^2.



Probs. 16–132/133

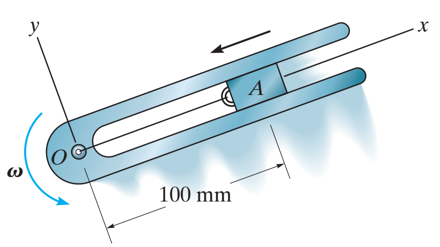
16–134. Rod AB rotates counterclockwise with a constant angular velocity $\omega = 3$ rad/s. Determine the velocity of point C located on the double collar when $\theta = 30°$. The collar consists of two pin-connected slider blocks which are constrained to move along the circular path and the rod AB.

16–135. Rod AB rotates counterclockwise with a constant angular velocity $\omega = 3$ rad/s. Determine the velocity and acceleration of point C located on the double collar when $\theta = 45°$. The collar consists of two pin-connected slider blocks which are constrained to move along the circular path and the rod AB.



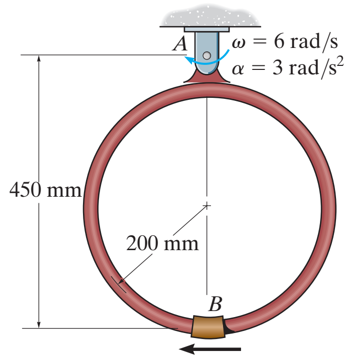
Probs. 16–134/135

***16–136.** Block A, which is attached to a cord, moves along the slot of a horizontal forked rod. At the instant shown, the cord is pulled down through the hole at O with an acceleration of $4\ \text{m/s}^2$ and its velocity is 2 m/s. Determine the acceleration of the block at this instant. The rod rotates about O with a constant angular velocity $\omega = 4$ rad/s.



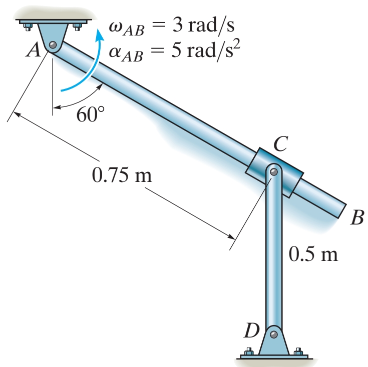
Prob. 16–136

16–137. Collar B moves to the left with a speed of 5 m/s, which is increasing at a constant rate of $1.5\ \text{m/s}^2$, relative to the hoop, while the hoop rotates with the angular velocity and angular acceleration shown. Determine the magnitudes of the velocity and acceleration of the collar at this instant.



Prob. 16–137

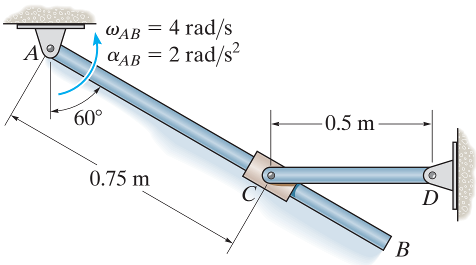
16–138. At the instant shown, rod AB has an angular velocity $\omega_{AB} = 3$ rad/s and an angular acceleration $\alpha_{AB} = 5\ \text{rad/s}^2$. Determine the angular velocity and angular acceleration of rod CD at this instant. The collar at C is pin connected to CD and slides over AB.



Prob. 16–138

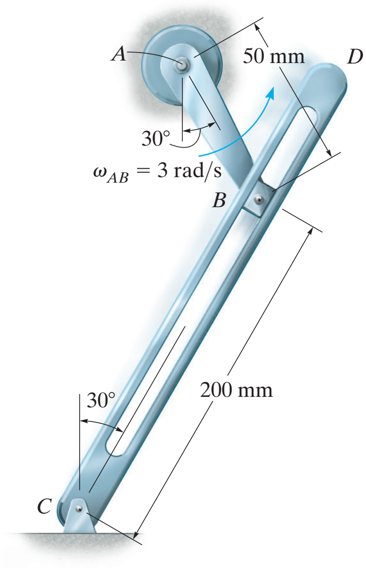
16

16–139. At the instant shown rod AB has an angular velocity $\omega_{AB} = 4$ rad/s and an angular acceleration $\alpha_{AB} = 2$ rad/s^2. Determine the angular velocity and angular acceleration of rod CD at this instant. The collar at C is pin connected to CD and slides freely along AB.



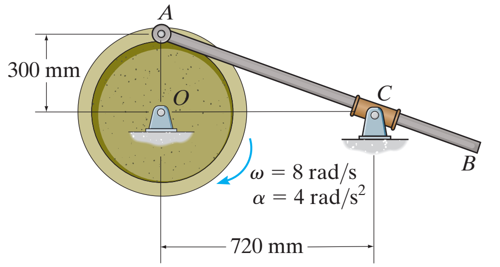
Prob. 16–139

***16–140.** The block B of the "quick-return" mechanism is confrined to move within the slot in member CD. If AB is rotating at a constant rate of $\omega_{AB} = 3$ rad/s, determine the angular velocity and angular acceleration of member CD at the instant shown.



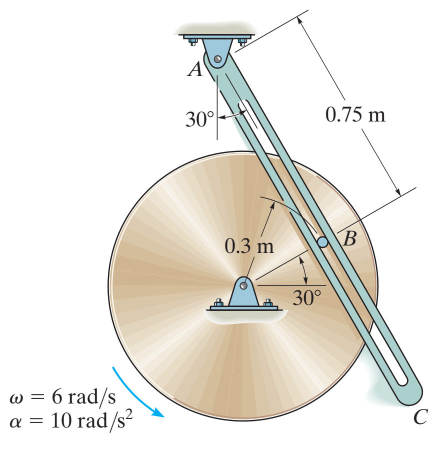
Prob. 16–140

16–141. The wheel is rotating with the angular velocity and angular acceleration at the instant shown. Determine the angular velocity and angular acceleration of the rod at this instant. The rod slides freely through the smooth collar.



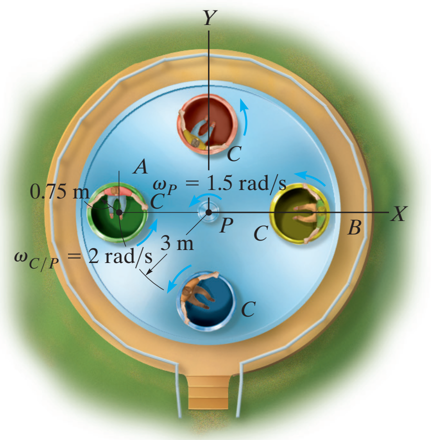
Prob. 16–141

16–142. The disk rotates with the angular motion shown. Determine the angular velocity and angular acceleration of the slottted link AC at this instant. The peg at B is fixed to the disk.



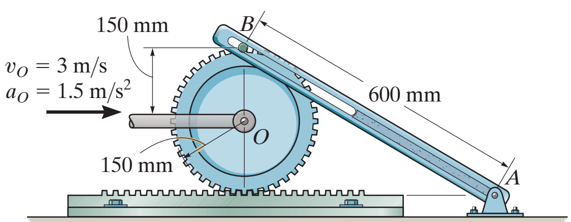
Prob. 16–142

16–143. A ride in an amusement park consists of a rotating platform P, having a constant angular velocity $\omega_P = 1.5$ rad/s, and four cars, C, mounted on the platform, which have constant angular velocities $\omega_{C/P} = 2$ rad/s measured relative to the platform. Determine the velocity and acceleration of the passenger at B at the instant shown.



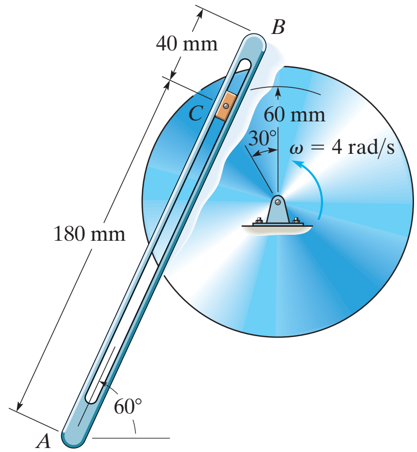
Prob. 16–143

***16–144.** Peg B on the gear slides freely along the slot in link AB. If the gear's center O moves with the velocity and acceleration shown, determine the angular velocity and angular acceleration of the link at this instant.



Prob. 16–144

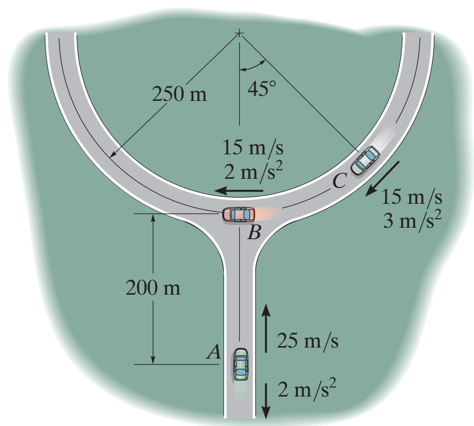
16–145. If the slider block C is fixed to the disk that has a constant counterclockwise angular velocity of 4 rad/s, determine the angular velocity and angular acceleration of the slotted arm AB at the instant shown.



Prob. 16–145

16–146. At the instant shown, car A travels with a speed of 25 m/s, which is decreasing at a constant rate of 2 m/s^2, while car C travels with a speed of 15 m/s, which is increasing at a constant rate of 3 m/s. Determine the velocity and acceleration of car A with respect to car C.

16–147. At the instant shown, car B travels with a speed of 15 m/s, which is increasing at a constant rate of 2 m/s^2, while car C travels with a speed of 15 m/s, which is increasing at a constant rate of 3 m/s^2. Determine the velocity and acceleration of car B with respect to car C.



Probs. 16–146/147

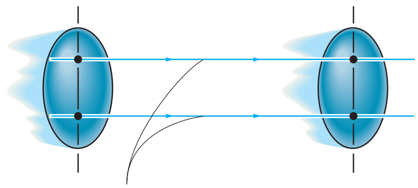
CHAPTER REVIEW

Rigid-Body Planar Motion

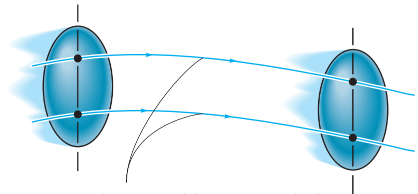
A rigid body undergoes three types of planar motion: translation, rotation about a fixed axis, and general plane motion.

Translation

When a body has rectilinear translation, all the particles of the body travel along parallel straight-line paths. If the paths have the same radius of curvature, then curvilinear translation occurs. Provided we know the motion of one of the particles, then the motion of all of the others is also known.



Path of rectilinear translation



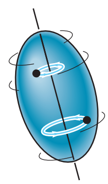
Path of curvilinear translation

Rotation about a Fixed Axis

For this type of motion, all of the particles move along circular paths. Here, all line segments in the body undergo the same angular displacement, angular velocity, and angular acceleration.

Once the angular motion of the body is known, then the velocity of any particle a distance r from the axis can be obtained.

The acceleration of any particle has two components. The tangential component accounts for the change in the magnitude of the velocity, and the normal component accounts for the change in the velocity's direction.



Rotation about a fixed axis

$$\omega = d\theta/dt \qquad\qquad \omega = \omega_0 + \alpha_c t$$

$$\alpha = d\omega/dt \qquad \text{or} \qquad \theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha_c t^2$$

$$\alpha\, d\theta = \omega\, d\omega \qquad\qquad \omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$$

Constant α_c

$$v = \omega r \qquad\qquad a_t = \alpha r \qquad a_n = \omega^2 r$$

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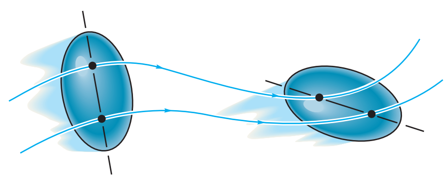
General Plane Motion

When a body undergoes general plane motion, it simultaneously translates and rotates. There are several methods for analyzing this motion.

Absolute Motion Analysis

If the motion of a point on a body or the angular motion of a line is known, then it may be possible to relate this motion to that of another point or line using an absolute motion analysis. To do so, linear position coordinates s or angular position coordinates θ are established (measured from a fixed point or line). These position coordinates are then related using the geometry of the body. The time derivative of this equation gives the relationship between the velocities and/or the angular velocities. A second time derivative relates the accelerations and/or the angular accelerations.



General plane motion

Relative Motion Using Translating Axes

General plane motion can also be analyzed using a relative-motion analysis between two points A and B located on the body. This method considers the motion in parts: first a translation of the selected base point A, then a relative "rotation" of the body about point A, which is measured from a translating axis. Since the relative motion is viewed as circular motion about the base point, point B will have a velocity $\mathbf{v}_{B/A}$ that is tangent to the circle. It also has two components of acceleration, $(\mathbf{a}_{B/A})_t$ and $(\mathbf{a}_{B/A})_n$. It is also important to realize that $\mathbf{a}_A$ and $\mathbf{a}_B$ will have tangential and normal components if these points move along curved paths.

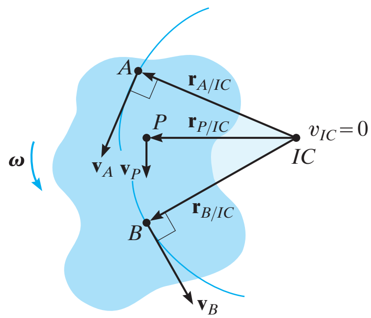
$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$$

Instantaneous Center of Zero Velocity

If the base point A is selected as having zero velocity, then the relative velocity equation becomes $\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/A}$.v In this case, motion appears as if the body rotates about an instantaneous axis passing through A.

The instantaneous center of rotation (IC) can be established provided the directions of the velocities of any two points on the body are known, or the velocity of a point and the angular velocity are known. Since a radial line r will always be perpendicular to each velocity, then the IC is at the point of intersection of these two radial lines. Its measured location is determined from the geometry of the body. Once it is established, then the velocity of any point P on the body can be determined from $v = \omega r$, where r extends from the IC to point P.



Relative Motion Using Rotating Axes

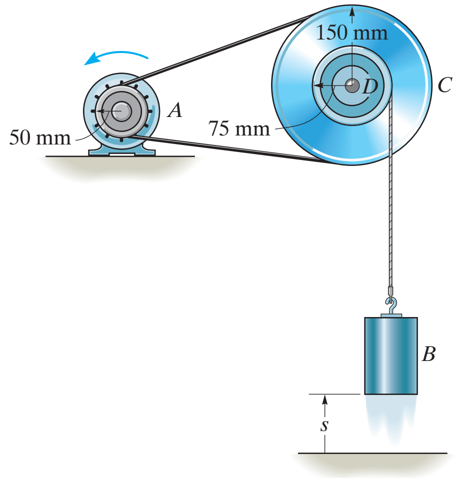
Problems that involve connected members that slide relative to one another or points not located on the same body can be analyzed using a relative-motion analysis referenced from a rotating frame. This gives rise to the term $2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz}$ that is called the Coriolis acceleration.

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz}$$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}$$

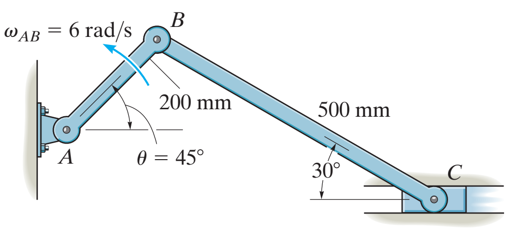
REVIEW PROBLEMS

R16–1. Starting at $(\omega_A)_0 = 3$ rad/s, when $\theta_A = 0$, $s = 0$, pulley A is given an angular acceleration $\alpha_A = (0.6\theta_A)$ rad/s^2, where θ_A is in radians. Determine the speed of block B when it has risen $s = 0.5$ m. The pulley has an inner hub D which is fixed to C and turns with it.



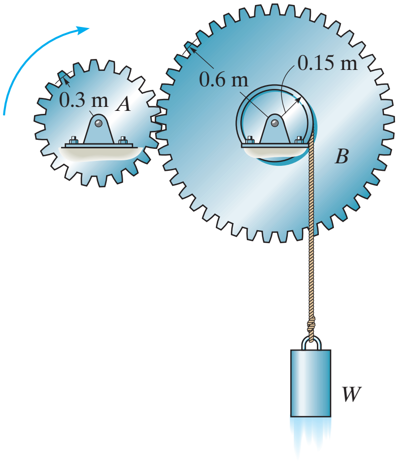
Prob. R16–1

R16–2. If bar AB has an angular velocity $\omega_{AB} = 6$ rad/s, determine the velocity of the slider block C at the instant shown.



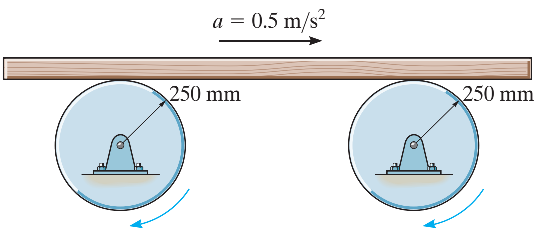
Prob. R16–2

R16–3. The hoisting gear A has an initial angular velocity of 60 rad/s and a constant deceleration of 1 rad/s^2. Determine the velocity and deceleration of the block W which is being hoisted by the hub on gear B when $t = 3$ s.



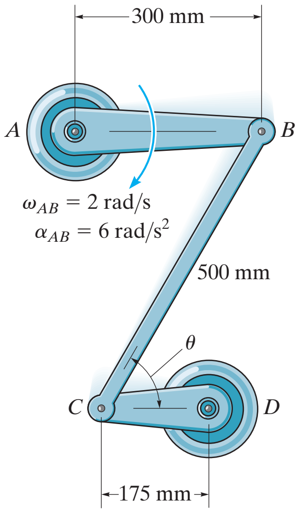
Prob. R16–3

R16–4. At the instant shown, the board has an acceleration of 0.5 m/s^2 to the right, while at the same instant points on the outer rim of each drum have an acceleration with a magnitude of 3 m/s^2. If the board does not slip on the drums, determine its speed due to the motion.



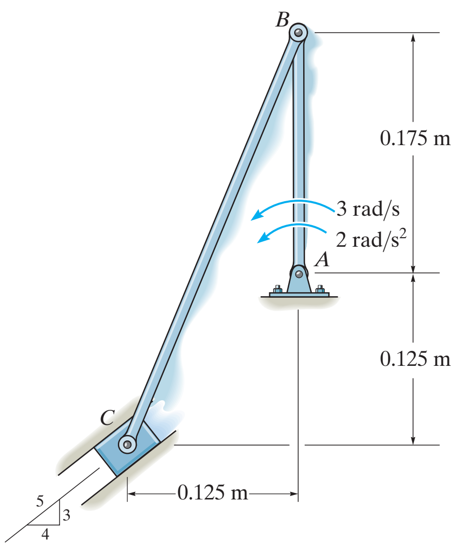
Prob. R16–4

R16–5. At the instant shown, link AB has an angular velocity $\omega_{AB} = 2$ rad/s and an angular acceleration $\alpha_{AB} = 6 \text{ rad/s}^2$. Determine the acceleration of the pin at C and the angular acceleration of link CB at this instant, when $\theta = 60°$.



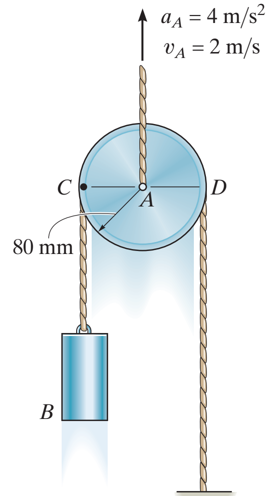
Prob. R16–5

R16–6. At the given instant member AB has the angular motions shown. Determine the velocity and acceleration of the slider block C at this instant.



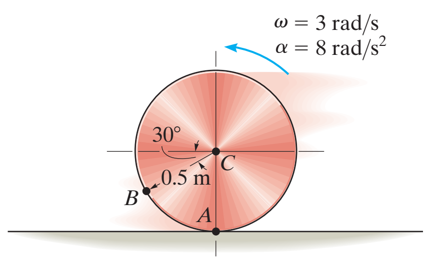
Prob. R16–6

R16–7. The center of the pulley is being lifted vertically with an acceleration of 4 m/s^2 at the instant it has a velocity of 2 m/s. If the cable does not slip on the pulley's surface, determine the accelerations of the cylinder B and point C on the pulley.



Prob. R16–7

R16–8. The disk is moving to the left such that it has an angular acceleration $\alpha = 8 \text{ rad/s}^2$ and angular velocity $\omega = 3$ rad/s at the instant shown. If it does not slip at A, determine the acceleration of point B.



Prob. R16–8

CHAPTER 17

Motorcycles and other vehicles can be subjected to severe dynamic loadings. In this chapter we will show how to determine these loadings for planar motion.

PLANAR KINETICS OF A RIGID BODY: FORCE AND ACCELERATION

Lecture Summary and Quiz, Example, and Problem-solving videos are available where this icon appears.

CHAPTER OBJECTIVES

- To introduce the methods used to determine the mass moment of inertia of a body.
- To develop the planar kinetic equations of motion for a symmetric rigid body.
- To discuss applications of these equations to bodies undergoing translation, rotation about a fixed axis, and general plane motion.

17.1 MASS MOMENT OF INERTIA

Since a body has a definite size and shape, an applied nonconcurrent force system can cause the body to both translate and rotate. The translational aspects of the motion were studied in Chapter 13 and are governed by the equation $\mathbf{F} = m\mathbf{a}$. It will be shown in the next section that the rotational aspects, caused by a moment $\mathbf{M}$, are governed by an equation of the form $\mathbf{M} = I\boldsymbol{\alpha}$. The symbol I in this equation is termed the mass moment of inertia. This property is a measure of the resistance of a body to angular acceleration ($\mathbf{M} = I\boldsymbol{\alpha}$) in the same way that mass is a measure of the body's resistance to acceleration ($\mathbf{F} = m\mathbf{a}$).

The flywheel on the engine of this tractor has a large moment of inertia about its axis of rotation. Once it is set into motion, it will be difficult to stop, and this in turn will prevent the engine from stalling and instead will allow it to maintain a constant power.

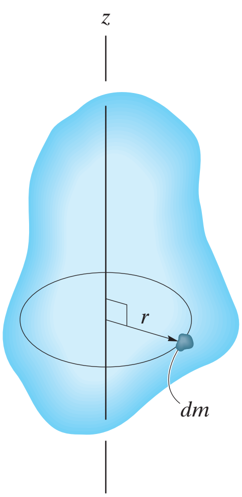


Fig. 17–1

17

We define the ***moment of inertia*** as the integral of the "second moment" about an axis of all the elements of mass dm which compose the body.* For example, the body's moment of inertia about the z axis in Fig. 17–1 is

$$I = \int_m r^2 \, dm \tag{17–1}$$

Here the "moment arm" r is the perpendicular distance from the z axis to the arbitrary element dm. Since the formulation involves r, the value of I will be different for each axis about which it is calculated. In the study of planar kinetics, the axis most often chosen passes through the body's mass center G and is always perpendicular to the plane of motion. The moment of inertia about this axis will be denoted as I_G. Since r is squared in Eq. 17–1, the mass moment of inertia is always a positive quantity. Common units used for its measurement are $\text{kg} \cdot \text{m}^2$.

If the body consists of material having a variable density, $\rho = \rho\,(x,y,z)$, the elemental mass dm of the body can be expressed in terms of its density and volume as $dm = \rho\, dV$, Fig. 17–2a. Substituting dm into Eq. 17–1, the body's moment of inertia is then calculated using volume elements for integration, i.e.,

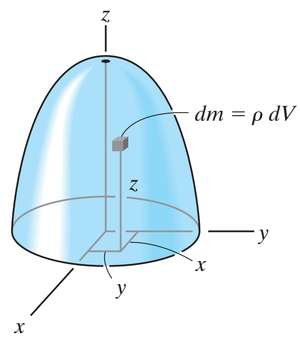
$$I = \int_V r^2 \rho \, dV \tag{17–2}$$

* Another property of the body, which measures the symmetry of the body's mass with respect to a coordinate system, is the product of inertia. This property applies to the three-dimensional motion of a body and will be discussed in Chapter 21.

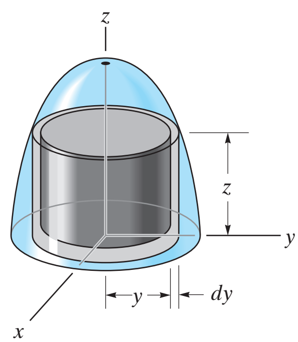
If ρ is *constant*, then this term may be factored out of the integral, and the integration is then simply a function of geometry,

$$I = \rho \int_V r^2 \, dV \qquad (17\text{–}3)$$

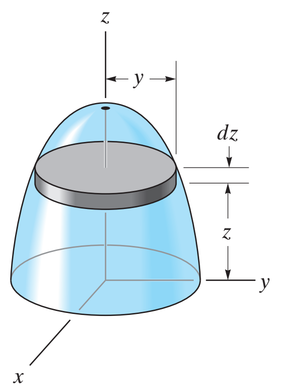
When the volume element chosen for integration has infinitesimal dimensions in all three directions, Fig. 17–2*a*, then the moment of inertia of the body must be determined using “triple integration.” However, for bodies that have symmetry about an axis, as in Fig. 17–2, the integration process can be simplified to a *single integration*. Here a shell element, Fig. 17–2*b*, or disk element, Fig. 17–2*c*, are often used for this purpose.



(a)



(b)



(c)

Fig. 17–2

PROCEDURE FOR ANALYSIS

To obtain the moment of inertia by integration, we will consider only symmetric bodies having volumes which are generated by revolving a curve about an axis, as in Fig. 17–2*a*. Two types of differential elements can be chosen.

Shell Element.

- If a *shell element* having a height z, radius $r = y$, and thickness dy is chosen, Fig. 17–2*b*, then the volume is $dV = (2\pi y)(z)dy$.
- This element may be used in Eq. 17–2 or 17–3 for determining the moment of inertia I_z of the body about the z axis, since the *entire element*, due to its “thinness,” lies at the *same* perpendicular distance $r = y$ from the z axis (see Example 17.1).

Disk Element.

- If a disk element having a radius y and a thickness dz is chosen, Fig. 17–2*c*, then the volume is $dV = (\pi y^2)dz$.
- This element is *finite* in the radial direction, and consequently its parts *do not* all lie at the *same radial distance* r from the z axis. As a result, Eq. 17–2 or 17–3 *cannot* be used to determine I_z directly. Instead, to perform the integration it is first necessary to determine the moment of inertia *of the element* about the z axis and then integrate this result (see Example 17.2).

Refer to the companion website for Lecture Summary and Quiz videos.

17

EXAMPLE 17.1

Determine the moment of inertia of the cylinder shown in Fig. 17–3*a* about the *z* axis. The density of the material, ρ, is constant.

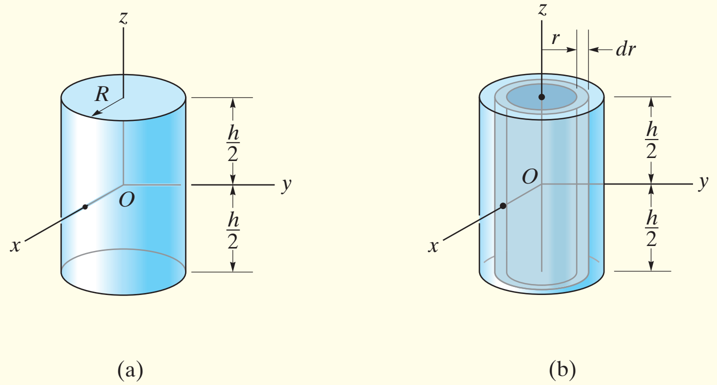


Fig. 17–3

SOLUTION

Shell Element. This problem will be solved using the shell element in Fig. 17–3*b* and a single integration. The volume of the element is $dV = (2\pi r)(h)\,dr$, so that its mass is $dm = \rho\,dV = \rho(2\pi hr\,dr)$. Since the entire element lies at the same distance r from the z axis, the moment of inertia of the element is

$$dI_z = r^2 dm = \rho 2\pi h r^3\,dr$$

Integrating over the entire region of the cylinder, we have

$$I_z = \int_m r^2\,dm = \rho 2\pi h \int_0^R r^3\,dr = \frac{\rho\pi}{2} R^4 h$$

The mass of the cylinder is

$$m = \int_m dm = \rho 2\pi h \int_0^R r\,dr = \rho\pi h R^2$$

And so,

$$I_z = \frac{1}{2} m R^2 \qquad \textit{Ans.}$$

17

EXAMPLE 17.2

If the density of the material is 2575 kg/m^3, determine the moment of inertia of the solid in Fig. 17–4*a* about the *y* axis.

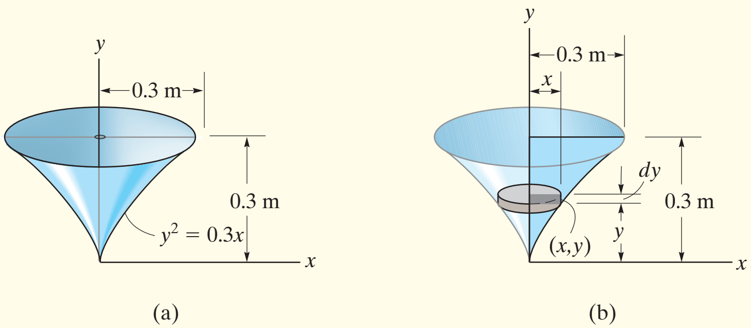


Fig. 17–4

SOLUTION

Disk Element. The moment of inertia will be found using a disk element, shown in Fig. 17–4*b*. Here the element intersects the curve at the arbitrary point (x,y) and has a mass

$$dm = \rho\, dV = \rho(\pi x^2)\, dy$$

Although all portions of the element are *not* located at the same distance from the *y* axis, it is still possible to determine the moment of inertia dI_y of the element about the *y* axis. In the preceding example it was shown that the moment of inertia of a cylinder about its longitudinal axis is $I = \frac{1}{2}mR^2$, where m and R are the mass and radius of the cylinder. Since the height is not involved in this formula, the disk can be thought of as a cylinder. Thus, for the disk element in Fig. 17–4*b*, we have

$$dI_y = \tfrac{1}{2}(dm)x^2 = \tfrac{1}{2}[\rho(\pi x^2)\, dy]x^2$$

Substituting $x = \dfrac{10}{3}y^2$, $\rho = 2575 \text{ kg/m}^3$, and integrating with respect to y, from $y = 0$ to $y = 0.3$ m, we get

$$I_y = \frac{\pi(2575 \text{ kg/m}^3)}{2}\int_0^{0.3 \text{ m}} x^4\, dy = \frac{\pi(2575)(10^4)}{2(3^4)}\int_0^{0.3 \text{ m}} y^8\, dy = 1.09 \text{ kg}\cdot\text{m}^2 \quad \textit{Ans.}$$

17

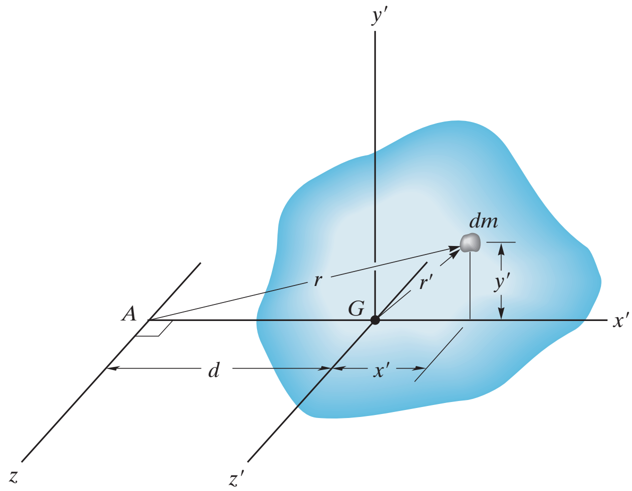


Fig. 17–5

Parallel-Axis Theorem. If the moment of inertia of the body about an axis passing through the body's mass center is known, then the moment of inertia about any other parallel axis can be determined by using the ***parallel-axis theorem***. This theorem can be derived by considering the body shown in Fig. 17–5. Here the z' axis passes through the mass center G, whereas the corresponding parallel z axis lies at a constant distance d away. Selecting the differential element of mass dm, which is located at point (x', y'), and using the Pythagorean theorem, $r^2 = (d + x')^2 + y'^2$, we can express the moment of inertia of the body about the z axis as

$$I = \int_m r^2\, dm = \int_m [(d + x')^2 + y'^2]\, dm$$

$$= \int_m (x'^2 + y'^2)\, dm + 2d \int_m x'\, dm + d^2 \int_m dm$$

Since $r'^2 = x'^2 + y'^2$, the first integral represents I_G. The second integral equals *zero*, since the z' axis passes through the body's mass center, i.e., $\int x'dm = \bar{x}'m = 0$ since $\bar{x}' = 0$. Finally, the third integral

represents the total mass m of the body. Hence, the moment of inertia about the z axis can be written as

$$I = I_G + md^2 \tag{17–4}$$

where

I_G = moment of inertia about the z' axis passing through the mass center G
m = mass of the body
d = perpendicular distance between the parallel z and z' axes

Radius of Gyration. Occasionally, the moment of inertia of a body about a specified axis is reported in handbooks using the ***radius of gyration,*** k. This is a geometrical property which has units of length. When it and the body's mass m are known, the body's moment of inertia is determined from the equation

$$I = mk^2 \quad \text{or} \quad k = \sqrt{\frac{I}{m}} \tag{17–5}$$

Note the similarity between the definition of k in this formula and r in the equation $dI = r^2\,dm$, which defines the moment of inertia of an elemental mass dm of the body about an axis.

Composite Bodies. If a body consists of a number of simple shapes such as disks, spheres, and rods, the moment of inertia of the body about any axis can be determined by adding algebraically the moments of inertia of all the composite shapes calculated about the axis. Algebraic addition is necessary since a composite part must be considered as a negative quantity if it has already been counted as a piece of another part—for example, a "hole" subtracted from a solid plate. The parallel-axis theorem is needed for the calculations if the center of mass of each composite part does not lie on the axis about which the moment of inertia is to be determined. For the calculation, then, $I = \Sigma(I_G + md^2)$. Here I_G for each of the composite parts is determined by integration, or for simple shapes, such as rods and disks, it can be found from a table, such as the one given on the inside back cover of this book.

Refer to the companion website for Lecture Summary and Quiz videos.

EXAMPLE 17.3

If the plate shown in Fig. 17–6*a* has a density of 8000 kg/m^3 and a thickness of 10 mm, determine its moment of inertia about an axis directed perpendicular to the page and passing through point O.

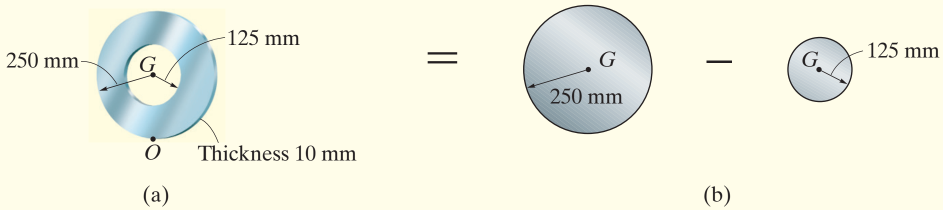


Fig. 17–6

SOLUTION

The plate consists of two composite parts, the 250-mm-radius disk *minus* a 125-mm-radius disk, Fig. 17–6*b*. The moment of inertia about O can be determined by calculating the moment of inertia of each of these parts about O and then adding the results *algebraically*. The calculations are performed by using the parallel-axis theorem in conjunction with the data listed in the table on the inside back cover.

Disk. The moment of inertia of a disk about the centroidal axis perpendicular to the plane of the disk is $I_G = \frac{1}{2}mr^2$. The mass center of the disk is located at a distance of 0.25 m from point O. Thus,

$$m_d = \rho_d V_d = 8000\ \text{kg/m}^3\,[\pi(0.25\ \text{m})^2(0.01\ \text{m})] = 15.71\ \text{kg}$$

$$\begin{aligned}(I_d)_O &= \tfrac{1}{2}m_d r_d^2 + m_d d^2\\ &= \tfrac{1}{2}(15.71\ \text{kg})(0.25\ \text{m})^2 + (15.71\ \text{kg})(0.25\ \text{m})^2\\ &= 1.473\ \text{kg}\cdot\text{m}^2\end{aligned}$$

Hole. For the 125-mm-radius disk (hole), we have

$$m_h = \rho_h V_h = 8000\ \text{kg/m}^3\,[\pi(0.125\ \text{m})^2(0.01\ \text{m})] = 3.927\ \text{kg}$$

$$\begin{aligned}(I_h)_O &= \tfrac{1}{2}m_h r_h^2 + m_h d^2\\ &= \tfrac{1}{2}(3.927\ \text{kg})(0.125\ \text{m})^2 + (3.927\ \text{kg})(0.25\ \text{m})^2\\ &= 0.276\ \text{kg}\cdot\text{m}^2\end{aligned}$$

The moment of inertia of the plate about point O is therefore

$$\begin{aligned}I_O &= (I_d)_O - (I_h)_O\\ &= 1.473\ \text{kg}\cdot\text{m}^2 - 0.276\ \text{kg}\cdot\text{m}^2\\ &= 1.20\ \text{kg}\cdot\text{m}^2\end{aligned}$$ *Ans.*

17

EXAMPLE 17.4

The pendulum in Fig. 17–7 is suspended from the pin at O and consists of two thin rods. Rod OA has a mass of 4.5 kg, and BC has a mass of 3.6 kg. Determine the moment of inertia of the pendulum about an axis passing through (a) point O, and (b) the mass center G of the pendulum.

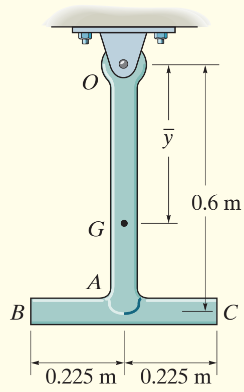


Fig. 17–7

SOLUTION

Part (a). Using the table on the inside back cover, the moment of inertia of rod OA about an axis perpendicular to the page and passing through point O of the rod is $I_O = \frac{1}{3}ml^2$. Hence,

$$(I_{OA})_O = \frac{1}{3}ml^2 = \frac{1}{3}(4.5\text{ kg})(0.6\text{ m})^2 = 0.54\text{ kg}\cdot\text{m}^2$$

This same value can be obtained using $I_G = \frac{1}{12}ml^2$ and the parallel-axis theorem.

$$(I_{OA})_O = \frac{1}{12}ml^2 + md^2 = \frac{1}{12}(4.5\text{ kg})(0.6\text{ m})^2 + (4.5\text{ kg})(0.3\text{ m})^2$$
$$= 0.54\text{ kg}\cdot\text{m}^2$$

For rod BC we have

$$(I_{BC})_O = \frac{1}{12}ml^2 + md^2 = \frac{1}{12}(3.6\text{ kg})(0.45\text{ m})^2 + (3.6\text{ kg})(0.6\text{ m})^2$$
$$= 1.357\text{ kg}\cdot\text{m}^2$$

The moment of inertia of the pendulum about O is therefore

$$I_O = 0.54 + 1.357 = 1.897 = 1.90\text{ kg}\cdot\text{m}^2 \qquad \textit{Ans.}$$

Part (b). The mass center G will be located relative to point O. Assuming this distance to be $\bar{y}$, Fig. 17–7, and using the formula for determining the mass center, we have

$$\bar{y} = \frac{\Sigma\tilde{y}m}{\Sigma m} = \frac{0.3(4.5) + 0.6(3.6)}{4.5 + 3.6} = 0.4333\text{ m}$$

The moment of inertia I_G may be found in the same manner as I_O, which requires successive applications of the parallel-axis theorem to transfer the moments of inertia of rods OA and BC to G. A more direct solution, however, involves using the result for I_O, i.e.,

$$I_O = I_G + md^2; \quad 1.897\text{ kg}\cdot\text{m}^2 = I_G + (8.1\text{ kg})(0.4333\text{ m})^2$$

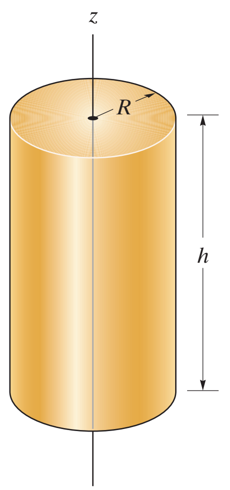
$$I_G = 0.376\text{ kg}\cdot\text{m}^2 \qquad \textit{Ans.}$$

Refer to the companion website for a self quiz of these Example problems.

17

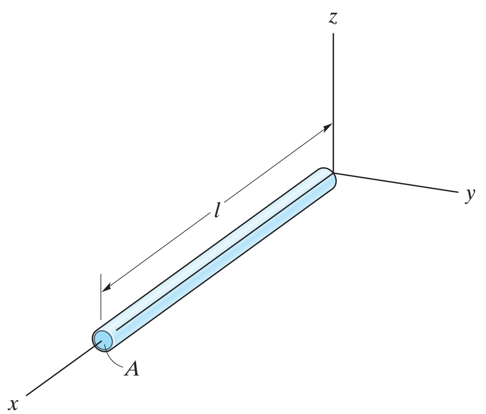
PROBLEMS

17–1. The solid cylinder has an outer radius R, height h, and is made from a material having a density that varies from its center as $\rho = k + ar^2$, where k and a are constants. Determine the mass of the cylinder and its moment of inertia about the z axis.



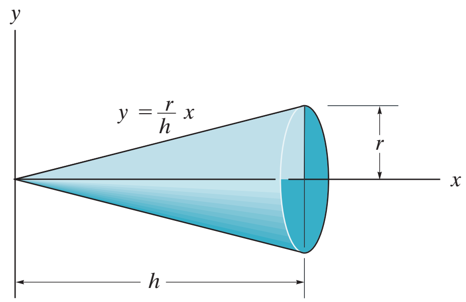
Prob. 17–1

17–2. Determine the moment of inertia I_y for the slender rod. The rod's density ρ and cross-sectional area A are constant. Express the result in terms of the rod's total mass m.



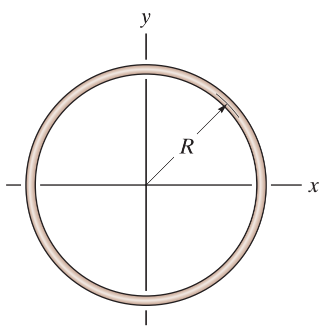
Prob. 17–2

17–3. The right circular cone is formed by revolving the shaded area around the x axis. Determine the moment of inertia I_x and express the result in terms of the total mass m of the cone. The cone has a constant density ρ.



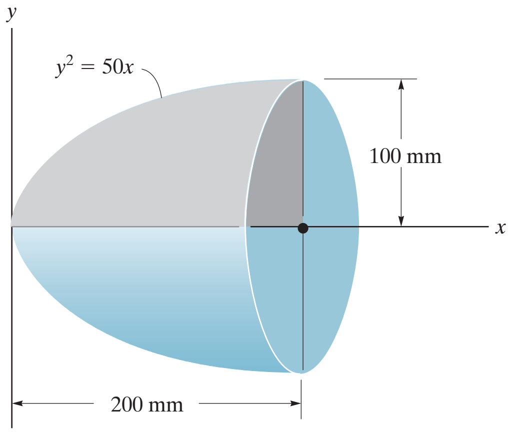
Prob. 17–3

***17–4.** Determine the moment of inertia of the thin ring about the z axis. The ring has a mass m.



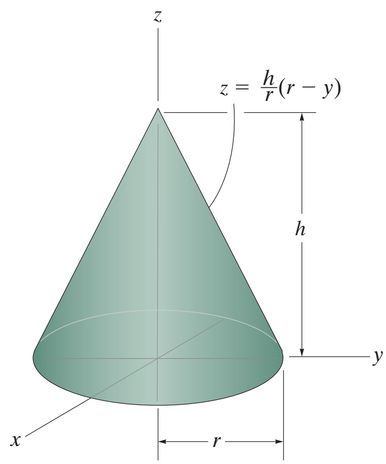
Prob. 17–4

17–5. The paraboloid is formed by revolving the shaded area around the x axis. Determine the radius of gyration k_x. The density of the material is $\rho = 5\ \mathrm{Mg/m^3}$.



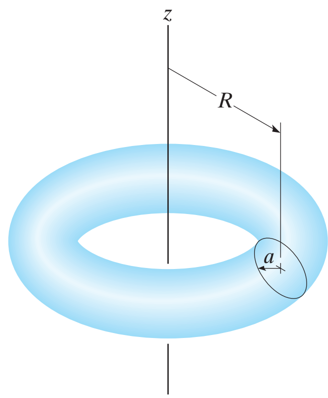
Prob. 17–5

17–6. Determine the mass moment of inertia I_z of the cone formed by revolving the shaded area around the z axis. The total density of the material is ρ. Express the results in terms of the mass m of the cone.



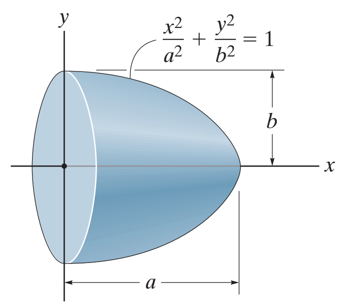
Prob. 17–6

17–7. Determine the moment of inertia I_z of the torus. The mass of the torus is m and the density ρ is constant. *Suggestion*: Use a shell element.



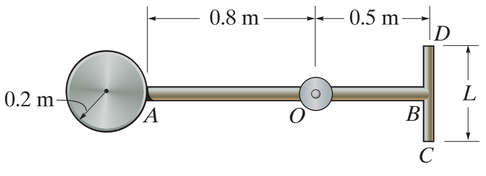
Prob. 17–7

***17–8** Determine the moment of inertia of the semi ellipsoid with respect to the x axis and express the result in terms of the mass m of the semi ellipsoid. The material has a constant density ρ.



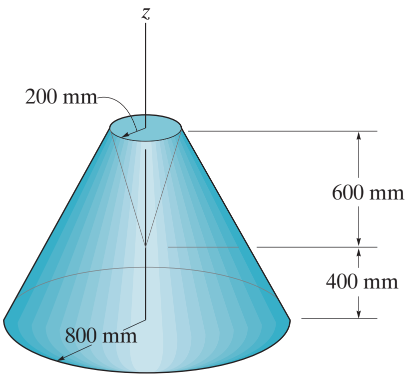
Prob. 17–8

17–9. The pendulum consists of a 2-kg disk and slender rods AB and DC which have a mass per unit length of 2 kg/m. Determine the length L of DC so that the center of mass is at the bearing O. What is the moment of inertia of this assembly about an axis perpendicular to the page and passing through point O?



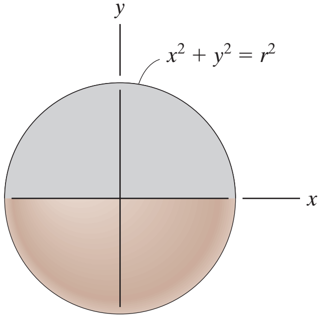
Prob. 17–9

17–10. Determine the moment of inertia I_z of the frustum of the cone, which has a conical depression. The material has a density of 200 kg/m^3.



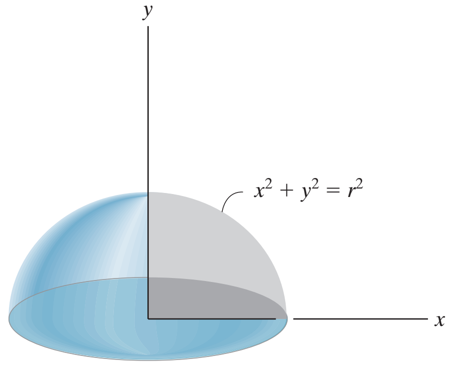
Prob. 17–10

17–11 The sphere is formed by revolving the shaded area around the x axis. Determine the moment of inertia I_x and express the result in terms of the total mass m of the sphere. The material has a constant density ρ.



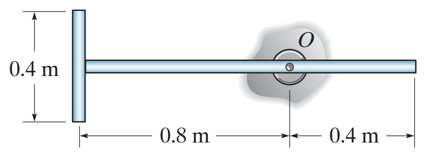
Prob. 17–11

***17–12.** The hemisphere is formed by rotating the shaded area around the y axis. Determine the moment of inertia I_y and express the result in terms of the total mass m of the hemisphere. The material has a constant density ρ.



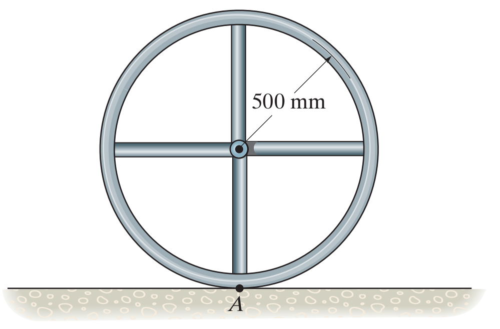
Prob. 17–12

17–13. The assembly is made of the slender rods that have a mass per unit length of 3 kg/m. Determine the mass moment of inertia of the assembly about an axis perpendicular to the page and passing through point O.



Prob. 17–13

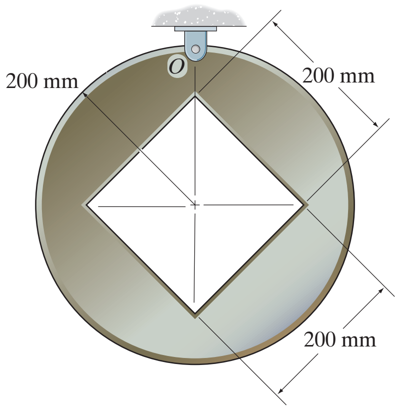
17–14. The wheel consists of a thin ring having a mass of 10 kg and four spokes made from slender rods and each having a mass of 2 kg. Determine the wheel's moment of inertia about an axis perpendicular to the page and passing through point A.



Prob. 17–14

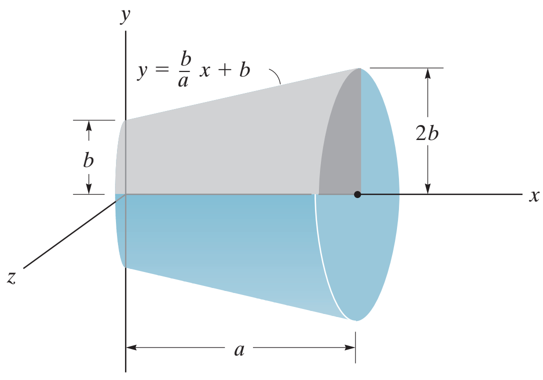
17

17–15. Determine the mass moment of inertia of the thin plate about an axis perpendicular to the page and passing through point O. The material has a mass per unit area of $20\ \text{kg/m}^2$.



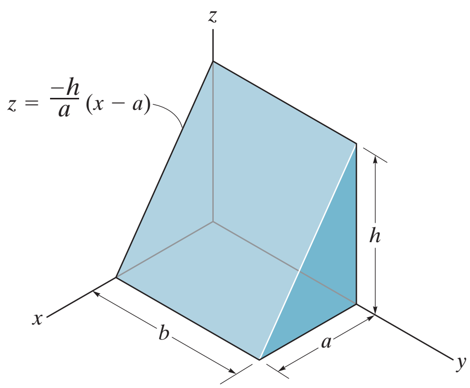
Prob. 17–15

*17–16. The frustum is formed by rotating the shaded area around the x axis. Determine the moment of inertia I_x and express the result in terms of the total mass m of the frustum. The frustum has a constant density ρ.



Prob. 17–16

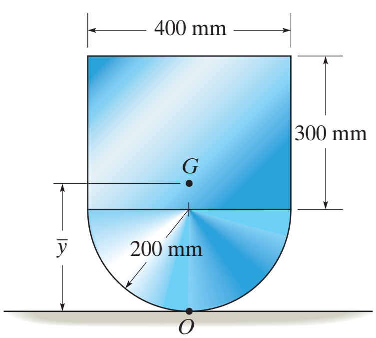
17–17. Determine the moment of inertia of the homogeneous triangular prism with respect to the y axis. Express the result in terms of the mass m of the prism. *Hint:* For integration, use thin plate elements parallel to the x–y plane and having a thickness dz.



Prob. 17–17

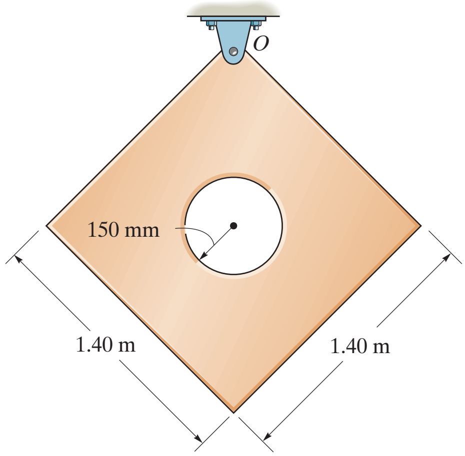
17–18. Determine the location $\bar{y}$ of the center of mass G of the assembly, and then calculate the moment of inertia about an axis perpendicular to the page and passing through G. The block has a mass of 3 kg and the semicylinder has a mass of 5 kg.

17–19. Determine the moment of inertia of the assembly about an axis perpendicular to the page and passing through point O. The block has a mass of 3 kg and the semicylinder has a mass of 5 kg.



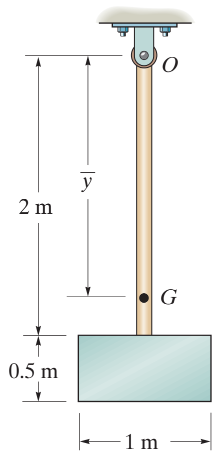
Probs. 17–18/19

***17–20.** Determine the moment of inertia about an axis perpendicular to the page and passing through the pin at O. The thin plate has a hole in its center. Its thickness is 50 mm, and the material has a density $\rho = 50\ \text{kg/m}^3$.



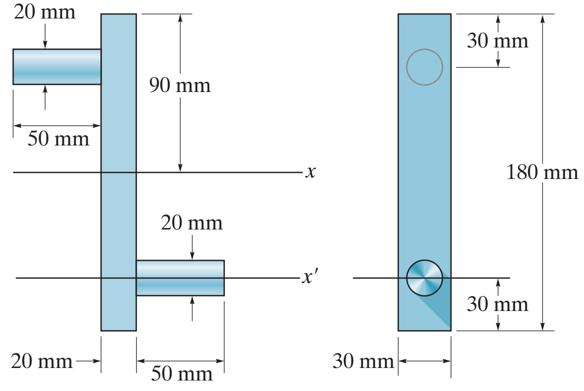
Prob. 17–20

17–21. The pendulum consists of the 3-kg slender rod and the 5-kg thin plate. Determine the location $\bar{y}$ of the center of mass G of the pendulum; then calculate the moment of inertia of the pendulum about an axis perpendicular to the page and passing through G.



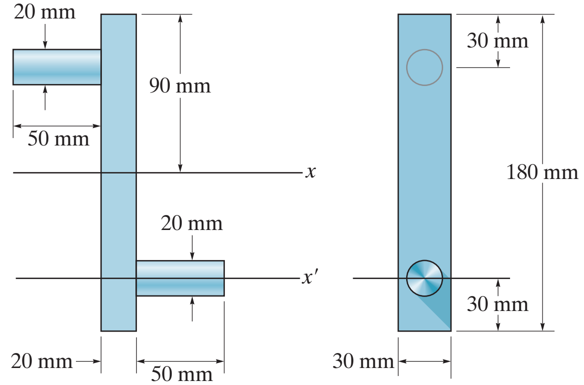
Prob. 17–21

17–22. Determine the moment of inertia of the overhung crank about the x' axis. The material is steel having a density of $\rho = 7.85\ \text{Mg/m}^3$.



Prob. 17–22

17–23 Determine the moment of inertia of the overhung crank about the x axis. The material is steel having a density of $\rho = 7.85\ \text{Mg/m}^3$.

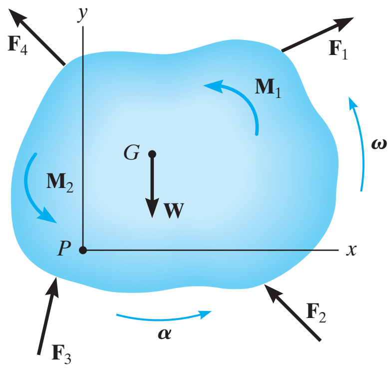


Prob. 17–23

17

17.2 PLANAR KINETIC EQUATIONS OF MOTION

In the following analysis we will limit our study of planar kinetics to rigid bodies which, along with their loadings, are considered to be *symmetrical* with respect to a fixed reference plane.* Since the motion of the body can be viewed within the reference plane, all the forces acting on the body can then be projected onto the plane. An example of an arbitrary body of this type is shown in Fig. 17–8*a*. Here the ***inertial frame of reference*** x, y, z has its origin *coincident* with the arbitrary point P in the body. By definition, these axes do not rotate and either are fixed or translate with constant velocity.



(a)

Fig. 17–8

Equation of Translational Motion. The external forces acting on the body in Fig. 17–8*a* represent the effect of gravitational, electrical, magnetic, or contact forces between adjacent bodies. Since this force system has been considered previously in Sec. 13.3 for the analysis of a system of particles, the resulting Eq. 13–5 can be used here for a rigid body, in which case,

$$\Sigma \mathbf{F} = m\mathbf{a}_G$$

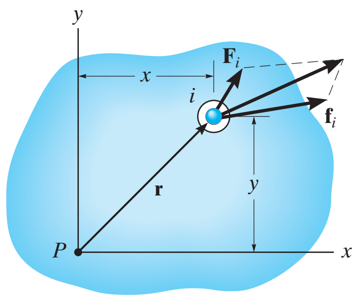
This equation is referred to as the ***translational equation of motion*** for the mass center of a rigid body. It states that *the sum of all the external forces acting on the body is equal to the body's mass times the acceleration of its mass center G.*

For motion of the body in the x–y plane, the translational equation of motion may be written in the form of two independent scalar equations, namely,

$$\Sigma F_x = m(a_G)_x$$
$$\Sigma F_y = m(a_G)_y$$

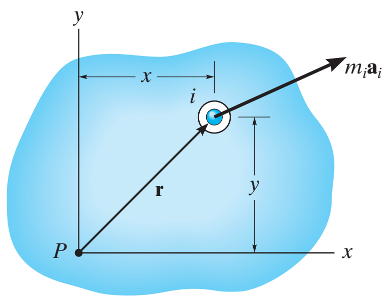
* By doing this, the rotational equation of motion reduces to a rather simplified form. The more general case of body shape and loading is considered in Chapter 21.

17



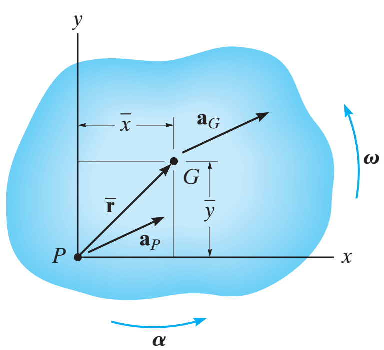
Particle free-body diagram

(b)



Particle kinetic diagram

(c)



(d)

Fig. 17–8 (cont.)

Equation of Rotational Motion. We will now determine the effects caused by the moments of the external force system calculated about an axis perpendicular to the plane of motion (the z axis) and passing through point P. As shown on the free-body diagram of the ith particle, Fig. 17–8b, $\mathbf{F}_i$ represents the *resultant external force* acting on the particle, and $\mathbf{f}_i$ is the *resultant of the internal forces* caused by interactions with adjacent particles. If the particle has a mass m_i and its acceleration is $\mathbf{a}_i$, then its kinetic diagram is shown in Fig. 17–8c. Summing moments about point P, we require

$$\mathbf{r} \times \mathbf{F}_i + \mathbf{r} \times \mathbf{f}_i = \mathbf{r} \times m_i \mathbf{a}_i$$

or

$$(\mathbf{M}_P)_i = \mathbf{r} \times m_i \mathbf{a}_i$$

The moments about P can also be expressed in terms of the acceleration of point P, Fig. 17–8d. If the body has an angular acceleration $\boldsymbol{\alpha}$ and angular velocity $\boldsymbol{\omega}$, then using Eq. 16–18 we have

$$\begin{aligned}(\mathbf{M}_P)_i &= m_i \mathbf{r} \times (\mathbf{a}_P + \boldsymbol{\alpha} \times \mathbf{r} - \omega^2 \mathbf{r}) \\ &= m_i[\mathbf{r} \times \mathbf{a}_P + \mathbf{r} \times (\boldsymbol{\alpha} \times \mathbf{r}) - \omega^2(\mathbf{r} \times \mathbf{r})]\end{aligned}$$

The last term is zero, since $\mathbf{r} \times \mathbf{r} = \mathbf{0}$. Expressing the vectors as Cartesian components and carrying out the cross-product operations yields

$$\begin{aligned}(M_P)_i\mathbf{k} &= m_i\{(x\mathbf{i} + y\mathbf{j}) \times [(a_P)_x\mathbf{i} + (a_P)_y\mathbf{j}] \\ &\quad + (x\mathbf{i} + y\mathbf{j}) \times [\alpha\mathbf{k} \times (x\mathbf{i} + y\mathbf{j})]\} \\ (M_P)_i\mathbf{k} &= m_i[-y(a_P)_x + x(a_P)_y + \alpha x^2 + \alpha y^2]\mathbf{k} \\ \circlearrowleft (M_P)_i &= m_i[-y(a_P)_x + x(a_P)_y + \alpha r^2]\end{aligned}$$

Letting $m_i \rightarrow dm$ and integrating over the entire mass m of the body, we obtain

$$\circlearrowleft \Sigma M_P = -\left(\int_m y\,dm\right)(a_P)_x + \left(\int_m x\,dm\right)(a_P)_y + \left(\int_m r^2 dm\right)\alpha$$

Here ΣM_P represents only the moment of the *external forces* acting on the body about point P. The resultant moment of the internal forces is zero, since for the entire body these forces occur in equal and opposite collinear pairs and thus the moment of each pair of forces about P cancels. The integrals in the first and second terms on the right are used to locate the body's center of mass G with respect to P, since $\bar{y}m = \int y\,dm$ and $\bar{x}m = \int x\,dm$. See Fig. 17–8d. The last integral represents the body's moment of inertia about the z axis, i.e., $I_P = \int r^2 dm$. Thus,

$$\circlearrowleft \Sigma M_P = -\bar{y}m(a_P)_x + \bar{x}m(a_P)_y + I_P\alpha \qquad (17\text{–}6)$$

It is possible to reduce this equation to a simpler form if point P coincides with the mass center G for the body. If this is the case, then $\bar{x} = \bar{y} = 0$, and therefore*

$$\Sigma M_G = I_G \alpha \qquad (17\text{–}7)$$

This ***rotational equation of motion*** states that *the sum of the moments of all the external forces about the body's mass center G is equal to the product of the moment of inertia of the body about an axis passing through G and the body's angular acceleration.*

Equation 17–6 can also be rewritten in terms of the x and y components of $\mathbf{a}_G$ and the body's moment of inertia I_G. If point G is located at $(\bar{x}, \bar{y})$, Fig. 17–8*d*, then by the parallel-axis theorem, $I_P = I_G + m(\bar{x}^2 + \bar{y}^2)$. Substituting into Eq. 17–6 and rearranging terms, we get

$$\circlearrowleft \Sigma M_P = \bar{y}m[-(a_P)_x + \bar{y}\alpha] + \bar{x}m[(a_P)_y + \bar{x}\alpha] + I_G\alpha \qquad (17\text{–}8)$$

From the kinematic diagram of Fig. 17–8*d*, $\mathbf{a}_P$ can be expressed in terms of $\mathbf{a}_G$ as

$$\mathbf{a}_G = \mathbf{a}_P + \boldsymbol{\alpha} \times \bar{\mathbf{r}} - \omega^2 \bar{\mathbf{r}}$$

$$(a_G)_x\mathbf{i} + (a_G)_y\mathbf{j} = (a_P)_x\mathbf{i} + (a_P)_y\mathbf{j} + \alpha\mathbf{k} \times (\bar{x}\mathbf{i} + \bar{y}\mathbf{j}) - \omega^2(\bar{x}\mathbf{i} + \bar{y}\mathbf{j})$$

Carrying out the cross product and equating the respective **i** and **j** components yields the two scalar equations

$$(a_G)_x = (a_P)_x - \bar{y}\alpha - \bar{x}\omega^2$$
$$(a_G)_y = (a_P)_y + \bar{x}\alpha - \bar{y}\omega^2$$

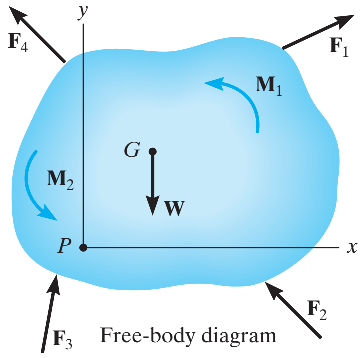
From these equations, $[-(a_P)_x + \bar{y}\alpha] = [-(a_G)_x - \bar{x}\omega^2]$ and $[(a_P)_y + \bar{x}\alpha] = [(a_G)_y + \bar{y}\omega^2]$. Substituting these results into Eq. 17–8 and simplifying gives

$$\circlearrowleft \Sigma M_P = -\bar{y}m(a_G)_x + \bar{x}m(a_G)_y + I_G\alpha \qquad (17\text{–}9)$$

This important result indicates that *when moments of the external forces shown on the free-body diagram are summed about point P, Fig. 17–8e, they are equivalent to the sum of the "kinetic moments" of the components of m$\mathbf{a}_G$ about P plus the "kinetic moment" of $I_G\boldsymbol{\alpha}$, Fig. 17–8f.* Note that $I_G\boldsymbol{\alpha}$ can be treated as a free vector and therefore act at any point. It is important to remember, however, that $m\mathbf{a}_G$ and $I_G\boldsymbol{\alpha}$ are not the same as a force or a couple moment. Instead, they are caused by the external effects of forces and couple moments acting on the body. As a result, we can therefore write Eq. 17–9 in a more general form as

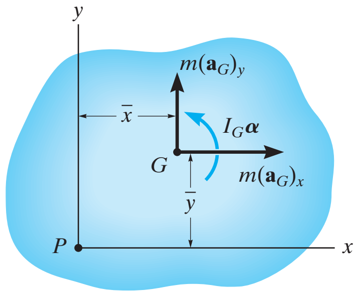
$$\Sigma M_P = \Sigma(\mathcal{M}_k)_P \qquad (17\text{–}10)$$

*It also reduces to a simpler form $\Sigma M_P = I_P\alpha$ if point P is a *fixed point* (see Eq. 17–16) or if the acceleration of point P is directed along the line PG.



Free-body diagram

(e)

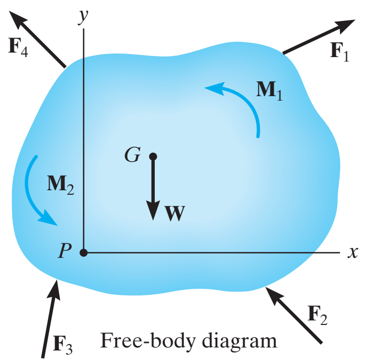


Kinetic diagram

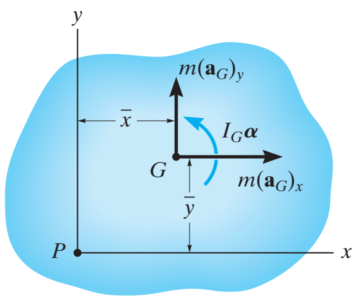
(f)

Fig. 17–8 (cont.)

17



(e)



Kinetic diagram

(f)

Fig. 17–8 (repeated)

General Application of the Equations of Motion. To summarize this analysis, three independent scalar equations can be written to describe the general plane motion of a symmetrical rigid body. They are

$$\Sigma F_x = m(a_G)_x$$
$$\Sigma F_y = m(a_G)_y$$
$$\Sigma M_G = I_G\alpha$$

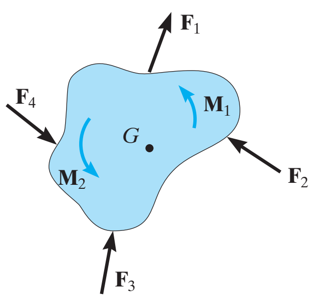
or

$$\Sigma M_P = \Sigma(\mathcal{M}_k)_P \qquad (17\text{–}11)$$

When applying these equations, one should always draw a free-body diagram, Fig. 17–8*e*, in order to account for the terms involved in ΣF_x, ΣF_y, ΣM_G, or ΣM_P. In some problems it may also be helpful to draw the kinetic diagram for the body, Fig. 17–8*f*. This diagram graphically accounts for the terms $m(\mathbf{a}_G)_x$, $m(\mathbf{a}_G)_y$, and $I_G\boldsymbol{\alpha}$. It is especially convenient when used to determine the moment of these three components when applying $\Sigma(\mathcal{M}_k)_P$.

17.3 EQUATIONS OF MOTION: TRANSLATION

When the rigid body in Fig. 17–9*a* undergoes a translation, all the particles of the body have the same acceleration. Furthermore, $\boldsymbol{\alpha} = \mathbf{0}$, in which case the rotational equation of motion applied at point G reduces to a simplified form, namely, $\Sigma M_G = 0$. Application of this and the force equations of motion will now be discussed for each of the two types of translation.

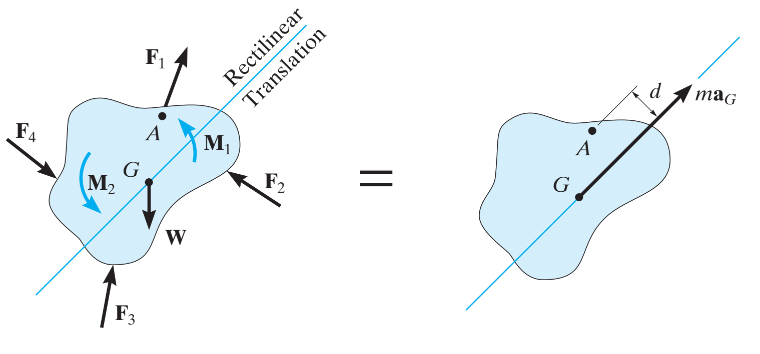


(a)

Fig. 17–9

Rectilinear Translation. When a body is subjected to rectilinear translation, all the particles of the body (slab) travel along parallel straight-line paths. The free-body and kinetic diagrams are shown in Fig. 17–9*b*. Since $I_G\boldsymbol{\alpha} = \mathbf{0}$, only $m\mathbf{a}_G$ is shown on the kinetic diagram. Hence, the equations of motion which apply in this case become

$$\boxed{\begin{aligned} \Sigma F_x &= m(a_G)_x \\ \Sigma F_y &= m(a_G)_y \\ \Sigma M_G &= 0 \end{aligned}} \qquad (17\text{–}12)$$



(b)

It is also possible to sum moments about some other point on or off the body, in which case the moment of $m\mathbf{a}_G$ must be taken into account. For example, if point A is chosen, which lies at a perpendicular distance d from the line of action of $m\mathbf{a}_G$, the following moment equation applies:

$$\circlearrowleft + \Sigma M_A = \Sigma(\mathcal{M}_k)_A; \qquad \Sigma M_A = (ma_G)d$$

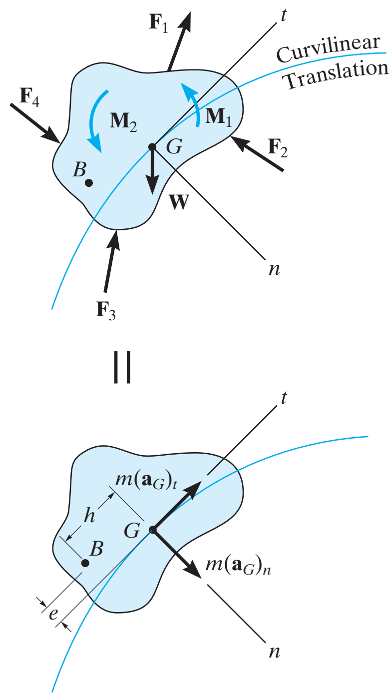
Here the sum of moments of the external forces and couple moments about A (ΣM_A, free-body diagram) equals the moment of $m\mathbf{a}_G$ about A ($\Sigma(\mathcal{M}_k)_A$, kinetic diagram).

Curvilinear Translation. When a rigid body is subjected to curvilinear translation, then all the particles of the body have the same accelerations as they travel along curved paths, as noted in Sec. 16.1. For analysis, it is often convenient to use an inertial coordinate system having an origin which coincides with the body's mass center at the instant considered, and axes which are oriented in the normal and tangential directions to the path of motion, Fig. 17–9*c*. The three scalar equations of motion are then

$$\begin{aligned} \Sigma F_n &= m(a_G)_n \\ \Sigma F_t &= m(a_G)_t \\ \Sigma M_G &= 0 \end{aligned} \qquad (17\text{–}13)$$

If moments are summed about another point such as B, Fig. 17–9*c*, then it is necessary to account for the moments, $\Sigma(\mathcal{M}_k)_B$, of the two components $m(\mathbf{a}_G)_n$ and $m(\mathbf{a}_G)_t$ about this point. From the kinetic diagram, h and e represent the perpendicular distances (or "moment arms") from B to the lines of action of the components. The required moment equation therefore becomes

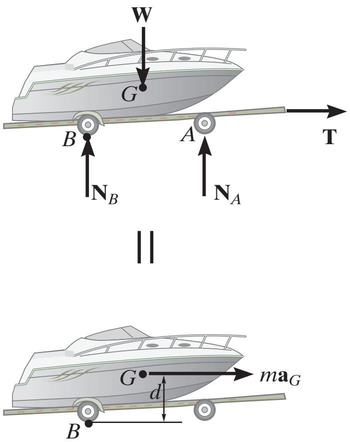
$$\circlearrowleft + \Sigma M_B = \Sigma(\mathcal{M}_k)_B; \qquad \Sigma M_B = e[m(a_G)_t] - h[m(a_G)_n]$$



(c)

Fig. 17–9 (cont.)

The free-body and kinetic diagrams are shown below. If moments are summed about the mass center, G, then $\Sigma M_G = 0$. However, if moments are summed about point B then $\circlearrowleft + \Sigma M_B = ma_G(d)$.



PROCEDURE FOR ANALYSIS

Kinetic problems involving rigid-body *translation* can be solved using the following procedure.

Free-Body Diagram.

- Establish the x, y or n, t inertial coordinate system and draw the free-body diagram in order to account for all the external forces and couple moments that act on the body.
- The direction and sense of the acceleration of the body's mass center $\mathbf{a}_G$ should be established.
- Identify the unknowns in the problem.
- If it is decided that the rotational equation of motion $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$ is to be used in the solution, then consider drawing the kinetic diagram, since it graphically accounts for the components $m(\mathbf{a}_G)_x$, $m(\mathbf{a}_G)_y$ or $m(\mathbf{a}_G)_t$, $m(\mathbf{a}_G)_n$ and is therefore convenient for "visualizing" the terms needed in the moment sum $\Sigma(\mathcal{M}_k)_P$.

Equations of Motion.

- Apply the three equations of motion in accordance with the established sign convention.
- To simplify the analysis, the moment equation $\Sigma M_G = 0$ can be replaced by the more general equation $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$, where point P is usually located at the intersection of the lines of action of as many unknown forces as possible.
- If the body is in contact with a *rough surface* and slipping occurs, use the friction equation $F = \mu_k N$. Remember, $\mathbf{F}$ always acts on the body so as to oppose the motion of the body relative to the surface it contacts.

Kinematics.

- Use kinematics to determine the velocity and position of the body.
- For rectilinear translation with *variable acceleration*

$$a_G = dv_G/dt \qquad a_G ds_G = v_G dv_G$$

- For rectilinear translation with *constant acceleration*

$$v_G = (v_G)_0 + a_G t \qquad v_G^2 = (v_G)_0^2 + 2a_G[s_G - (s_G)_0]$$

$$s_G = (s_G)_0 + (v_G)_0 t + \tfrac{1}{2}a_G t^2$$

- For curvilinear translation

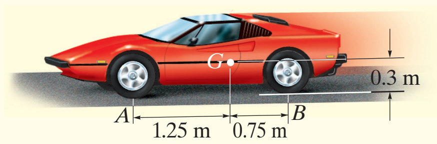
$$(a_G)_n = v_G^2/\rho$$

$$(a_G)_t = dv_G/dt \qquad (a_G)_t\, ds_G = v_G\, dv_G$$

Refer to the companion website for Lecture Summary and Quiz videos.

EXAMPLE 17.5

The car shown in Fig. 17–10*a* has a mass of 2 Mg and a center of mass at *G*. Determine the acceleration if the rear "driving" wheels are always slipping, whereas the front wheels are free to rotate. Neglect the mass of the wheels. The coefficient of kinetic friction between the wheels and the road is $\mu_k = 0.25$.



(a)

SOLUTION I

Free-Body Diagram. As shown in Fig. 17–10*b*, the rear-wheel frictional force $\mathbf{F}_B$ pushes the car forward, and since slipping occurs, $F_B = 0.25N_B$. The frictional forces acting on the front wheels are zero, since these wheels have negligible mass.* There are three unknowns in this problem, N_A, N_B, and a_G.

Equations of Motion. Here we will sum moments about the mass center. The car (point *G*) accelerates to the left, i.e., in the negative *x* direction, Fig. 17–10*b*.

$$\overset{+}{\rightarrow}\Sigma F_x = m(a_G)_x; \qquad -0.25N_B = -(2000\text{ kg})a_G \qquad (1)$$

$$+\uparrow\Sigma F_y = m(a_G)_y; \qquad N_A + N_B - 2000(9.81)\text{ N} = 0 \qquad (2)$$

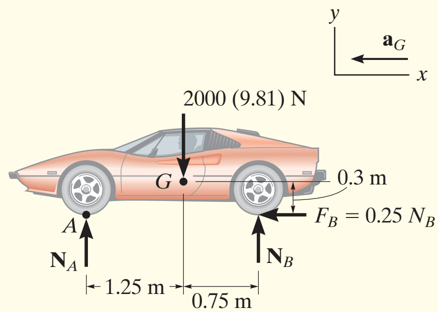
$$\circlearrowleft+\Sigma M_G = 0; \quad -N_A(1.25\text{ m}) - 0.25N_B(0.3\text{ m}) + N_B(0.75\text{ m}) = 0 \quad (3)$$

Solving,

$$a_G = 1.59\text{ m/s}^2 \leftarrow \qquad \textit{Ans.}$$

$$N_A = 6.88\text{ kN}$$

$$N_B = 12.7\text{ kN}$$



(b)

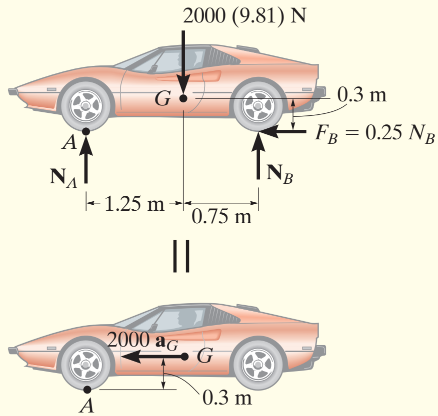
SOLUTION II

Free-Body and Kinetic Diagrams. If the "moment" equation is applied about point *A*, then the unknown N_A will be eliminated from the equation. To "visualize" the moment of $m\mathbf{a}_G$ about *A*, we will include the kinetic diagram as part of the analysis, Fig. 17–10*c*.

Equation of Motion.

$$\circlearrowleft+\Sigma M_A = \Sigma(\mathcal{M}_k)_A; \qquad N_B(2\text{ m}) - [2000(9.81)\text{ N}](1.25\text{ m}) = (2000\text{ kg})a_G(0.3\text{ m})$$

Solving this and Eq. 1 for a_G leads to a simpler solution than that obtained from Eqs. 1 to 3.



(c)

Fig. 17–10

*With negligible wheel mass, $I\alpha = 0$ and the frictional force at *A* required to turn the wheel is zero. If the wheels' mass were included, then the solution would be more involved, since a general-plane-motion analysis of the wheels would have to be considered (see Sec. 17.5).

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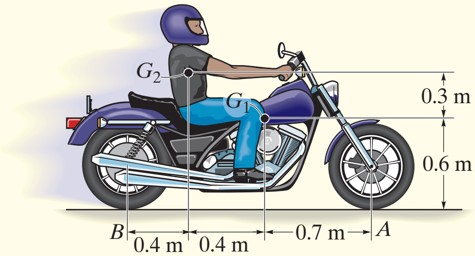
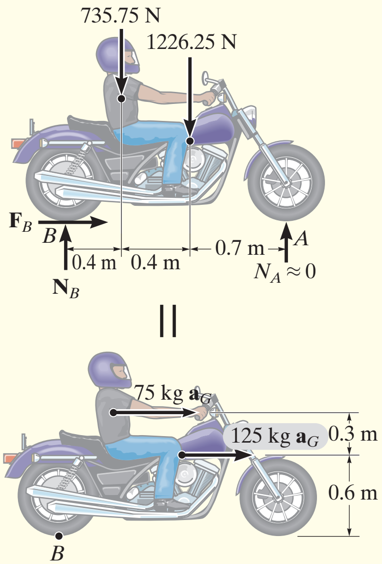
EXAMPLE 17.6

The motorcycle shown in Fig. 17–11*a* has a mass of 125 kg and a center of mass at G_1, while the rider has a mass of 75 kg and a center of mass at G_2. Determine the minimum coefficient of static friction between the wheels and the pavement in order for the rider to do a "wheely," i.e., lift the front wheel off the ground as shown in the photo. What acceleration is necessary to do this? Neglect the mass of the wheels and assume that the front wheel is free to roll.



(a)



(b)

Fig. 17–11

SOLUTION

Free-Body and Kinetic Diagrams. In this problem we will consider both the motorcycle and the rider as a single *system*. It is possible first to determine the location of the center of mass for this "system" by using the equations $\bar{x} = \Sigma\tilde{x}m/\Sigma m$ and $\bar{y} = \Sigma\tilde{y}m/\Sigma m$. Here, however, we will consider the weight and mass of the motorcycle and rider separately as shown on the free-body and kinetic diagrams, Fig. 17–11*b*. Both of these parts move with the *same* acceleration. We have assumed that the front wheel is *about* to leave the ground, so that the normal reaction $N_A \approx 0$. The three unknowns in the problem are N_B, F_B, and a_G.

Equations of Motion.

$$\overset{+}{\rightarrow}\Sigma F_x = m(a_G)_x; \qquad F_B = (75\text{ kg} + 125\text{ kg})a_G$$

$$+\uparrow\Sigma F_y = m(a_G)_y; \qquad N_B - 735.75\text{ N} - 1226.25\text{ N} = 0$$

$$\circlearrowleft+\Sigma M_B = \Sigma(\mathcal{M}_k)_B; \qquad -(735.75\text{ N})(0.4\text{ m}) - (1226.25\text{ N})(0.8\text{ m}) = -(75\text{ kg }a_G)(0.9\text{ m}) - (125\text{ kg }a_G)(0.6\text{ m})$$

Solving,

$$a_G = 8.95\text{ m/s}^2 \rightarrow \qquad \textit{Ans.}$$

$$N_B = 1962\text{ N}$$

$$F_B = 1790\text{ N}$$

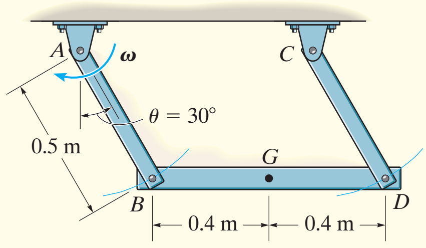
Thus the minimum coefficient of static friction is

$$(\mu_s)_{\min} = \frac{F_B}{N_B} = \frac{1790\text{ N}}{1962\text{ N}} = 0.912 \qquad \textit{Ans.}$$

17

EXAMPLE 17.7

The 100-kg beam BD shown in Fig. 17-12a is suspended from two rods having negligible mass. Determine the force developed in each rod if at the instant $\theta = 30°, \omega = 6$ rad/s.



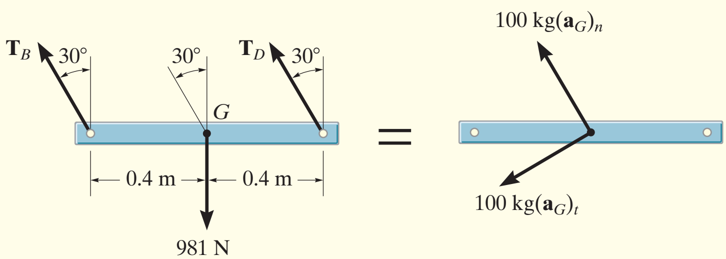
(a)

SOLUTION

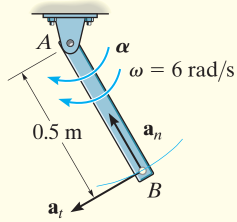
Free-Body and Kinetic Diagrams. The beam moves with curvilinear translation since all points on the beam move along circular paths, each path having the same radius of 0.5 m, but different centers of curvature. Using normal and tangential coordinates, the free-body and kinetic diagrams for the beam are shown in Fig. 17–12b. Because of the translation, G has the same motion as the pin at B, which is connected to both the rod and the beam. Note that the tangential component of acceleration acts downward to the left due to the clockwise direction of $\boldsymbol{\alpha}$, Fig. 17–12c. Furthermore, the normal component of acceleration is always directed toward the center of curvature (toward point A for rod AB). Since the angular velocity of AB is 6 rad/s when $\theta = 30°$, then

$$(a_G)_n = \omega^2 r = (6 \text{ rad/s})^2(0.5 \text{ m}) = 18 \text{ m/s}^2$$

The three unknowns are T_B, T_D, and $(a_G)_t$.



(b)



(c)

Fig. 17–12

Equations of Motion.

$$+\nwarrow \Sigma F_n = m(a_G)_n; \quad T_B + T_D - 981 \cos 30° \text{ N} = 100 \text{ kg}(18 \text{ m/s}^2) \quad (1)$$

$$+\swarrow \Sigma F_t = m(a_G)_t; \quad 981 \sin 30° = 100 \text{ kg}(a_G)_t \quad (2)$$

$$\circlearrowleft + \Sigma M_G = 0; \quad -(T_B \cos 30°)(0.4 \text{ m}) + (T_D \cos 30°)(0.4 \text{ m}) = 0 \quad (3)$$

Simultaneous solution of these three equations gives

$$T_B = T_D = 1.32 \text{ kN} \qquad \textit{Ans.}$$

$$(a_G)_t = 4.905 \text{ m/s}^2$$

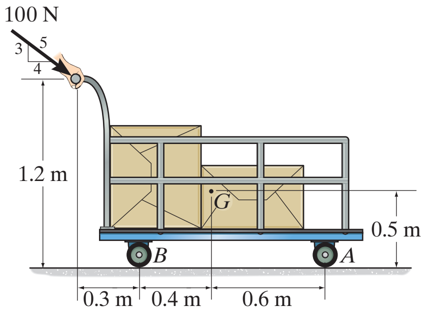
NOTE: It is also possible to apply the equations of motion along horizontal and vertical x, y axes, but the solution becomes more involved.

Refer to the companion website for a self quiz of these Example problems.

17

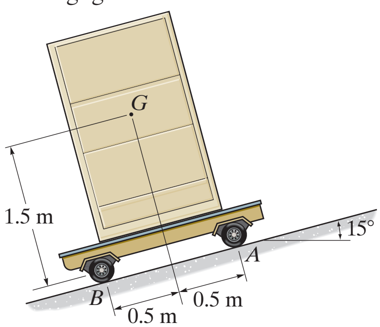
FUNDAMENTAL PROBLEMS

F17–1. The cart and its load have a total mass of 100 kg. Determine the acceleration of the cart and the normal reactions on the pair of wheels at A and B. Neglect the mass of the wheels.



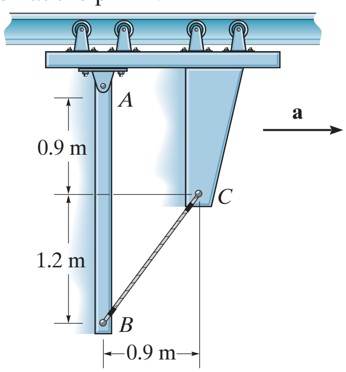
Prob. F17–1

F17–2. If the 80-kg cabinet is allowed to roll down the inclined plane, determine the acceleration of the cabinet and the normal reactions on the pair of rollers at A and B that have negligible mass.



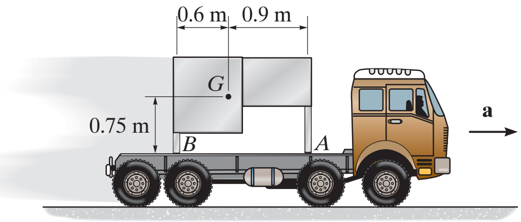
Prob. F17–2

F17–3. The 10-kg link AB is pinned to a moving frame at A and held in a vertical position by means of a string BC which can support a maximum tension of 50 N. Determine the maximum acceleration of the frame without breaking the string. What are the corresponding components of reaction at the pin A?



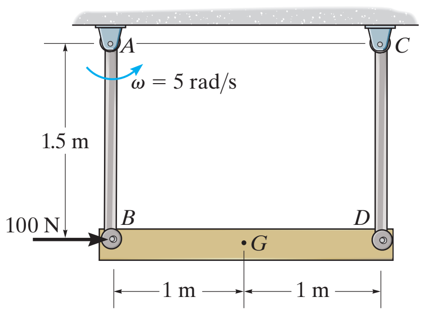
Prob. F17–3

F17–4. Determine the maximum acceleration of the truck without causing the 100-kg assembly to move relative to the truck. Also what is the corresponding normal reaction on legs A and B? The assembly has a mass center at G and the coefficient of static friction between its legs and the truck bed is $\mu_s = 0.2$.



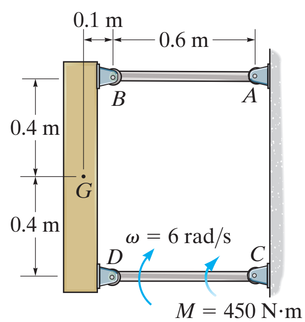
Prob. F17–4

F17–5. At the instant shown both rods of negligible mass swing with a counterclockwise angular velocity of $\omega = 5$ rad/s, while the 50-kg bar is subjected to the 100-N horizontal force. Determine the tension developed in the rods and the angular acceleration of the rods at this instant.



Prob. F17–5

F17–6. At the instant shown, rod CD rotates with an angular velocity of $\omega = 6$ rad/s. If it is subjected to a couple moment $M = 450$ N·m, determine the force developed in rod AB, the horizontal and vertical component of reaction on pin D, and the angular acceleration of rod CD at this instant. The bar has a mass of 50 kg and center of mass at G. Neglect the mass of rods AB and CD.



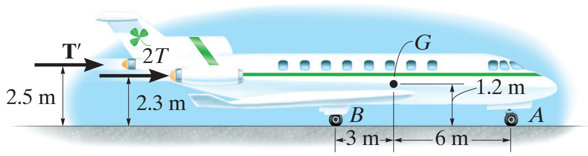
Prob. F17–6

17

PROBLEMS

All solutions must include a free-body diagram and a kinetic diagram.

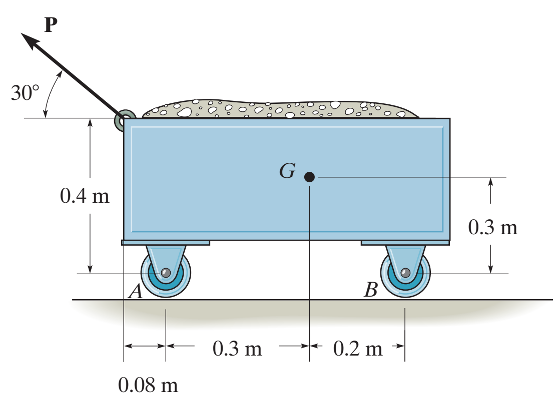
***17–24.** The jet aircraft has a total mass of 22 Mg and a center of mass at G. Initially at takeoff the engines provide a thrust $2T = 4$ kN and $T' = 1.5$ kN. Determine the acceleration of the plane and the normal reactions on the nose wheel at A and each of the *two* wing wheels located at B. Neglect the mass of the wheels and, due to low velocity, neglect any lift caused by the wings.



Prob. 17–24

17–25. A force of $P = 300$ N is applied to the 60-kg cart. Determine the reactions at both the wheels at A and both the wheels at B. Also, what is the acceleration of the cart? The mass center of the cart is at G.

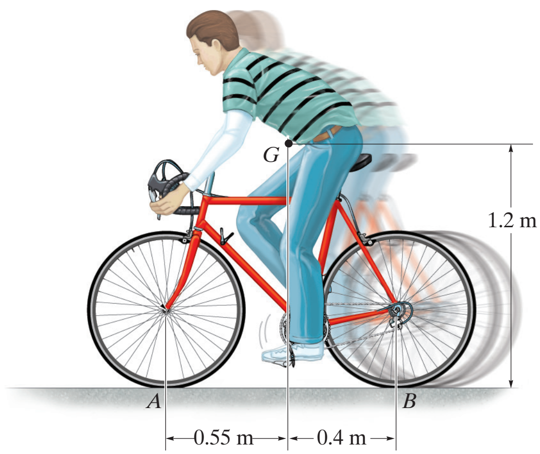
17–26. Determine the largest force **P** that can be applied to the 60-kg cart, without causing one of the wheel reactions, either at A or at B, to be zero. Also, what is the acceleration of the cart? The mass center of the cart is at G.



Probs. 17–25/26

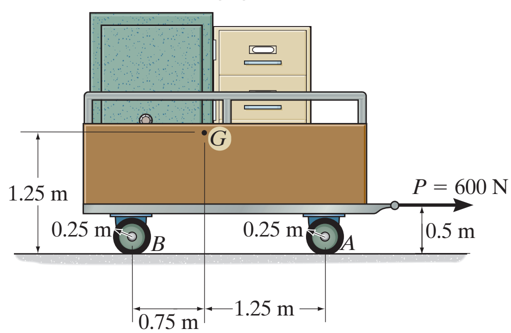
17–27. The bicycle and rider have a mass of 80 kg with center of mass located at G. If the coefficient of kinetic friction at the rear tire is $\mu_B = 0.8$, determine the normal reactions at the tires A and B, and the deceleration of the rider, when the rear wheel locks for braking. What is the normal reaction at the rear wheel when the bicycle is traveling at constant velocity and the brakes are not applied? Neglect the mass of the wheels.

***17–28.** The bicycle and rider have a mass of 80 kg with center of mass located at G. Determine the minimum coefficient of kinetic friction between the road and the wheels so that the rear wheel B starts to lift off the ground when the rider applies the brakes to the front wheel. Neglect the mass of the wheels.



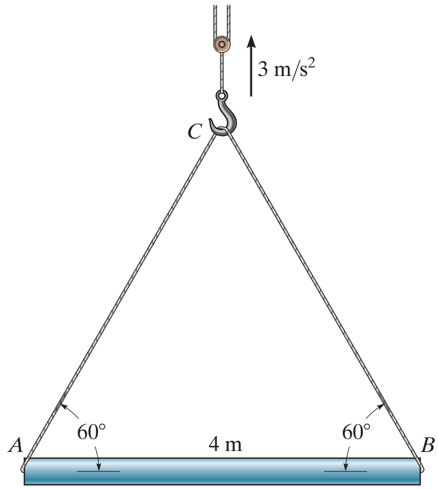
Probs. 17–27/28

17–29. The trailer with its load has a mass of 150-kg and a center of mass at G. If it is subjected to a horizontal force of $P = 600$ N, determine the trailer's acceleration and the normal force on the pair of wheels at A and at B. The wheels are free to roll and have negligible mass.



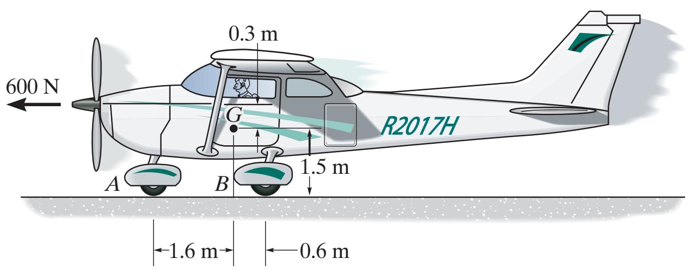
Prob. 17–29

17–30. The uniform girder AB has a mass of 8 Mg. Determine the internal axial, shear, and bending-moment loadings at the center of the girder if a crane gives it an upward acceleration of $3\ \text{m/s}^2$.



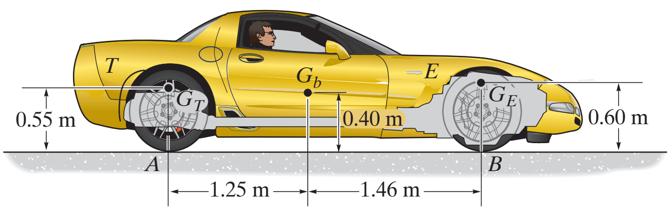
Prob. 17–30

17–31. At the *start* of takeoff, the propeller on the 2-Mg plane exerts a horizontal thrust of 600 N on the plane. Determine the plane's acceleration and the vertical reactions at the nose wheel A and each of the *two* wing wheels B. Neglect the lifting force of the wings since the plane is originally at rest. The mass center is at G.



Prob. 17–31

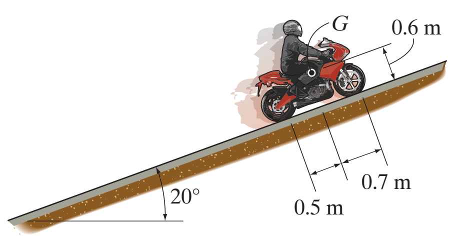
***17–32.** The sports car has been designed so that the 260-kg engine E and 165-kg transmission T have been placed over the front and rear wheels, respectively. Their mass centers are located at G_E and G_T. The mass center of the remaining 512-kg body and frame is located at G_b. If power is supplied to the rear wheels only, determine the shortest time it takes for the car to reach a speed of 80 km/h, starting from rest. The front wheels are free rolling, and the coefficient of static friction between the powered wheels and the road is $\mu_s = 0.4$. Neglect the mass of the wheels and driver.



Prob. 17–32

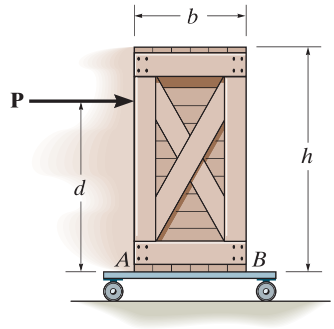
17–33. The motorcycle and rider have a total mass of 900 kg and mass center at G. If it is traveling up the slope with a constant acceleration of $3\ \text{m/s}^2$, determine the normal reactions on the front and rear wheels. Also, what is the required friction force developed on the rear wheel? The front wheel is free to roll.

17–34. The motorcycle and rider have a total mass of 900 kg and mass center at G. Determine the minimum coefficient of static friction between the rear wheel and the slope in order for the rider to perform a "wheely," that is, to begin to lift the front wheel off the road. Also, what is the corresponding acceleration? The front wheel is free to roll.



Probs. 17–33/34

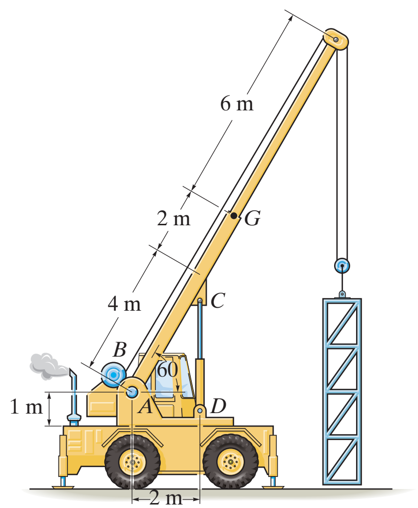
17–35. The crate of mass m is supported on a cart of negligible mass. Determine the maximum force P that can be applied a distance d from the cart bottom without causing the crate to tip on the cart.



Prob. 17–35

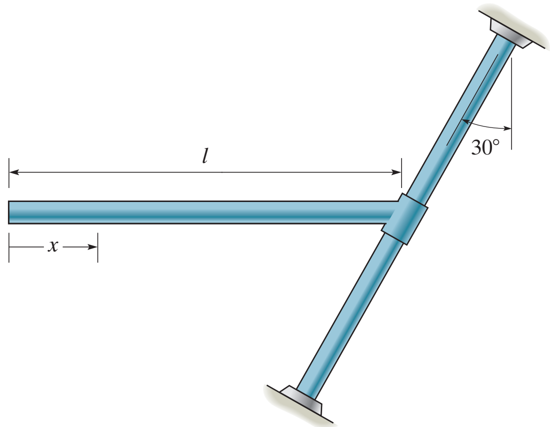
***17–36.** The assembly has a mass of 8 Mg and is hoisted using the boom and pulley system. If the winch at B draws in the cable with an acceleration of 2 m/s^2, determine the compressive force in the hydraulic cylinder needed to support the boom. The boom has a mass of 2 Mg and mass center at G.

17–37. The assembly has a mass of 4 Mg and is hoisted using the winch at B. Determine the greatest acceleration of the assembly so that the compressive force in the hydraulic cylinder supporting the boom does not exceed 180 kN. What is the tension in the supporting cable? The boom has a mass of 2 Mg and mass center at G.



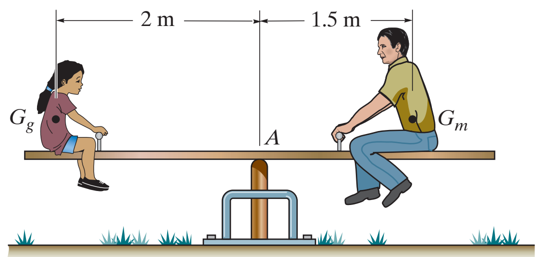
Probs. 17–36/37

17–38. The bar has a weight per length w and is supported by the smooth collar. If it is released from rest, determine the internal normal force, shear force, and bending moment in the bar as a function of x.



Prob. 17–38

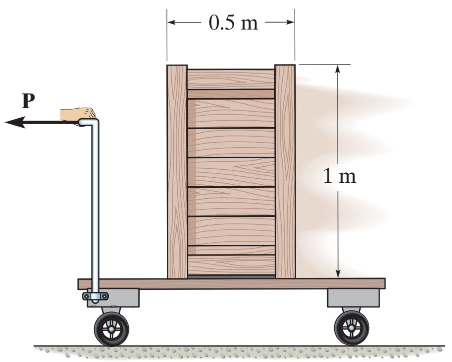
17–39. A 75-kg man and 40-kg girl sit on the seesaw, which has negligible mass. At the instant the man lifts his feet from the ground, determine their accelerations if each sits upright, i.e., they do not rotate. The centers of mass of the man and girl are at G_m and G_g, respectively.



Prob. 17–39

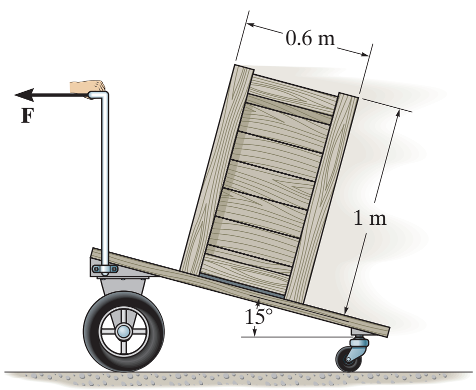
***17–40.** The 150-kg uniform crate rests on the 10-kg cart. Determine the maximum force *P* that can be applied to the handle without causing the crate to tip on the cart. Slipping does not occur.

17–41. The 150-kg uniform crate rests on the 10-kg cart. Determine the maximum force *P* that can be applied to the handle without causing the crate to slip or tip on the cart. The coefficient of static friction between the crate and cart is $\mu_s = 0.2$.



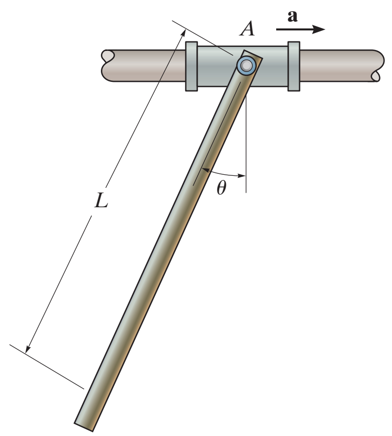
Probs. 17–40/41

17–42. The uniform crate has a mass of 50 kg and rests on the cart having an inclined surface. Determine the smallest acceleration that will cause the crate either to tip or slip relative to the cart. What is the magnitude of this acceleration? The coefficient of static friction between the crate and cart is $\mu_s = 0.5$.



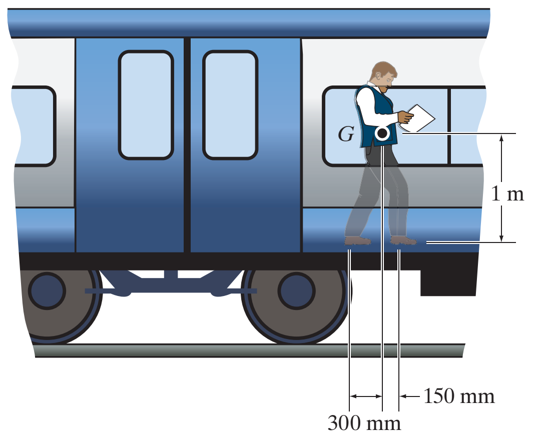
Prob. 17–42

17–43. The uniform bar of mass *m* is pin connected to the collar, which slides along the smooth horizontal rod. If the collar is given a constant acceleration of **a**, determine the bar's inclination angle θ. Neglect the collar's mass.



Prob. 17–43

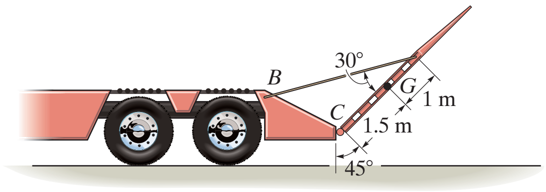
***17–44.** A man stands in the aisle of a moving train. If the coefficient of static friction between his shoes and the floor is $\mu_s = 0.5$, and he has a mass of 85 kg, with center of mass at *G*, determine the smallest train acceleration to the right to cause the man to either slide or tip. Assume the man holds himself rigid.



Prob. 17–44

17–45. The drop gate at the end of the trailer has a mass of 1.25 Mg and mass center at G. If it is supported by the cable AB and hinge at C, determine the tension in the cable when the truck begins to accelerate at $5\ \text{m/s}^2$. Also, what are the horizontal and vertical components of reaction at the hinge C?

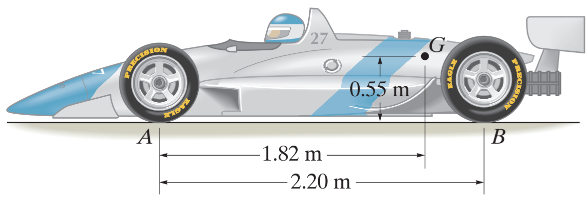
17–46. The drop gate at the end of the trailer has a mass of 1.25 Mg and mass center at G. If it is supported by the cable AB and hinge at C, determine the maximum deceleration of the truck so that the gate does not begin to rotate forward. What are the horizontal and vertical components of reaction at the hinge C?



Probs. 17–45/46

17–47. Determine the greatest possible acceleration of the 975-kg race car so that its front tires do not leave the ground and none of the tires slip on the track. The coefficients of static and kinetic friction are $\mu_s = 0.8$ and $\mu_k = 0.6$, respectively. Neglect the mass of the tires. The car has rear-wheel drive and the front tires are free to roll.

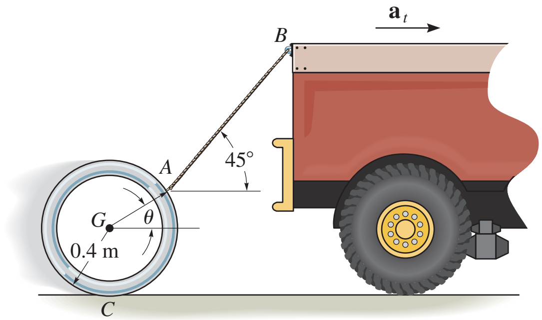
***17–48.** Determine the greatest possible acceleration of the 975-kg race car so that its front wheels do not leave the ground and none of the tires slip on the track. The coefficients of static and kinetic friction are $\mu_s = 0.8$ and $\mu_k = 0.6$, respectively. Neglect the mass of the tires. The car has four-wheel drive.



Probs. 17–47/48

17–49. The pipe has a mass of 800 kg and is being towed behind the truck. If the acceleration of the truck is $a_t = 0.5\ \text{m/s}^2$, determine the angle θ and the tension in the cable. The coefficient of kinetic friction between the pipe and the ground is $\mu_k = 0.1$.

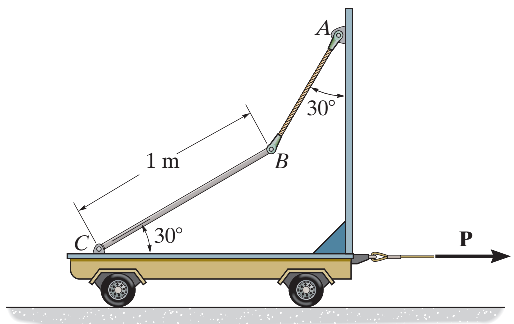
17–50. The pipe has a mass of 800 kg and is being towed behind a truck. If the angle $\theta = 30°$, determine the acceleration of the truck and the tension in the cable. The coefficient of kinetic friction between the pipe and the ground is $\mu_k = 0.1$.



Probs. 17–49/50

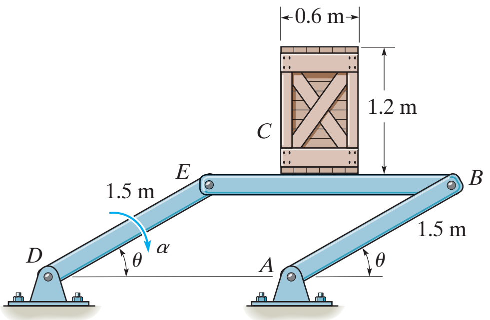
17–51. If the cart's mass is 30 kg and it is subjected to a horizontal force of $P = 90$ N, determine the tension in cord AB and the horizontal and vertical components of reaction on end C of the uniform 15-kg rod BC.

***17–52.** If the cart's mass is 30 kg, determine the horizontal force P that should be applied to the cart so that the cord AB just becomes slack. The uniform rod BC has a mass of 15 kg.



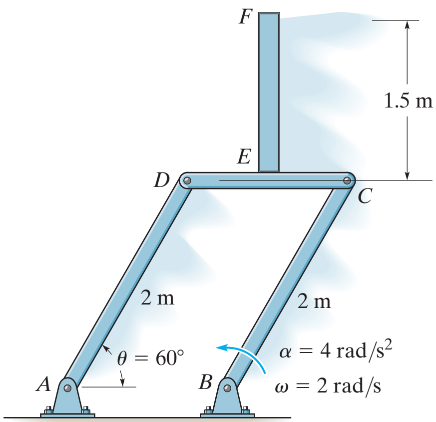
Probs. 17–51/52

17–53. The 100-kg uniform crate C rests on the elevator floor where the coefficient of static friction is $\mu_s = 0.4$. Determine the largest initial angular acceleration α, starting from rest at $\theta = 90°$, without causing the crate to slip. No tipping occurs.



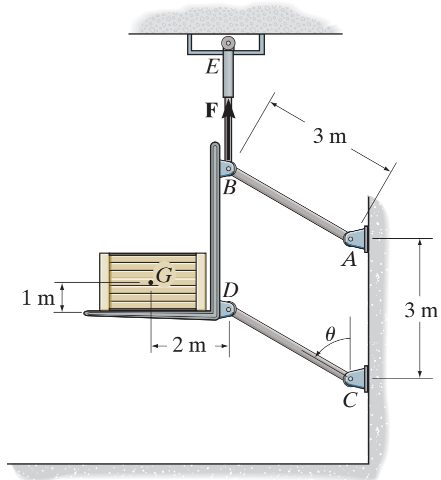
Prob. 17–53

17–54. The two uniform 4-kg bars DC and EF are fixed (welded) together at E. Determine the normal force N_E, shear force V_E, and moment M_E, which DC exerts on EF at E if at the instant $\theta = 60°$ BC has an angular velocity $\omega = 2\ \text{rad/s}$ and an angular acceleration $\alpha = 4\ \text{rad/s}^2$ as shown.



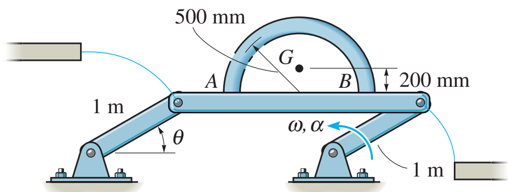
Prob. 17–54

17–55. If the hydraulic cylinder BE exerts a vertical force of $F = 1.5$ kN on the platform, determine the force developed in links AB and CD at the instant $\theta = 90°$. The platform is at rest when $\theta = 45°$. Neglect the mass of the links and the platform. The 200-kg crate does not slip on the platform.



Prob. 17–55

***17–56.** The arched pipe has a mass of 80 kg and rests on the surface of the platform for which the coefficient of static friction is $\mu_s = 0.3$. Determine the greatest angular acceleration α of the platform, starting from rest when $\theta = 45°$, without causing the pipe to slip on the platform.



Prob. 17–56

17.4 EQUATIONS OF MOTION: ROTATION ABOUT A FIXED AXIS

Consider the rigid body (or slab) shown in Fig. 17–13*a*, which is constrained to rotate in the vertical plane about a fixed axis perpendicular to the page and passing through the pin at O. The angular velocity and angular acceleration are caused by the external force and couple moment system acting on the body. Because the body's center of mass G moves around a circular path, the acceleration of this point is best represented by its tangential and normal components. The tangential component of acceleration has a magnitude of $(a_G)_t = \alpha r_G$ and must act in a direction which is consistent with the body's angular acceleration $\boldsymbol{\alpha}$. The magnitude of the normal component of acceleration is $(a_G)_n = \omega^2 r_G$. This component is always directed from point G towards O, regardless of the rotational sense of $\boldsymbol{\omega}$.

The free-body and kinetic diagrams for the body are shown in Fig. 17–13*b*. The two components $m(\mathbf{a}_G)_t$ and $m(\mathbf{a}_G)_n$, shown on the kinetic diagram, are associated with the tangential and normal components of acceleration of the body's mass center. The $I_G\boldsymbol{\alpha}$ vector acts in the same direction as $\boldsymbol{\alpha}$ and has a magnitude of $I_G\alpha$. From the derivation given in Sec. 17.2, the equations of motion which apply to the body can be written in the form

$$\begin{aligned} \Sigma F_n &= m(a_G)_n = m\omega^2 r_G \\ \Sigma F_t &= m(a_G)_t = m\alpha r_G \\ \Sigma M_G &= I_G\alpha \end{aligned} \qquad (17\text{–}14)$$

The moment equation can be replaced by a moment summation about any arbitrary point P on or off the body provided one accounts for the moments $\Sigma(\mathcal{M}_k)_P$ produced by $I_G\boldsymbol{\alpha}$, $m(\mathbf{a}_G)_t$, and $m(\mathbf{a}_G)_n$ about the point.

Moment Equation About Point O. Often it is convenient to sum moments about the pin at O in order to eliminate the unknown force $\mathbf{F}_O$. From the kinetic diagram, Fig. 17–13*b*, this requires

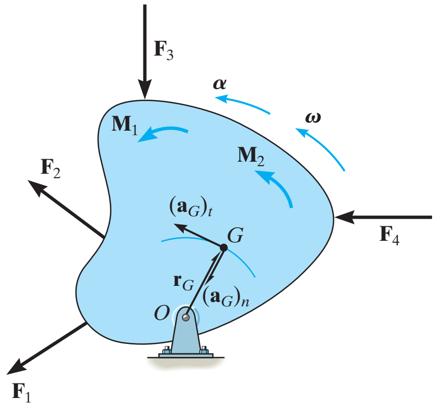
$$\circlearrowleft + \Sigma M_O = \Sigma(\mathcal{M}_k)_O; \qquad \Sigma M_O = r_G m(a_G)_t + I_G\alpha \qquad (17\text{–}15)$$

The moment of $m(\mathbf{a}_G)_n$ is not included here since the line of action of this vector passes through O. Substituting $(a_G)_t = r_G\alpha$, we may rewrite the above equation as $\circlearrowleft + \Sigma M_O = (I_G + mr_G^2)\alpha$. From the parallel-axis theorem, $I_O = I_G + md^2$, and therefore the term in parentheses represents the moment of inertia of the body about the fixed axis of rotation passing through O.* Consequently, we can also write the three equations of motion for the body as

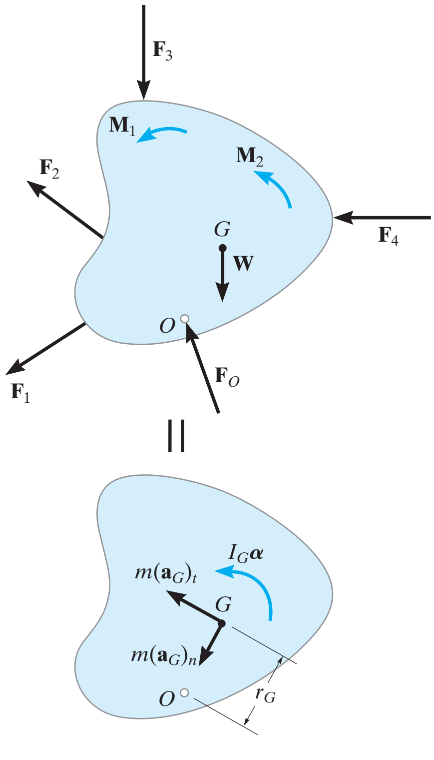
$$\begin{aligned} \Sigma F_n &= m(a_G)_n = m\omega^2 r_G \\ \Sigma F_t &= m(a_G)_t = m\alpha r_G \\ \Sigma M_O &= I_O\alpha \end{aligned} \qquad (17\text{–}16)$$

When using these equations, remember that "$I_O\alpha$" accounts for the "moment" of both $m(\mathbf{a}_G)_t$ and $I_G\boldsymbol{\alpha}$ about point O, Fig. 17–13*b*. In other words, $\Sigma M_O = \Sigma(\mathcal{M}_k)_O = I_O\alpha$.

*The result $\Sigma M_O = I_O\alpha$ can also be obtained *directly* from Eq. 17–6 by selecting point P to coincide with O, realizing that $(a_P)_x = (a_P)_y = 0$.

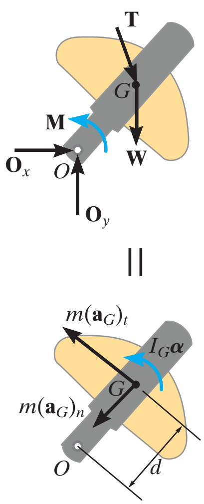


(a)



(b)

Fig. 17–13



The crank on the oil-pumping rig undergoes rotation about a fixed axis which is caused by a driving torque **M** of the motor. The loadings shown on the free-body diagram cause the effects shown on the kinetic diagram. If moments are summed about the mass center, G, then $\Sigma M_G = I_G\alpha$. However, if moments are summed about point O, noting that $(a_G)_t = \alpha d$, then $\circlearrowright + \Sigma M_O = I_G\alpha + m(a_G)_t d + m(a_G)_n(0) = (I_G + md^2)\alpha = I_O\alpha$.

Refer to the companion website for Lecture Summary and Quiz videos.

PROCEDURE FOR ANALYSIS

Kinetic problems which involve the rotation of a body about a fixed axis can be solved using the following procedure.

Free-Body Diagram.

- Establish the inertial n, t coordinate system and specify the direction and sense of the accelerations $(\mathbf{a}_G)_n$ and $(\mathbf{a}_G)_t$ and the angular acceleration $\boldsymbol{\alpha}$ of the body. Recall that $(\mathbf{a}_G)_t$ must act in a direction which is in accordance with the rotational sense of $\boldsymbol{\alpha}$, whereas $(\mathbf{a}_G)_n$ always acts toward the axis of rotation, point O.
- Draw the free-body diagram to account for all the external forces and couple moments that act on the body.
- Determine the moment of inertia I_G or I_O.
- Identify the unknowns in the problem.
- If it is decided that the rotational equation of motion $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$ is to be used, i.e., P is a point other than G or O, then consider drawing the kinetic diagram in order to help "visualize" the "moments" developed by the components $m(\mathbf{a}_G)_n$, $m(\mathbf{a}_G)_t$, and $I_G\boldsymbol{\alpha}$ when writing the terms for the moment sum $\Sigma(\mathcal{M}_k)_P$.

Equations of Motion.

- Apply the three equations of motion in accordance with the established sign convention.
- If moments are summed about the body's mass center, G, then $\Sigma M_G = I_G\alpha$, since $(m\mathbf{a}_G)_t$ and $(m\mathbf{a}_G)_n$ create no moment about G.
- If moments are summed about the pin support O on the axis of rotation, then $(m\mathbf{a}_G)_n$ creates no moment about O, and it can be shown that $\Sigma M_O = I_O\alpha$.

Kinematics.

- Use kinematics if a complete solution cannot be obtained strictly from the equations of motion.
- If the angular acceleration is variable, use

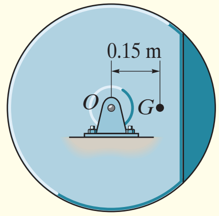
$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} \qquad \alpha\, d\theta = \omega\, d\omega$$

- If the angular acceleration is constant, use

$$\omega = \omega_0 + \alpha_c t$$
$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha_c t^2$$
$$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$$

EXAMPLE 17.8

The unbalanced 25-kg flywheel shown in Fig. 17–14a has a radius of gyration of $k_G = 0.18$ m about an axis passing through its mass center G. If it is released from rest, determine the horizontal and vertical components of reaction at the pin O at the instant.



(a)

SOLUTION

Free-Body and Kinetic Diagrams. Since G moves in a circular path, it will have both normal and tangential components of acceleration. Also, since $\boldsymbol{\alpha}$, which is caused by the flywheel's weight, acts clockwise, the tangential component of acceleration must act downward. Why? Since $\omega = 0$, only $m(a_G)_t = m\alpha r_G$ and $I_G\boldsymbol{\alpha}$ are shown on the kinetic diagram in Fig. 17–14b. Here, the moment of inertia about G is

$$I_G = mk_G^2 = (25\text{ kg})(0.18\text{ m})^2 = 0.81\text{ kg}\cdot\text{m}^2$$

The three unknowns are O_n, O_t, and α.

Equations of Motion.

$$\overset{+}{\leftarrow}\Sigma F_n = m\omega^2 r_G; \qquad O_n = 0 \qquad \textit{Ans.}$$

$$+\downarrow\Sigma F_t = m\alpha r_G; \qquad -O_t + 25(9.81)\text{ N} = (25\text{ kg})(\alpha)(0.15\text{ m}) \qquad (1)$$

$$\curvearrowright+\Sigma M_G = I_G\alpha; \qquad O_t(0.15\text{ m}) = (0.81\text{ kg}\cdot\text{m}^2)\alpha$$

Solving,

$$\alpha = 26.8\text{ rad/s}^2 \quad O_t = 145\text{ N} \qquad \textit{Ans.}$$

Moments can also be summed about point O in order to eliminate $\mathbf{O}_n$ and $\mathbf{O}_t$ and thereby obtain a direct solution for $\boldsymbol{\alpha}$, Fig. 17–14b. This can be done in one of two ways.

$$\curvearrowright+\Sigma M_O = \Sigma(\mathcal{M}_k)_O;$$

$$[25(9.81)\text{ N}](0.15\text{ m}) = (0.81\text{ kg}\cdot\text{m}^2)\alpha + \left[(25\text{ kg})\alpha(0.15\text{ m})\right](0.15\text{ m})$$

$$245.25\text{ N}(0.15\text{ m}) = 1.3725\alpha \qquad (2)$$

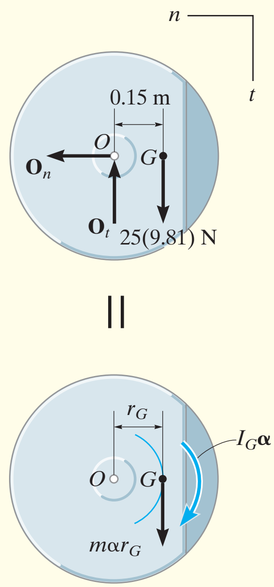
If $\Sigma M_O = I_O\alpha$ is applied, then by the parallel-axis theorem the moment of inertia of the flywheel about O is

$$I_O = I_G + mr_G^2 = 0.81 + (25)(0.15)^2 = 1.3725\text{ kg}\cdot\text{m}^2$$

Hence,

$$\curvearrowright+\Sigma M_O = I_O\alpha; \quad (245.25\text{ N})(0.15\text{ m}) = (1.3725\text{ kg}\cdot\text{m}^2)\alpha$$

which is the same as Eq. 2. Solving for α and substituting into Eq. 1 yields the answer for O_t obtained previously.

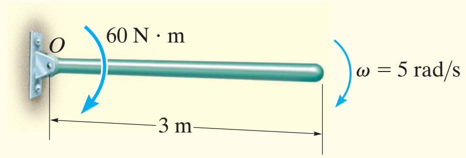


(b)

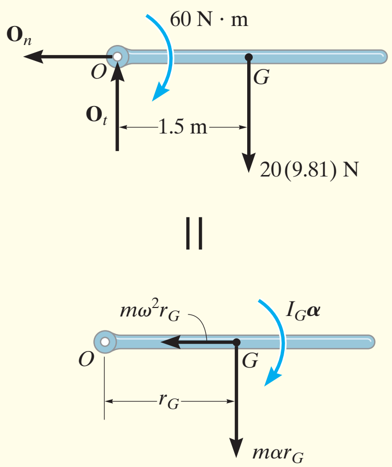
Fig. 17–14

EXAMPLE 17.9

At the instant shown in Fig. 17–15*a*, the 20-kg slender rod has an angular velocity of $\omega = 5$ rad/s. Determine the angular acceleration and the horizontal and vertical components of reaction of the pin on the rod at this instant.



(a)



(b)

Fig. 17–15

SOLUTION

Free-Body and Kinetic Diagrams. Fig. 17–15*b*. As shown on the kinetic diagram, point G moves around a circular path and so it has two components of acceleration. It is important that the tangential component $a_t = \alpha r_G$ act downward since it must be in accordance with the rotational sense of $\boldsymbol{\alpha}$. The three unknowns are O_n, O_t, and α.

Equation of Motion.

$$\overset{+}{\leftarrow}\Sigma F_n = m\omega^2 r_G; \qquad O_n = (20\text{ kg})(5\text{ rad/s})^2(1.5\text{ m})$$

$$+\downarrow\Sigma F_t = m\alpha r_G; \qquad -O_t + 20(9.81)\text{ N} = (20\text{ kg})(\alpha)(1.5\text{ m})$$

$$\circlearrowright+\Sigma M_G = I_G\alpha; \qquad O_t(1.5\text{ m}) + 60\text{ N}\cdot\text{m} = \left[\tfrac{1}{12}(20\text{ kg})(3\text{ m})^2\right]\alpha$$

Solving

$$O_n = 750\text{ N} \quad O_t = 19.05\text{ N} \quad \alpha = 5.90\text{ rad/s}^2 \qquad \textit{Ans.}$$

A more direct solution to this problem would be to sum moments about point O to eliminate O_n and O_t and obtain a *direct solution* for α. Here,

$$\circlearrowright+\Sigma M_O = \Sigma(\mathcal{M}_k)_O; \qquad 60\text{ N}\cdot\text{m} + 20(9.81)\text{ N}(1.5\text{ m}) =$$

$$\left[\tfrac{1}{12}(20\text{ kg})(3\text{ m})^2\right]\alpha + [20\text{ kg}(\alpha)(1.5\text{ m})](1.5\text{ m})$$

$$\alpha = 5.90\text{ rad/s}^2 \qquad \textit{Ans.}$$

Also, since $I_O = \frac{1}{3}ml^2$ for a slender rod, we can apply

$$\circlearrowright+\Sigma M_O = I_O\alpha; \quad 60\text{ N}\cdot\text{m} + 20(9.81)\text{ N}(1.5\text{ m}) = \left[\tfrac{1}{3}(20\text{ kg})(3\text{ m})^2\right]\alpha$$

$$\alpha = 5.90\text{ rad/s}^2 \qquad \textit{Ans.}$$

NOTE: By comparison, the last equation provides the simplest solution for α and *does not* require use of the kinetic diagram.

17

EXAMPLE 17.10

The drum shown in Fig. 17–16a has a mass of 60 kg and a radius of gyration $k_O = 0.25$ m. A cord of negligible mass is wrapped around the periphery of the drum and attached to a block having a mass of 20 kg. If the block is released, determine the drum's angular acceleration.

SOLUTION I

Free-Body Diagram. Here we will consider the drum and block separately, Fig. 17–16b. Assuming the block accelerates downward at **a**, then it creates a counterclockwise angular acceleration $\boldsymbol{\alpha}$ of the drum. The moment of inertia of the drum is

$$I_O = mk_O^2 = (60\text{ kg})(0.25\text{ m})^2 = 3.75\text{ kg}\cdot\text{m}^2$$

There are five unknowns, namely O_x, O_y, T, a, and α.

Equations of Motion. Moments will be summed about O to eliminate O_x and O_y. For the drum and block, we have

$$\circlearrowleft +\Sigma M_O = I_O\alpha; \qquad T(0.4\text{ m}) = (3.75\text{ kg}\cdot\text{m}^2)\alpha \qquad (1)$$

$$+\uparrow \Sigma F_y = m(a_G)_y; \qquad -20(9.81)\text{ N} + T = -(20\text{ kg})a \qquad (2)$$

Kinematics. Since the point of contact A between the cord and drum has a tangential component of acceleration **a**, then

$$\circlearrowleft + a = \alpha r; \qquad a = \alpha(0.4\text{ m}) \qquad (3)$$

Solving the above equations,

$$T = 106\text{ N} \quad a = 4.52\text{ m/s}^2$$

$$\alpha = 11.3\text{ rad/s}^2 \circlearrowleft \qquad \textit{Ans.}$$

SOLUTION II

Free-Body and Kinetic Diagrams. The cable tension T can be eliminated from the analysis by considering the drum and block as a single system, Fig. 17–16c. The kinetic diagram is shown since moments will be summed about point O.

Equations of Motion. Using Eq. 3 and applying the moment equation about O to eliminate the unknowns O_x and O_y, we have

$$\circlearrowleft +\Sigma M_O = \Sigma(\mathcal{M}_k)_O; \qquad [20(9.81)\text{ N}]\,(0.4\text{ m}) =$$

$$(3.75\text{ kg}\cdot\text{m}^2)\alpha + [20\text{ kg}(\alpha\, 0.4\text{ m})](0.4\text{ m})$$

$$\alpha = 11.3\text{ rad/s}^2 \circlearrowleft \qquad \textit{Ans.}$$

NOTE: If the block were removed and a force of 20(9.81) N were applied to the cord, show that $\alpha = 20.9\text{ rad/s}^2$. This value is larger than our result since the block has an inertia, or resistance to acceleration.

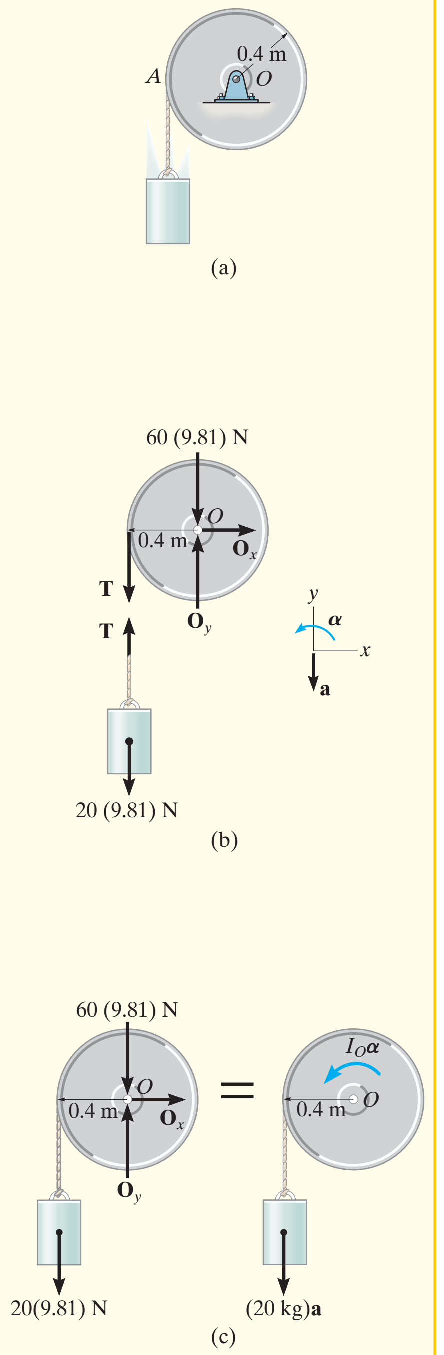


Fig. 17–16

17

EXAMPLE 17.11

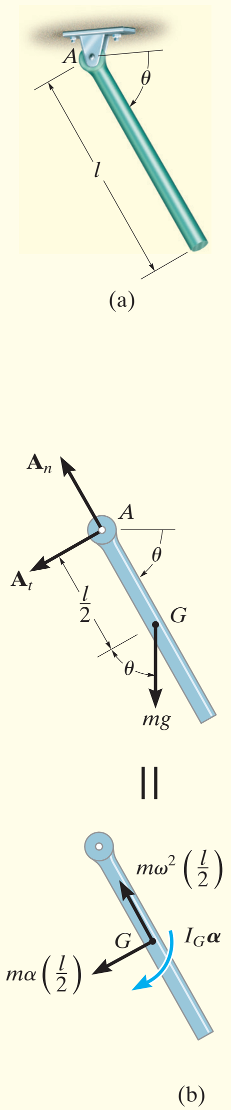


Fig. 17–17

The slender rod shown in Fig. 17–17*a* has a mass m and length l and is released from rest when $\theta = 0°$. Determine the horizontal and vertical components of force which the pin at A exerts on the rod at the instant $\theta = 90°$.

SOLUTION

Free-Body and Kinetic Diagrams. The free-body diagram for the rod in the general position θ is shown in Fig. 17–17*b*. For convenience, the force components at A are shown acting in the n and t directions. Note that $\boldsymbol{\alpha}$ acts clockwise and so $(\mathbf{a}_G)_t$ acts in the $+t$ direction.

The moment of inertia of the rod about point A is $I_A = \frac{1}{3}ml^2$. For a given angle θ there are four unknowns, namely, A_n, A_t, ω, and α.

Equations of Motion. Moments will be summed about A in order to eliminate A_n and A_t.

$$+\nwarrow\Sigma F_n = m\omega^2 r_G; \qquad A_n - mg\sin\theta = m\omega^2(l/2) \qquad (1)$$

$$+\swarrow\Sigma F_t = m\alpha r_G; \qquad A_t + mg\cos\theta = m\alpha(l/2) \qquad (2)$$

$$\curvearrowright+\Sigma M_A = I_A\alpha; \qquad mg\cos\theta(l/2) = \left(\tfrac{1}{3}ml^2\right)\alpha \qquad (3)$$

Kinematics. As shown by Eq. 3, α is not constant; rather, it depends on the position θ of the rod. The necessary fourth equation is obtained using kinematics, where α and ω can be related to θ by the equation

$$(\curvearrowright+) \qquad \omega\, d\omega = \alpha\, d\theta \qquad (4)$$

Note that the positive clockwise direction for this equation agrees with that of Eq. 3. This is important since we are seeking a simultaneous solution.

In order to solve for ω at $\theta = 90°$, eliminate α from Eqs. 3 and 4, which yields

$$\omega\, d\omega = (1.5g/l)\cos\theta\, d\theta$$

Since $\omega = 0$ at $\theta = 0°$, we have

$$\int_0^{\omega} \omega\, d\omega = (1.5g/l)\int_{0°}^{90°} \cos\theta\, d\theta$$

$$\omega^2 = 3g/l$$

Substituting this value into Eq. 1 with $\theta = 90°$ and solving Eqs. 1 to 3 yields

$$\alpha = 0$$

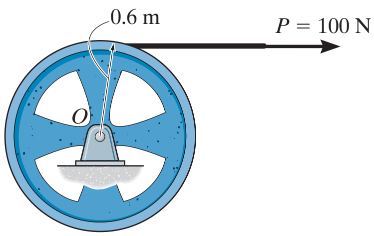
$$A_t = 0 \qquad A_n = 2.5mg \qquad \textit{Ans.}$$

NOTE: If $\Sigma M_A = \Sigma(\mathcal{M}_k)_A$ is used, one must account for the moments of $I_G\boldsymbol{\alpha}$ and $m(\mathbf{a}_G)_t$ about A.

Refer to the companion website for a self quiz of these Example problems.

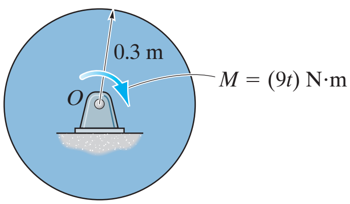
FUNDAMENTAL PROBLEMS

F17–7. The 100-kg wheel has a radius of gyration about its center O of $k_O = 500$ mm. If the wheel starts from rest, determine its angular velocity in $t = 3$ s.



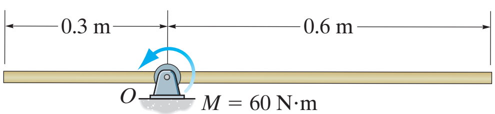
Prob. F17–7

F17–8. The 50-kg disk is subjected to the couple moment of $M = (9t)$ N·m, where t is in seconds. Determine the angular velocity of the disk when $t = 4$ s starting from rest.



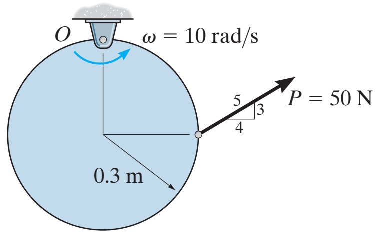
Prob. F17–8

F17–9. At the instant shown, the uniform 30-kg slender rod has a counterclockwise angular velocity of $\omega = 6$ rad/s. Determine the tangential and normal components of reaction of pin O on the rod and the angular acceleration of the rod at this instant.



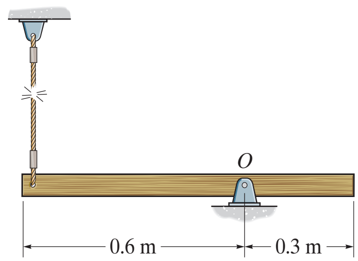
Prob. F17–9

F17–10. At the instant shown, the 30-kg disk has a counterclockwise angular velocity of $\omega = 10$ rad/s. Determine the tangential and normal components of reaction of the pin O on the disk and the angular acceleration of the disk at this instant.



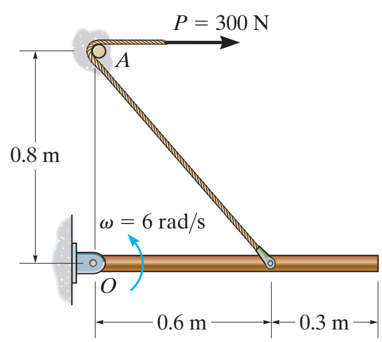
Prob. F17–10

F17–11. The uniform slender rod has a mass of 15 kg. Determine the horizontal and vertical components of reaction at the pin O, and the angular acceleration of the rod just after the cord is cut.



Prob. F17–11

F17–12. The uniform 30-kg slender rod is being lifted by the cord that passes over the small smooth peg at A. If the rod has a counterclockwise angular velocity of $\omega = 6$ rad/s at the instant shown, determine the tangential and normal components of reaction at the pin O and the angular acceleration of the rod.



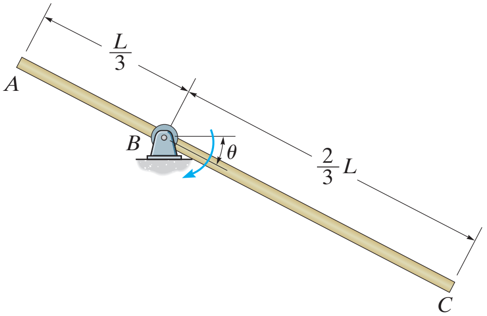
Prob. F17–12

17

PROBLEMS

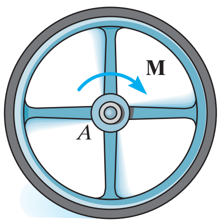
All solutions must include a free-body diagram.

17–57. The uniform slender rod has a mass m. If it is released from rest when $\theta = 0°$, determine the magnitude of the reactive force exerted on it by pin B when $\theta = 90°$.



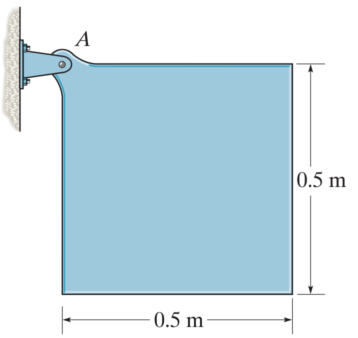
Prob. 17–57

17–58. The 10-kg wheel has a radius of gyration $k_A = 200$ mm. If the wheel is subjected to a moment $M = (5t)$ N · m, where t is in seconds, determine its angular velocity when $t = 3$ s starting from rest. Also, calculate the reactions which the fixed pin A exerts on the wheel during the motion.



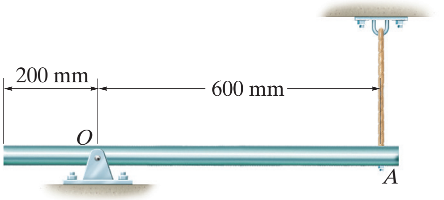
Prob. 17–58

17–59. The uniform 24-kg plate is released from rest at the position shown. Determine its initial angular acceleration and the horizontal and vertical reactions at the pin A.



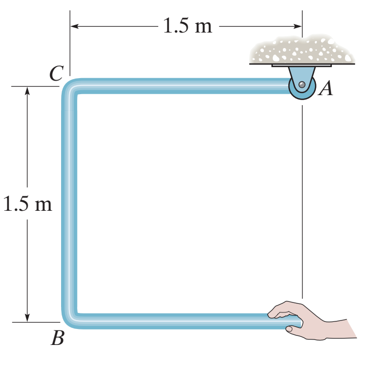
Prob. 17–59

***17–60.** The uniform slender rod has a mass of 5 kg. If the cord at A is cut, determine the reaction at the pin O, (a) when the rod is still in the horizontal position, and (b) when the rod swings to the vertical position.



Prob. 17–60

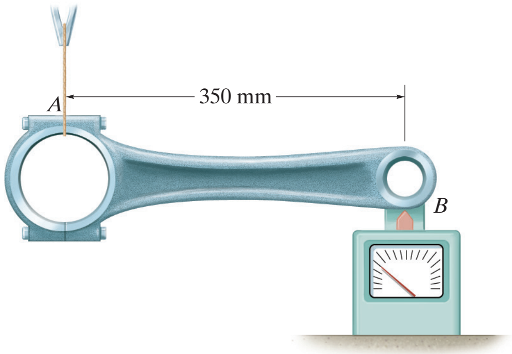
17–61. The bent rod has a mass of 2 kg/m. If it is released from rest in the position shown, determine its initial angular acceleration and the horizontal and vertical components of reaction at A.



Prob. 17–61

17

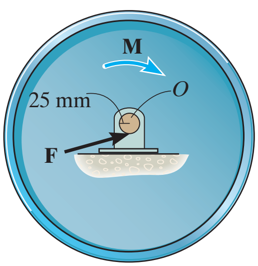
17–62. In order to experimentally determine the moment of inertia I_G of a 4-kg connecting rod, the rod is suspended horizontally at A by a cord and at B by a bearing and piezoelectric sensor, an instrument used for measuring force. Under these equilibrium conditions, the force at B is measured as 14.6 N. If at the instant the cord is released the reaction at B is measured as 9.3 N, determine the value of I_G. The support at B does not move when the measurement is taken. For the calculation, the horizontal location of G must be determined.



Prob. 17–62

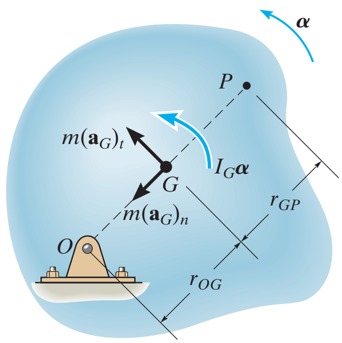
17–63. A motor supplies a constant torque $M = 2\ \text{N} \cdot \text{m}$ to a 50-mm-diameter shaft O connected to the center of the 30-kg flywheel. The resultant bearing friction **F**, which the bearing exerts on the shaft, acts tangent to the shaft and has a magnitude of 50 N. Determine how long the torque must be applied to the shaft to increase the flywheel's angular velocity from 4 rad/s to 15 rad/s. The flywheel has radius of gyration $k_O = 0.15$ m about its center O.

***17–64.** If the motor of Prob. 17–63 is disengaged from the shaft once the flywheel is rotating at 15 rad/s, so that $M = 0$, determine how long it will take before the resultant bearing friction $F = 50$ N stops the flywheel from rotating.



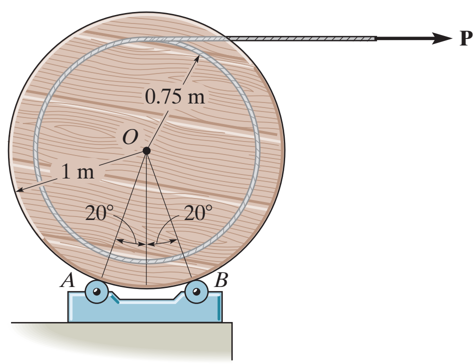
Probs. 17–63/64

17–65. The kinetic diagram representing the general rotational motion of a rigid body about a fixed axis passing through O is shown in the figure. Show that $I_G\boldsymbol{\alpha}$ may be eliminated by moving the vectors $m(\mathbf{a}_G)_t$ and $m(\mathbf{a}_G)_n$ to point P, located a distance $r_{GP} = k_G^2/r_{OG}$ from the center of mass G of the body. Here k_G represents the radius of gyration of the body about an axis passing through G. The point P is called the *center of percussion* of the body.



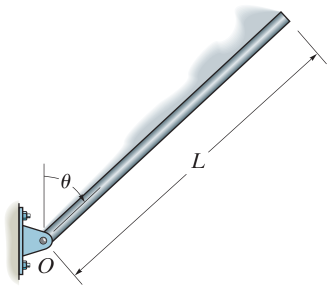
Prob. 17–65

17–66. If a horizontal force of $P = 100$ N is applied to the 300-kg reel of cable, determine its initial angular acceleration. The reel rests on rollers at A and B and has a radius of gyration of $k_O = 0.6$ m.



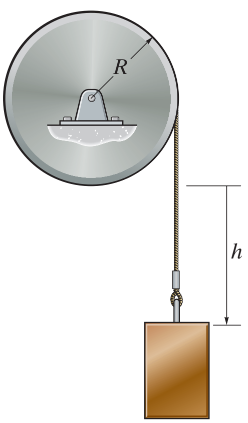
Prob. 17–66

17–67. The rod has a length L and mass m. If it is released from rest when $\theta \approx 0°$, determine its angular velocity as a function of θ. Also express the horizontal and vertical components of reaction at the pin O as a function of θ.



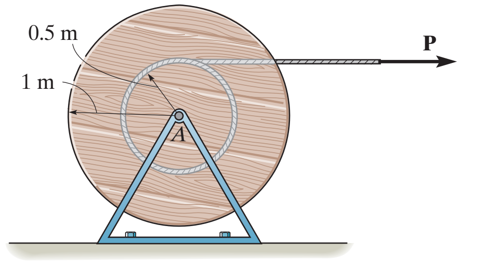
Prob. 17–67

***17–68.** The disk has a mass M and a radius R. If the block of mass m is attached to the cord, determine the angular velocity of the disk when the block is released from rest and falls a distance h.



Prob. 17–68

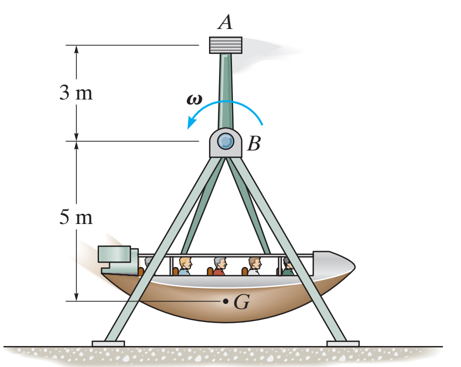
17–69. The reel of cable has a mass of 400 kg and a radius of gyration of $k_A = 0.75$ m. Determine its angular velocity when $t = 2$ s, starting from rest, if the force $P = (20t^2 + 80)$ N is applied, where t is in seconds. Neglect the mass of the unwound cable, and assume it is always at a radius of 0.5 m.



Prob. 17–69

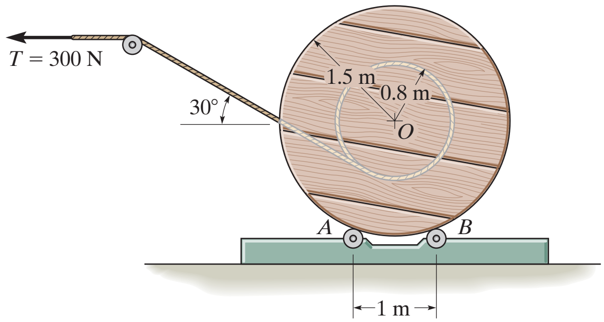
17–70. The passengers, the gondola, and its swing frame have a total mass of 50 Mg, a mass center at G, and a radius of gyration $k_B = 3.5$ m. Additionally, the 3-Mg steel block at A can be considered as a point of concentrated mass. Determine the horizontal and vertical components of reaction at pin B if the gondola swings freely at $\omega = 1$ rad/s when it reaches its lowest point as shown. Also, what is the gondola's angular acceleration at this instant?

17–71. The passengers, the gondola, and its swing frame have a total mass of 50 Mg, a mass center at G, and a radius of gyration $k_B = 3.5$ m. Additionally, the 3-Mg steel block at A can be considered as a point of concentrated mass. Determine the angle θ to which the gondola will swing before it stops momentarily, if it has an angular velocity of $\omega = 1$ rad/s at its lowest point.



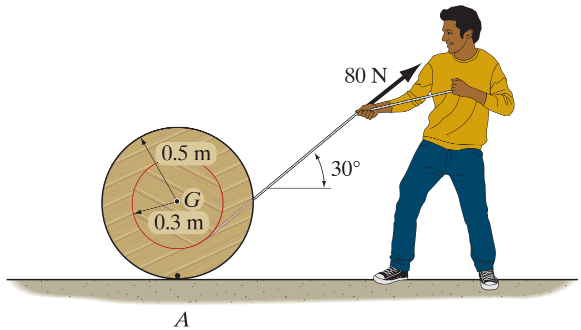
Probs. 17–70/71

*17–72. Cable is unwound from a spool supported on small rollers at A and B by exerting a force $T = 300$ N on the cable. Determine the time needed to unravel 5 m of cable from the spool if the spool and cable have a total mass of 600 kg and a radius of gyration of $k_O = 1.2$ m. For the calculation, neglect the mass of the cable being unwound and the mass of the rollers at A and B. The rollers turn with no friction.



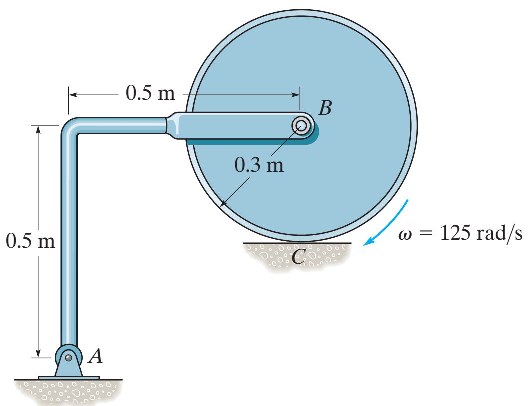
Prob. 17–72

17–74. The man pulls up on the cord with a force of 80 N. Determine the angular acceleration of the spool if there is no slipping at A. The spool has a mass of 60 kg and a radius of gyration about its mass center G of $k_G = 0.35$ m.



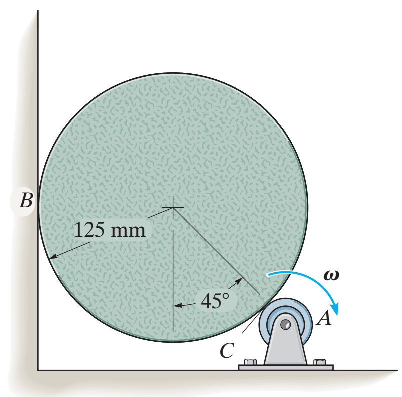
Prob. 17–74

17–73. The 30-kg disk is originally spinning at $\omega = 125$ rad/s. If it is placed on the ground, for which the coefficient of kinetic friction is $\mu_C = 0.5$, determine the time required for the motion to stop. What are the horizontal and vertical components of force which the member AB exerts on the pin at A during this time? Neglect the mass of AB.



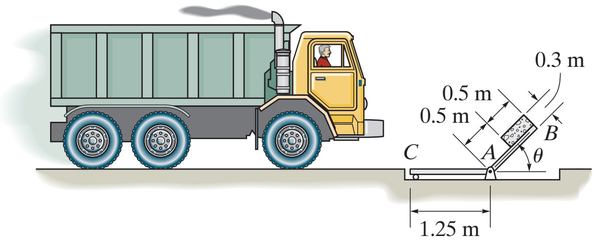
Prob. 17–73

17–75. The 5-kg cylinder is initially at rest when it is placed in contact with the wall B and the rotor at A. If the rotor always maintains a constant clockwise angular velocity $\omega = 6$ rad/s, determine the initial angular acceleration of the cylinder. The coefficient of kinetic friction at the contacting surfaces B and C is $\mu_k = 0.2$.



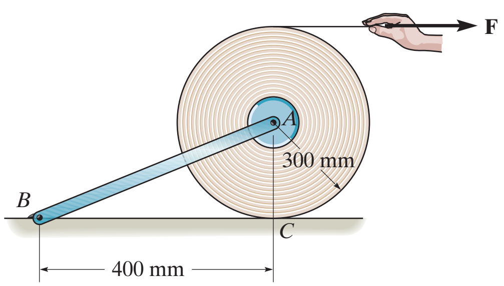
Prob. 17–75

***17–76.** The device acts as a pop-up barrier to prevent the passage of a vehicle. It consists of a 100-kg steel plate AC and a 200-kg counterweight solid concrete block located as shown. Determine the moment of inertia of the plate and block about the hinged axis through A. Neglect the mass of the supporting arms AB. Also, determine the initial angular acceleration of the assembly when it is released from rest at $\theta = 45°$.



Prob. 17–76

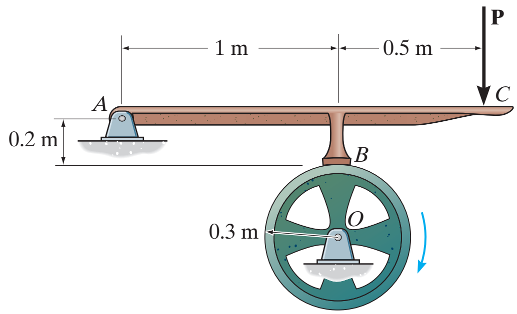
17–77. The 20-kg roll of paper has a radius of gyration $k_A = 120$ mm about an axis passing through point A. It is pin supported at both ends by two brackets AB. The roll rests on the floor, for which the coefficient of kinetic friction is $\mu_k = 0.2$. If a horizontal force $F = 60$ N is applied to the end of the paper, determine the initial angular acceleration of the roll as the paper unrolls.



Prob. 17–77

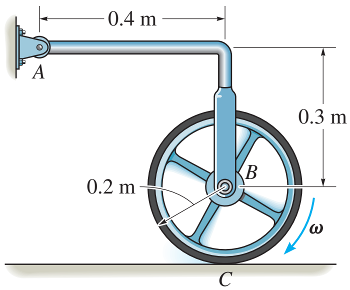
17–78. The 50-kg flywheel has a radius of gyration about its center of the mass of $k_O = 250$ mm. It rotates with a constant angular velocity of 1200 rev/min before the brake is applied. If the coefficient of kinetic friction between the brake pad B and the wheel's rim is $\mu_k = 0.5$, and a force of $P = 300$ N is applied to the braking mechanism's handle, determine the time required to stop the wheel.

17–79. The 50-kg flywheel has a radius of gyration about its center of mass of $k_O = 250$ mm. It rotates with a constant angular velocity of 1200 rev/min before the brake is applied. If the coefficient of kinetic friction between the brake pad B and the wheel's rim is $\mu_k = 0.5$, determine the constant force P that must be applied to the braking mechanism's handle in order to stop the wheel in 100 revolutions.



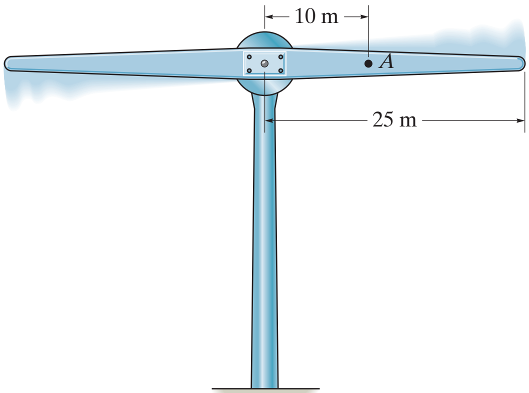
Probs. 17–78/79

***17–80.** The wheel has a mass of 25 kg and a radius of gyration $k_B = 0.15$ m. It is originally spinning at $\omega = 40$ rad/s. If it is placed on the ground, for which the coefficient of kinetic friction is $\mu_C = 0.5$, determine the time required for the motion to stop. What are the horizontal and vertical components of reaction which the pin at A exerts on AB during this time? Neglect the mass of AB.



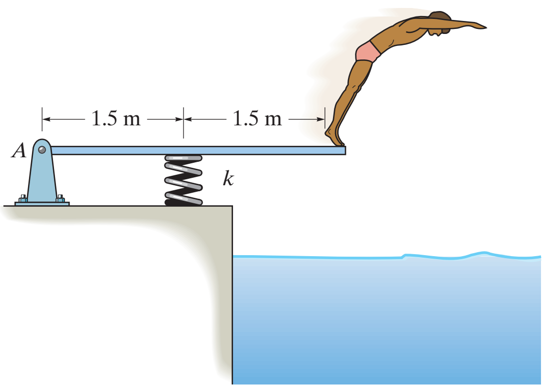
Prob. 17–80

17–81. The turbine consists of a rotor which is powered from a torque applied at its center. At the instant the rotor is horizontal it has an angular velocity of 15 rad/s and a clockwise angular acceleration of 8 rad/s^2. Determine the internal normal force, shear force, and moment at a section through A. Assume the rotor is a 50-m-long slender rod, having a mass of 3 kg/m.



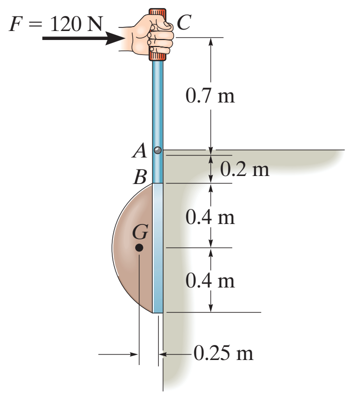
Prob. 17–81

17–82. Determine the angular acceleration of the 25-kg diving board, and the horizontal and vertical components of reaction at the pin A, the instant the man dives off. Assume that the board is uniform and rigid, and that at the instant he dives off, the spring is compressed a maximum amount of 200 mm, $\omega = 0$, and the board is horizontal. Take $k = 7$ kN/m.



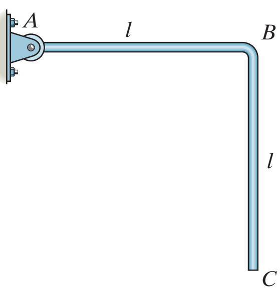
Prob. 17–82

17–83. The furnace cover has a mass of 20 kg and a radius of gyration $k_G = 0.25$ m about its mass center G. If an operator applies a force $F = 120$ N to the handle in order to open the cover, determine the cover's initial angular acceleration and the horizontal and vertical components of reaction which the pin at A exerts on the cover at the instant the cover begins to open. Neglect the mass of the handle BAC in the calculation.



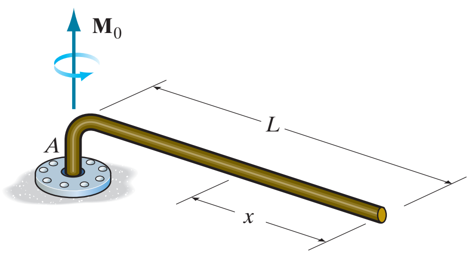
Prob. 17–83

***17–84.** The two-bar assembly is released from rest in the position shown. Determine the initial bending moment at the fixed joint B. Each bar has a mass m and length l.



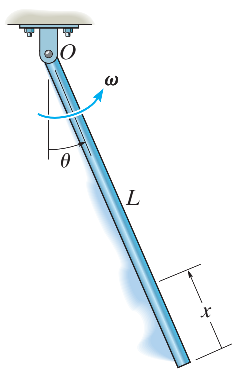
Prob. 17–84

17–85. The uniform rod of length L has a mass of m. If the couple moment $\mathbf{M}_0$ is applied to its end, determine the internal normal force and shear force, and the bending moment in the rod as a function of x just after $\mathbf{M}_0$ is applied. The bearing at A allows free rotation.



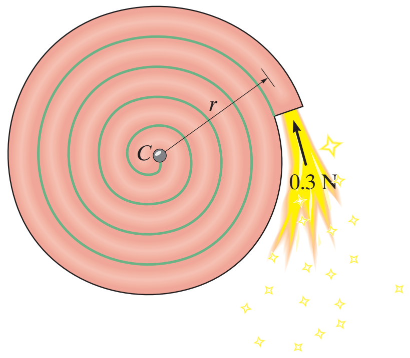
Prob. 17–85

17–86. The bar has a weight per length of w. If it is rotating in the vertical plane at a constant rate $\boldsymbol{\omega}$ about point O, determine the internal normal force, shear force, and moment as a function of x and θ.



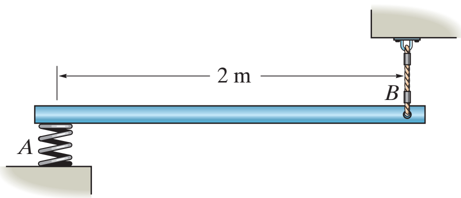
Prob. 17–86

17–87. The "Catherine wheel" is a firework that consists of a coiled tube of powder which is pinned at its center. If the powder burns at a constant rate of 20 g/s such that the exhaust gases always exert a force having a constant magnitude of 0.3 N, directed tangent to the wheel, determine the angular velocity of the wheel when 75% of the mass is burned off. Initially, the wheel is at rest and has a mass of 100 g and a radius of $r = 75$ mm. For the calculation, consider the wheel to always be a thin disk.



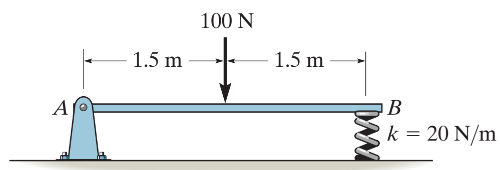
Prob. 17–87

***17–88.** The 4-kg slender rod is supported horizontally by a spring at A and a cord at B. Determine the angular acceleration of the rod and the acceleration of the rod's mass center at the instant the cord at B is cut. *Hint*: The stiffness of the spring is not needed for the calculation.



Prob. 17–88

17–89. The 4-kg slender rod is initially supported horizontally by a spring at B and pin at A. Determine the angular acceleration of the rod and the acceleration of the rod's mass center at the instant the 100-N force is applied.



Prob. 17–89

17.5 EQUATIONS OF MOTION: GENERAL PLANE MOTION

The rigid body (or slab) shown in Fig. 17–18*a* is subjected to general plane motion caused by the externally applied force and couple-moment system. The free-body and kinetic diagrams for the body are shown in Fig. 17–18*b*. If an x and y inertial coordinate system is established as shown, the three equations of motion are

$$\begin{aligned}\Sigma F_x &= m(a_G)_x \\ \Sigma F_y &= m(a_G)_y \\ \Sigma M_G &= I_G\alpha\end{aligned} \qquad (17\text{–}17)$$

In some problems it may be convenient to sum moments about a point P other than G in order to eliminate as many unknown forces as possible from the moment summation. When used for this more general case, the three equations of motion are

$$\begin{aligned}\Sigma F_x &= m(a_G)_x \\ \Sigma F_y &= m(a_G)_y \\ \Sigma M_P &= \Sigma(\mathcal{M}_k)_P\end{aligned} \qquad (17\text{–}18)$$

Here $\Sigma(\mathcal{M}_k)_P$ represents the moment sum of $I_G\boldsymbol{\alpha}$ and $m\mathbf{a}_G$ (or its components) about P.

Moment Equation About the *IC*. Some types of problems involve a uniform disk, or body of circular shape, that rolls on a rough surface *without slipping*, Fig. 17–19. If we sum the moments about the instantaneous center of zero velocity, then $\Sigma(\mathcal{M}_k)_{IC}$ becomes $I_{IC}\alpha$, so that

$$\Sigma M_{IC} = I_{IC}\alpha \qquad (17\text{–}19)$$

This result compares with $\Sigma M_O = I_O\alpha$, which is used for a body pinned at point O, Eq. 17–16. See Prob. 17–91.

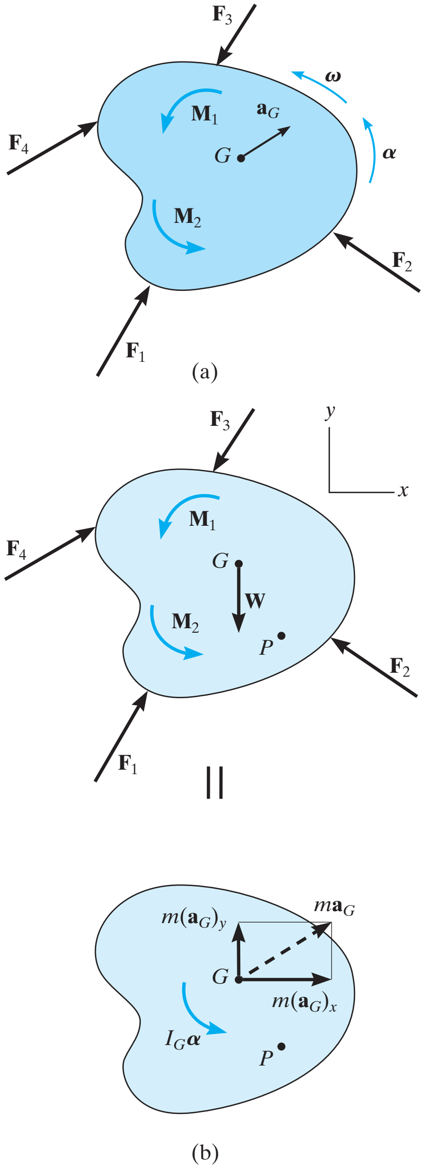


Fig. 17–18

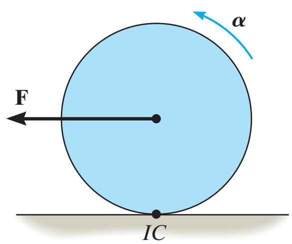
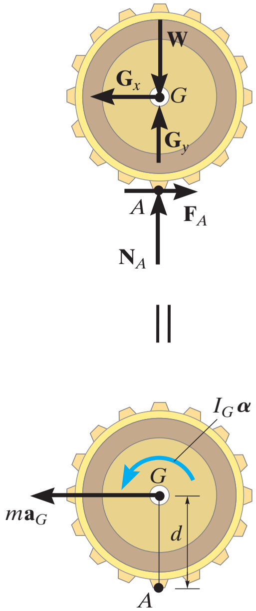


Fig. 17–19



As the soil compactor, or "sheep's foot roller" moves forward, the roller has general plane motion. The forces shown on its free-body diagram cause the effects shown on the kinetic diagram. If moments are summed about the mass center, G, then $\Sigma M_G = I_G\alpha$. However, if moments are summed about point A (the IC) then $\circlearrowright + \Sigma M_A = I_G\alpha + (ma_G)d = I_A\alpha$ since $a_G = \alpha d$.

PROCEDURE FOR ANALYSIS

Kinetic problems involving general plane motion of a rigid body can be solved using the following procedure.

Free-Body Diagram.

- Establish the x, y inertial coordinate system and draw the free-body diagram for the body.
- Specify the direction and sense of the acceleration of the mass center, $\mathbf{a}_G$, and the angular acceleration $\boldsymbol{\alpha}$ of the body.
- Determine the moment of inertia I_G.
- Identify the unknowns in the problem.
- If it is decided that the rotational equation of motion $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$ is to be used, then consider drawing the kinetic diagram in order to help "visualize" the "moments" developed by the components $m(\mathbf{a}_G)_x$, $m(\mathbf{a}_G)_y$, and $I_G\boldsymbol{\alpha}$ when writing the terms in the moment sum $\Sigma(\mathcal{M}_k)_P$.

Equations of Motion.

- Apply the three equations of motion in accordance with the established sign convention.

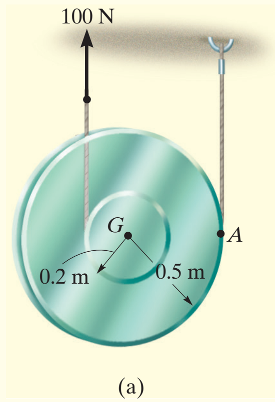
Kinematics.

- Use kinematics if a complete solution cannot be obtained strictly from the equations of motion.
- If the body's motion is constrained due to its supports, additional equations may be obtained by using $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$, which relates the accelerations of any two points A and B on the body.
- When a wheel, disk, cylinder, or ball *rolls without slipping*, then $a_G = \alpha r$.

Refer to the companion website for Lecture Summary and Quiz videos.

EXAMPLE 17.12

Determine the angular acceleration of the spool in Fig. 17–20*a*. The spool has a mass of 8 kg and a radius of gyration of $k_G = 0.35$ m. The cords of negligible mass are wrapped around its inner hub and outer rim.



(a)

SOLUTION I

Free-Body and Kinetic Diagrams. Fig. 17–20*b*. The 100-N force causes $\mathbf{a}_G$ to act upward. Also, $\boldsymbol{\alpha}$ acts clockwise, since the spool winds around the cord at *A*.

The moment of inertia of the spool about its mass center is

$$I_G = mk_G^2 = 8\text{ kg}(0.35\text{ m})^2 = 0.980\text{ kg}\cdot\text{m}^2$$

There are three unknowns T, a_G, and α.

Equations of Motion.

$$+\uparrow \Sigma F_y = m(a_G)_y; \qquad T + 100\text{ N} - 78.48\text{ N} = (8\text{ kg})a_G \qquad (1)$$

$$\circlearrowleft + \Sigma M_G = I_G\alpha; \qquad 100\text{ N}(0.2\text{ m}) - T(0.5\text{ m}) = (0.980\text{ kg}\cdot\text{m}^2)\alpha \qquad (2)$$

Kinematics. A complete solution is obtained if kinematics is used to relate a_G to α. In this case the spool "rolls without slipping" on the cord at *A*. Hence, we can use the results of Example 16.4 or 16.15, so that

$$(\circlearrowleft +)\ a_G = \alpha r; \qquad a_G = \alpha\,(0.5\text{ m}) \qquad (3)$$

Solving Eqs. 1 to 3, we have

$$\alpha = 10.3\text{ rad/s}^2 \qquad \textit{Ans.}$$

$$a_G = 5.16\text{ m/s}^2$$

$$T = 19.8\text{ N}$$

SOLUTION II

Equations of Motion. We can eliminate the unknown T by summing moments about point *A*. From the free-body and kinetic diagrams, Figs. 17–20*b*, we have

$$\circlearrowleft + \Sigma M_A = \Sigma(\mathcal{M}_k)_A; \quad 100\text{ N}(0.7\text{ m}) - 78.48\text{ N}(0.5\text{ m})$$
$$= (0.980\text{ kg}\cdot\text{m}^2)\alpha + [(8\text{ kg})a_G](0.5\text{ m})$$

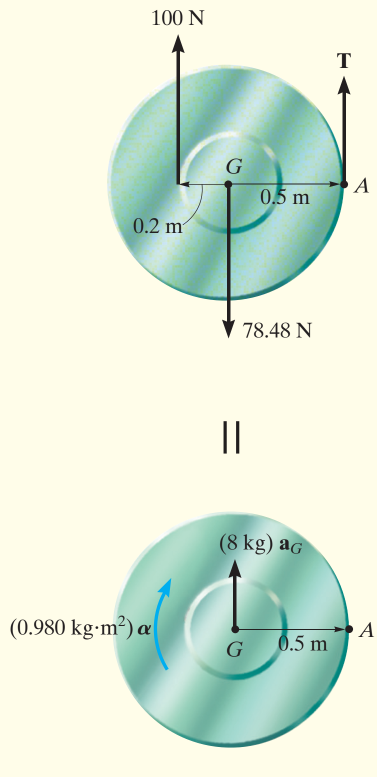
Using Eq. 3,

$$\alpha = 10.3\text{ rad/s}^2 \qquad \textit{Ans.}$$

SOLUTION III

Equations of Motion. The simplest way to solve this problem is to realize that point *A* is the *IC* for the spool. Then Eq. 17–19 applies.

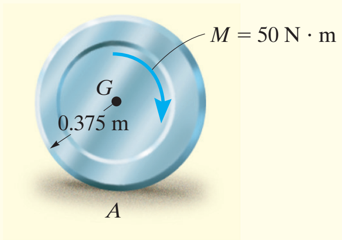
$$\circlearrowleft + \Sigma M_A = I_A\alpha; \quad (100\text{ N})(0.7\text{ m}) - (78.48\text{ N})(0.5\text{ m})$$
$$= [0.980\text{ kg}\cdot\text{m}^2 + (8\text{ kg})(0.5\text{ m})^2]\alpha$$
$$\alpha = 10.3\text{ rad/s}^2 \qquad \textit{Ans.}$$



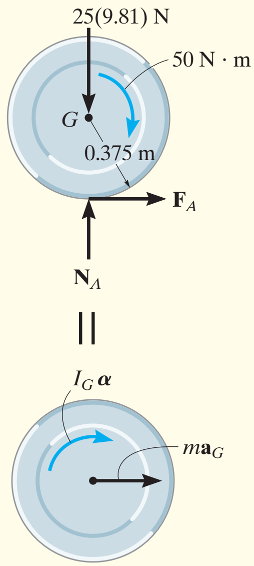
(b)

Fig. 17–20

EXAMPLE 17.13



(a)



(b)

Fig. 17–21

The 25-kg wheel shown in Fig. 17–21*a* has a radius of gyration $k_G = 0.21$ m. If a 50-N · m couple moment is applied to the wheel, determine the acceleration of its mass center G. The coefficients of static and kinetic friction between the wheel and the plane at A are $\mu_s = 0.3$ and $\mu_k = 0.25$, respectively.

SOLUTION

Free-Body and Kinetic Diagrams. By inspection of Fig. 17–21*b*, it is seen that the couple moment causes the wheel to have a clockwise angular acceleration of $\boldsymbol{\alpha}$. As a result, the acceleration of the mass center, $\mathbf{a}_G$, is directed to the right. The moment of inertia is

$$I_G = mk_G^2 = 25 \text{ kg}(0.21 \text{ m})^2 = 1.1025 \text{ kg} \cdot \text{m}^2$$

The unknowns are N_A, F_A, a_G, and α.

Equations of Motion.

$$\overset{+}{\rightarrow} \Sigma F_x = m(a_G)_x; \qquad F_A = (25\text{kg})a_G \qquad (1)$$

$$+\uparrow \Sigma F_y = m(a_G)_y; \qquad N_A - 25(9.81) \text{ N} = 0 \qquad (2)$$

$$\curvearrowright + \Sigma M_G = I_G\alpha; \quad 50 \text{ N} \cdot \text{m} - 0.375 \text{ m}(F_A) = (1.1025 \text{ kg} \cdot \text{m}^2)\alpha \qquad (3)$$

A fourth equation is needed for a complete solution.

Kinematics (No Slipping). If no slipping occurs, then

$$(\curvearrowright +) \qquad a_G = (0.375 \text{ m})\alpha \qquad (4)$$

Solving Eqs. 1 to 4,

$$N_A = 245 \text{ N} \qquad F_A = 102 \text{ N}$$
$$\alpha = 10.8 \text{ rad/s}^2 \qquad a_G = 4.06 \text{ m/s}^2$$

This solution requires $F_A \leq \mu_s N_A$. However, since $102 \text{ N} > 0.3(25)(9.81) \text{ N} = 73.6 \text{ N}$, the wheel slips as it rolls.

(Slipping). Equation 4 is not valid, and so $F_A = \mu_k N_A$, or

$$F_A = 0.25 N_A \qquad (5)$$

Solving Eqs. 1 to 3 and 5 yields

$$N_A = 245 \text{ N} \qquad F_A = 61.3 \text{ N}$$
$$\alpha = 24.5 \text{ rad/s}^2$$
$$a_G = 2.45 \text{ m/s}^2 \rightarrow$$

Ans.

EXAMPLE 17.14

The uniform slender pole shown in Fig. 17–22*a* has a mass of 100 kg. If the coefficients of static and kinetic friction between the end of the pole and the surface are $\mu_s = 0.3$ and $\mu_k = 0.25$, respectively, determine the pole's angular acceleration at the instant the 400-N horizontal force is applied. The pole is originally at rest.

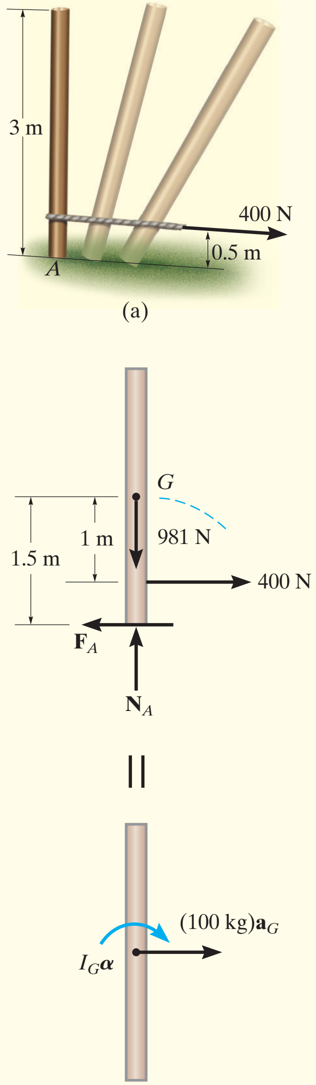


Fig. 17–22

SOLUTION

Free-Body and Kinetic Diagrams. Figure 17–22*b*. The path of motion of the mass center G will be along an unknown curved path having a radius of curvature ρ, which is initially on a vertical line. However, there is no normal or y component of acceleration since the pole is originally at rest, i.e., $\mathbf{v}_G = \mathbf{0}$, so that $(a_G)_y = v_G^2/\rho = 0$. We will assume the mass center accelerates to the right and that the pole has a clockwise angular acceleration of $\boldsymbol{\alpha}$. The unknowns are N_A, F_A, a_G, and α.

Equation of Motion.

$$\overset{+}{\rightarrow}\Sigma F_x = m(a_G)_x; \qquad 400\text{ N} - F_A = (100\text{ kg})a_G \qquad (1)$$

$$+\uparrow\Sigma F_y = m(a_G)_y; \qquad N_A - 981\text{ N} = 0 \qquad (2)$$

$$\curvearrowright +\Sigma M_G = I_G\alpha; \; F_A(1.5\text{ m}) - (400\text{ N})(1\text{ m}) = [\tfrac{1}{12}(100\text{ kg})(3\text{ m})^2]\alpha \qquad (3)$$

A fourth equation is needed for a complete solution.

Kinematics (No Slipping). With this assumption, point A acts as a "pivot" so that α is clockwise, then a_G is directed to the right.

$$a_G = \alpha r_{AG}; \qquad a_G = (1.5\text{ m})\,\alpha \qquad (4)$$

Solving Eqs. 1 to 4 yields

$$N_A = 981\text{ N} \qquad F_A = 300\text{ N}$$

$$a_G = 1\text{ m/s}^2 \qquad \alpha = 0.667\text{ rad/s}^2$$

The assumption of no slipping requires $F_A \le \mu_s N_A$. However, $300\text{ N} > 0.3(981\text{ N}) = 294\text{ N}$ and so the pole slips at A.

(Slipping). For this case Eq. 4 does *not* apply. Instead the frictional equation $F_A = \mu_k N_A$ must be used. Hence,

$$F_A = 0.25 N_A \qquad (5)$$

Solving Eqs. 1 to 3 and 5 simultaneously yields

$$N_A = 981\text{ N} \qquad F_A = 245\text{ N} \qquad a_G = 1.55\text{ m/s}^2$$

$$\alpha = -0.428\text{ rad/s}^2 = 0.428\text{ rad/s}^2 \curvearrowleft \qquad \textit{Ans.}$$

17

EXAMPLE 17.15

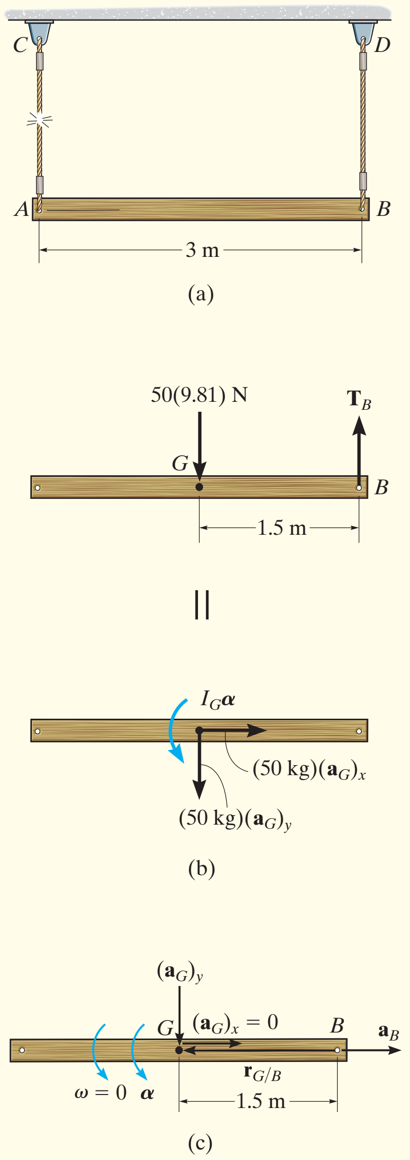


Fig. 17–23

The uniform 50-kg bar in Fig. 17–23*a* is held in the equilibrium position by cords *AC* and *BD*. Determine the tension in *BD* and the angular acceleration of the bar immediately after *AC* is cut.

SOLUTION

Free-Body and Kinetic Diagrams. Fig. 17–23*b*. There are four unknowns, T_B, $(a_G)_x$, $(a_G)_y$, and α.

Equations of Motion.

$$\overset{+}{\rightarrow}\Sigma F_x = m(a_G)_x; \qquad 0 = 50\text{ kg}\,(a_G)_x$$

$$(a_G)_x = 0$$

$$+\uparrow\Sigma F_y = m(a_G)_y; \qquad T_B - 50(9.81)\text{ N} = -50\text{ kg}\,(a_G)_y \tag{1}$$

$$\circlearrowright +\Sigma M_G = I_G\alpha; \qquad T_B(1.5\text{ m}) = \left[\frac{1}{12}(50\text{ kg})(3\text{ m})^2\right]\alpha \tag{2}$$

Kinematics. Since the bar is at rest just after the cable is cut, then its angular velocity and the velocity of point *B* at this instant are equal to zero. Thus $(a_B)_n = v_B^2/\rho_{BD} = 0$. Therefore, $\mathbf{a}_B$ only has a tangential component, which is directed along the *x* axis, Fig. 17–23*c*. Applying the relative acceleration equation to points *G* and *B*,

$$\mathbf{a}_G = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{G/B} - \omega^2\mathbf{r}_{G/B}$$

$$-(a_G)_y\mathbf{j} = a_B\mathbf{i} + (\alpha\mathbf{k}) \times (-1.5\mathbf{i}) - \mathbf{0}$$

$$-(a_G)_y\mathbf{j} = a_B\mathbf{i} - 1.5\alpha\mathbf{j}$$

Equating the **i** and **j** components of both sides of this equation,

$$0 = a_B$$

$$(a_G)_y = 1.5\alpha \tag{3}$$

Solving Eqs. 1 through 3 yields

$$\alpha = 4.905\text{ rad/s}^2 \qquad \textit{Ans.}$$

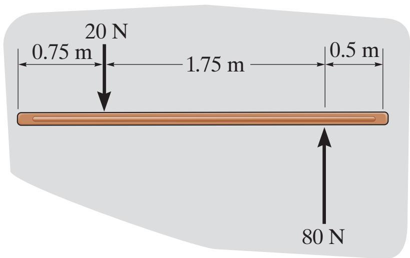
$$T_B = 123\text{ N} \qquad \textit{Ans.}$$

$$(a_G)_y = 7.36\text{ m/s}^2$$

Refer to the companion website for a self quiz of these Example problems.

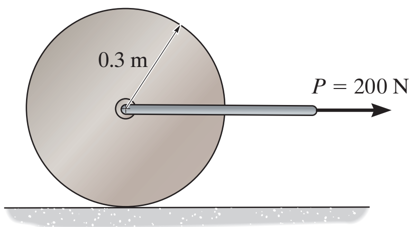
FUNDAMENTAL PROBLEMS

F17–13. The uniform 60-kg slender bar is initially at rest on a smooth *horizontal plane* when the forces are applied. Determine the acceleration of the bar's mass center and the angular acceleration of the bar at this instant.



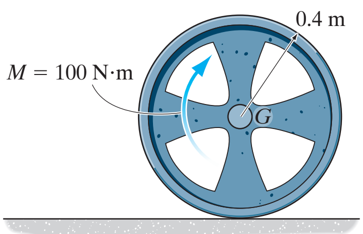
Prob. F17–13

F17–14. The 100-kg cylinder rolls without slipping on the horizontal plane. Determine the acceleration of its mass center and its angular acceleration.



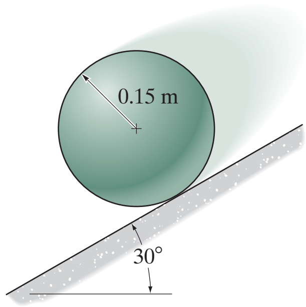
Prob. F17–14

F17–15. The 20-kg wheel has a radius of gyration about its center G of $k_G = 300$ mm. When the wheel is subjected to the couple moment, it slips as it rolls. Determine the angular acceleration of the wheel and the acceleration of G. The coefficient of kinetic friction between the wheel and the plane is $\mu_k = 0.5$.



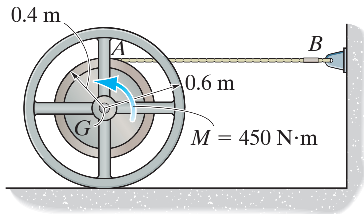
Prob. F17–15

F17–16. The 20-kg sphere rolls down the inclined plane without slipping. Determine the angular acceleration of the sphere and the acceleration of its mass center.



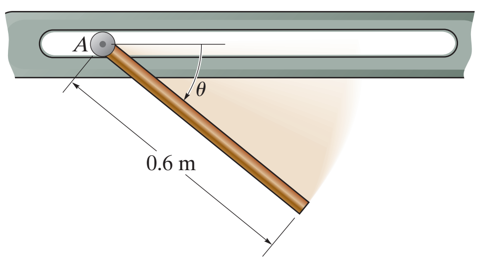
Prob. F17–16

F17–17. The 200-kg spool has a radius of gyration about its mass center of $k_G = 300$ mm. If the couple moment is applied to the spool and the coefficient of kinetic friction between the spool and the ground is $\mu_k = 0.2$, determine the angular acceleration of the spool, the acceleration of G, and the tension in the cable.



Prob. F17–17

F17–18. The 12-kg slender rod is pinned to a small roller A that slides freely along the slot. If the rod is released from rest at $\theta = 0°$, determine the angular acceleration of the rod and the acceleration of the roller immediately after the release.

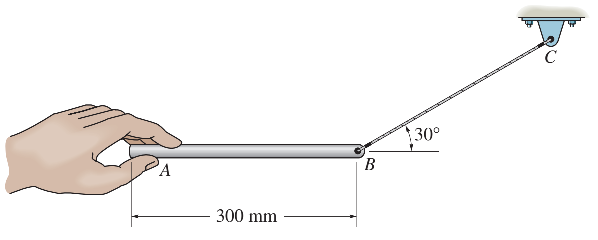


Prob. F17–18

PROBLEMS

All solutions must include a free-body diagram and a kinetic diagram.

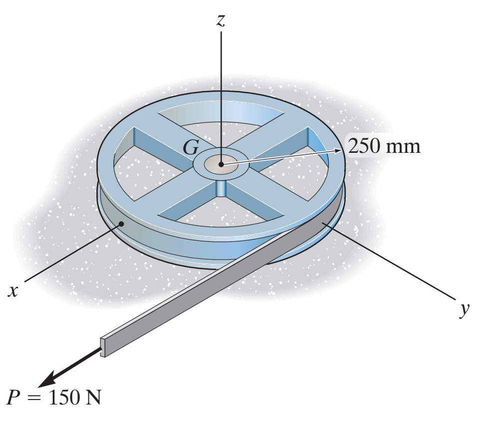
17–90. The 2-kg slender bar is supported by cord BC and then released from rest at A. Determine the initial angular acceleration of the bar and the tension in the cord.



Prob. 17–90

17–91. If the disk in Fig. 17–19 *rolls without slipping*, show that when moments are summed about the instantaneous center of zero velocity, IC, it is possible to use the moment equation $\Sigma M_{IC} = I_{IC}\,\alpha$, where I_{IC} represents the moment of inertia of the disk calculated about the instantaneous axis of zero velocity.

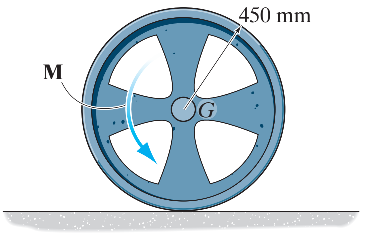
***17–92.** The 75-kg wheel has a radius of gyration about the z axis of $k_z = 150$ mm. If the belt of negligible mass is subjected to a force of $P = 150$ N, determine the acceleration of the mass center and the angular acceleration of the wheel. The surface is smooth and the wheel is free to slide.



Prob. 17–92

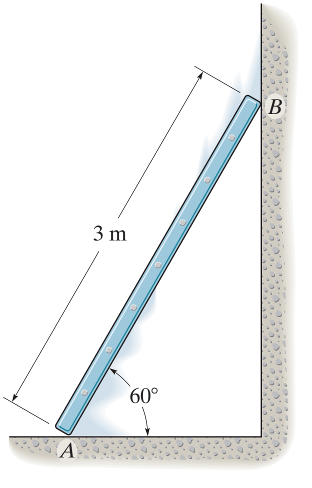
17–93. The 75-kg wheel has a radius of gyration about its mass center of $k_G = 375$ mm. If it is subjected to a torque of $M = 100\ \text{N}\cdot\text{m}$, determine its angular acceleration. The coefficients of static and kinetic friction between the wheel and the ground are $\mu_s = 0.2$ and $\mu_k = 0.15$, respectively.

17–94. The 75-kg wheel has a radius of gyration about its mass center of $k_G = 375$ mm. If it is subjected to a torque of $M = 150\ \text{N}\cdot\text{m}$, determine its angular acceleration. The coefficients of static and kinetic friction between the wheel and the ground are $\mu_s = 0.2$ and $\mu_k = 0.15$, respectively.



Probs. 17–93/94

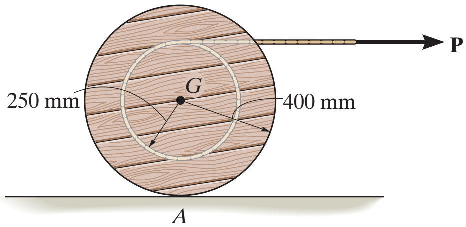
17–95. The slender 12-kg bar has a clockwise angular velocity of $\omega = 2$ rad/s when it is in the position shown. Determine its angular acceleration and the normal reactions of the smooth surface A and B at this instant.



Prob. 17–95

17

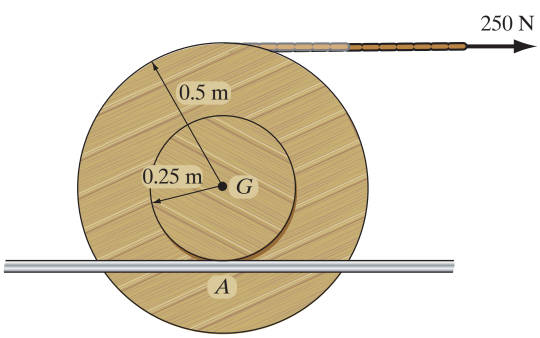
***17–96.** The spool has a mass of 100 kg and a radius of gyration $k_G = 0.3$ m. If the coefficients of static and kinetic friction at A are $\mu_s = 0.2$ and $\mu_k = 0.15$, respectively, and $P = 600$ N, determine the angular acceleration of the spool.



Prob. 17–96

17–97. The spool has a mass of 100 kg and radius of gyration about its mass center of $k_G = 0.3$ m. Determine its angular acceleration when the force of 250 N is applied to the cable. The spool rolls without slipping.

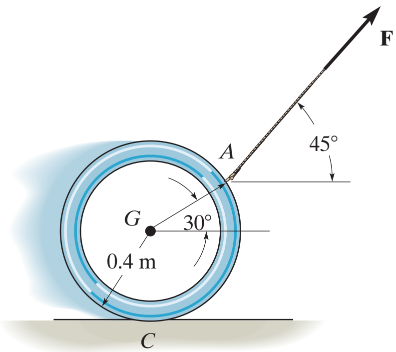
17–98. The spool has a mass of 100 kg and radius of gyration about its mass center of $k_G = 0.3$ m. Determine the smallest coefficient of static friction that will prevent slipping on the rail at A. What is the angular acceleration of the spool?



Probs. 17–97/98

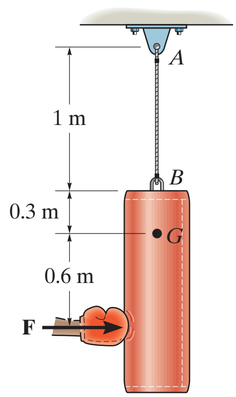
17–99. A force of $F = 10$ N is applied to the 10-kg ring as shown. If slipping does not occur, determine the ring's initial angular acceleration, and the acceleration of its mass center, G. Neglect the thickness of the ring.

***17–100.** If the coefficient of static friction at C is $\mu_s = 0.3$, determine the largest force F that can be applied to the 5-kg ring without causing it to slip. Neglect the thickness of the ring.



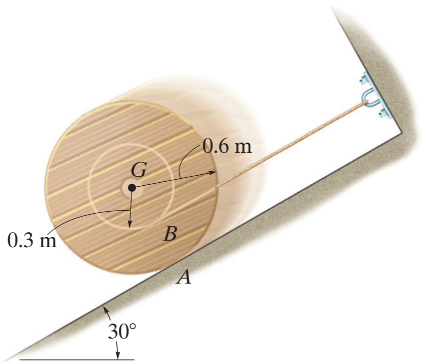
Probs. 17–99/100

17–101. The 20-kg punching bag has a radius of gyration about its center of mass G of $k_G = 0.4$ m. If it is initially at rest and is subjected to a horizontal force $F = 30$ N, determine the initial angular acceleration of the bag and the tension in the supporting cable AB.



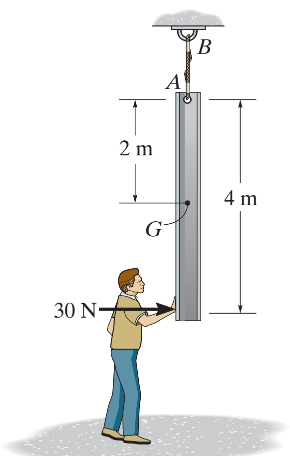
Prob. 17–101

17–102. The spool has a mass of 75 kg and a radius of gyration $k_G = 0.380$ m. It rests on the inclined surface for which the coefficient of kinetic friction is $\mu_k = 0.15$. If the spool is released from rest and slips at A, determine the initial tension in the cord and the angular acceleration of the spool.



Prob. 17–102

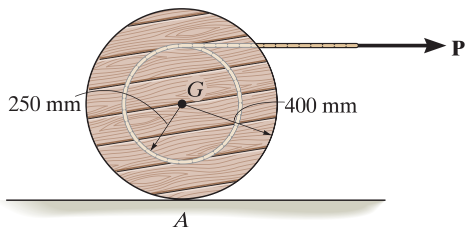
17–103. The slender, 200-kg beam is suspended by a cable at its end as shown. If a man pushes on its other end with a horizontal force of 30 N, determine the initial acceleration of its mass center G, the beam's angular acceleration, the tension in the cable AB, and the initial acceleration of the end A.



Prob. 17–103

***17–104.** The spool has a mass of 100 kg and a radius of gyration of $k_G = 0.3$ m. If the coefficients of static and kinetic friction at A are $\mu_s = 0.2$ and $\mu_k = 0.15$, respectively, determine the angular acceleration of the spool if $P = 50$ N.

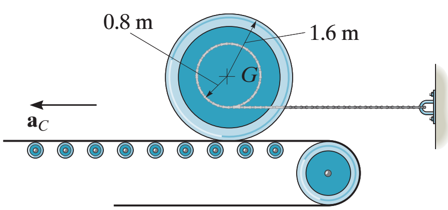
17–105. Solve Prob. 17–104 if the cord and force $P = 50$ N are directed vertically upwards.



Probs. 17–104/105

17–106. The spool has a mass of 500 kg and a radius of gyration $k_G = 1.30$ m. It rests on the surface of a conveyor belt for which the coefficient of static friction is $\mu_s = 0.5$ and the coefficient of kinetic friction is $\mu_k = 0.4$. If the conveyor accelerates at $a_C = 1\ \text{m/s}^2$, determine the initial tension in the wire and the angular acceleration of the spool. The spool is originally at rest.

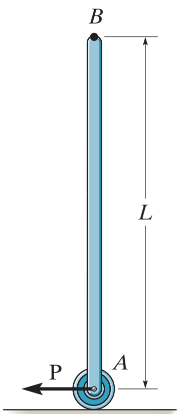
17–107. The spool has a mass of 500 kg and a radius of gyration $k_G = 1.30$ m. It rests on the surface of a conveyor belt for which the coefficient of static friction is $\mu_s = 0.5$. Determine the greatest acceleration a_C of the conveyor so that the spool will not slip. Also, what are the initial tension in the wire and the angular acceleration of the spool? The spool is originally at rest.



Probs. 17–106/107

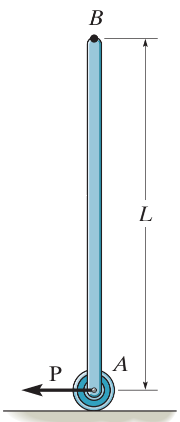
17

***17–108.** The uniform bar of mass m and length L is balanced in the vertical position when the horizontal force **P** is applied to the roller at A. Determine the bar's initial angular acceleration and the acceleration of its top point B.



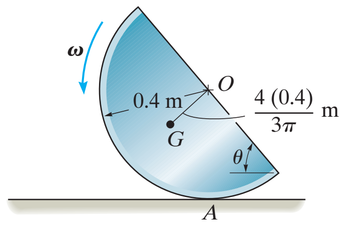
Prob. 17–108

17–109. Solve Prob. 17–108 if the roller is removed and the coefficient of kinetic friction at the ground is μ_k.



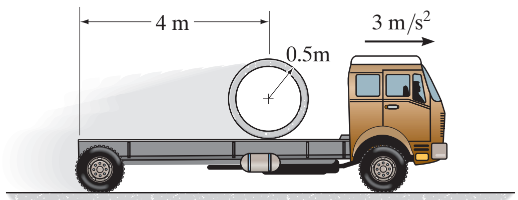
Prob. 17–109

17–110. The semicircular disk having a mass of 10 kg is rotating at $\omega = 4$ rad/s at the instant $\theta = 60°$. If the coefficient of static friction at A is $\mu_s = 0.5$, determine if the disk slips at this instant.



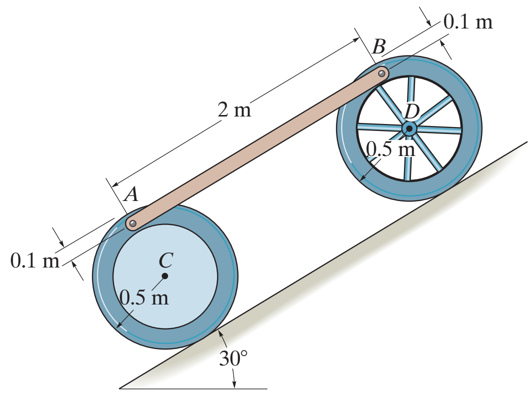
Prob. 17–110

17–111. The 500-kg concrete culvert has a mean radius of 0.5 m. If the truck has an acceleration of $3\,\text{m/s}^2$, determine the culvert's angular acceleration. Assume that the culvert does not slip on the truck bed, and neglect its thickness.



Prob. 17–111

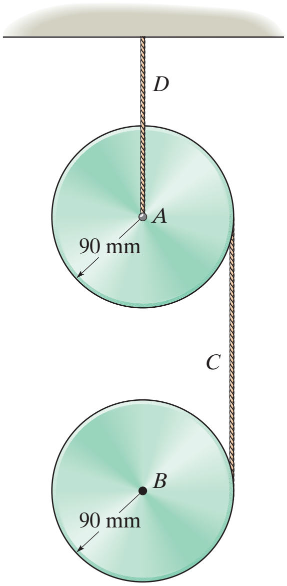
***17–112.** Wheel C has a mass of 60 kg and a radius of gyration of 0.4 m, whereas wheel D has a mass of 40 kg and a radius of gyration of 0.35 m. Determine the angular acceleration of each wheel at the instant shown. Neglect the mass of the link and assume the assembly does not slip on the plane.



Prob. 17–112

17

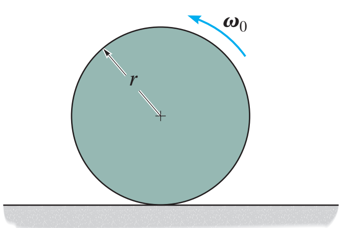
17–113. A cord is wrapped around each of the two 10-kg disks. If they are released from rest determine the angular acceleration of each disk and the tension in the cord C. Neglect the mass of the cord.



Prob. 17–113

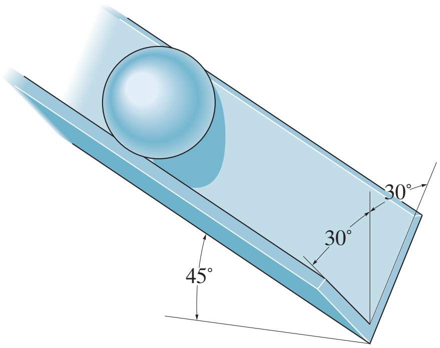
17–114. The uniform disk of mass m is rotating with an angular velocity of ω_0 when it is placed on the floor. Determine the initial angular acceleration of the disk and the acceleration of its mass center. The coefficient of kinetic friction between the disk and the floor is μ_k.

17–115. The uniform disk of mass m is rotating with an angular velocity of ω_0 when it is placed on the floor. Determine the time before it starts to roll without slipping. What is the angular velocity of the disk at this instant? The coefficient of kinetic friction between the disk and the floor is μ_k.



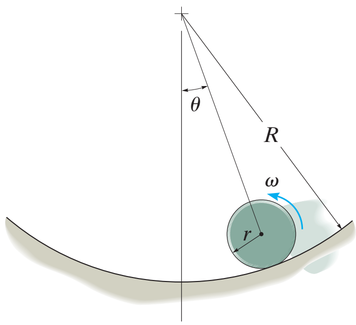
Probs. 17–114/115

***17–116.** The solid ball of radius r and mass m rolls without slipping down the 60° trough. Determine its angular acceleration.



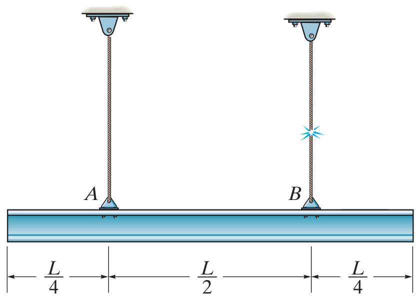
Prob. 17–116

17–117. The disk of mass m and radius r rolls without slipping on the circular path. Determine the normal force which the path exerts on the disk and the disk's angular acceleration if at the instant shown the disk has an angular velocity of $\boldsymbol{\omega}$.



Prob. 17–117

17–118. The uniform beam has a weight W. If it is originally at rest while being supported at A and B by cables, determine the tension in cable A if cable B suddenly fails. Assume the beam is a slender rod.



Prob. 17–118

17

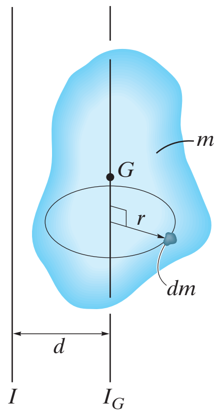
CHAPTER REVIEW

Moment of Inertia

The moment of inertia is a measure of the resistance of a body to a change in its angular velocity. It is defined by $I = \int r^2 dm$ and will be different for each axis about which it is calculated.

Many bodies are composed of simple shapes. If this is the case, then tabular values of I can be used, such as the ones given on the inside back cover of this book. To obtain the moment of inertia of a composite body about any specified axis, the moment of inertia of each part is determined about the axis and the results are added together. Doing this often requires use of the parallel-axis theorem.

$$I = I_G + md^2$$



Planar Equations of Motion

The equations of motion define the translational and rotational motion of a rigid body. In order to account for all of the terms in these equations, a free-body diagram should always accompany their application, and for some problems, it may also be convenient to draw the kinetic diagram which shows $m\mathbf{a}_G$ and $I_G\boldsymbol{\alpha}$.

$$\Sigma F_x = m(a_G)_x$$
$$\Sigma F_y = m(a_G)_y$$
$$\Sigma M_G = 0$$

Rectilinear translation

$$\Sigma F_n = m(a_G)_n$$
$$\Sigma F_t = m(a_G)_t$$
$$\Sigma M_G = 0$$

Curvilinear translation

$$\Sigma F_n = m(a_G)_n = m\omega^2 r_G$$
$$\Sigma F_t = m(a_G)_t = m\alpha r_G$$
$$\Sigma M_G = I_G\alpha \text{ or } \Sigma M_O = I_O\alpha$$

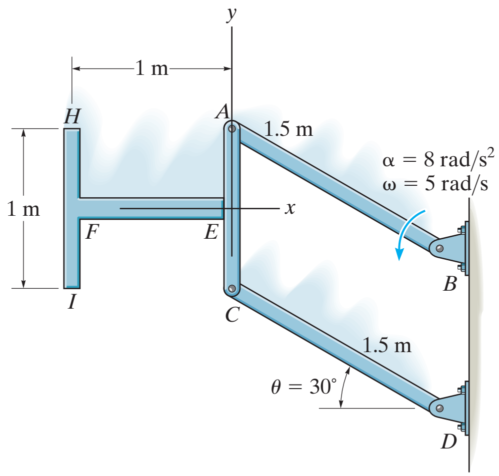
Rotation About a Fixed Axis

$$\Sigma F_x = m(a_G)_x$$
$$\Sigma F_y = m(a_G)_y$$
$$\Sigma M_G = I_G\alpha \text{ or } \Sigma M_P = \Sigma(\mathcal{M}_k)_P$$

General Plane Motion

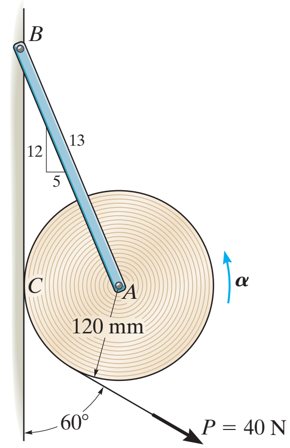
REVIEW PROBLEMS

R17–1. The two 1.5-kg rods EF and HI are fixed (welded) to the link AC at E. Determine the internal axial force E_x, shear force E_y, and moment M_E, which the bar AC exerts on FE at E if at the instant $\theta = 30°$ link AB has an angular velocity $\omega = 5$ rad/s and an angular acceleration $\alpha = 8$ rad/s^2 as shown.



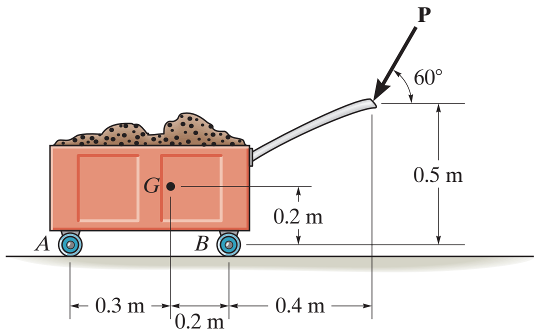
Prob. R17–1

R17–2. A 20-kg roll of paper, originally at rest, is pin supported at its ends to bracket AB. The roll rests against a wall for which the coefficient of kinetic friction at C is $\mu_C = 0.3$. If a force of 40 N is applied uniformly to the end of the sheet, determine the initial angular acceleration of the roll and the tension in the bracket as the paper unwraps. For the calculation, treat the roll as a cylinder.



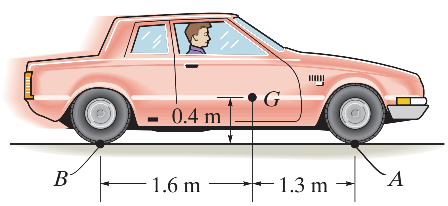
Prob. R17–2

R17–3. The handcart has a mass of 200 kg and center of mass at G. Determine the normal reactions at *each* of the wheels at A and B if a force $P = 50$ N is applied to the handle. Neglect the mass and rolling resistance of the wheels.



Prob. R17–3

R17–4. The car has a mass of 1.50 Mg and a mass center at G. Determine the maximum acceleration it can have if power is supplied only to the rear wheels. Neglect the mass of the wheels in the calculation, and assume that the wheels that do not receive power are free to roll. Also, assume that slipping of the powered wheels occurs, where the coefficient of kinetic friction is $\mu_k = 0.3$.



Prob. R17–4

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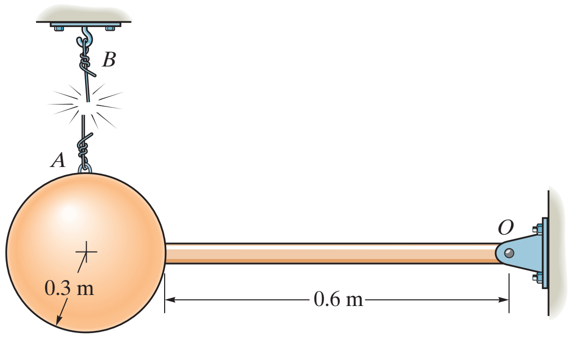
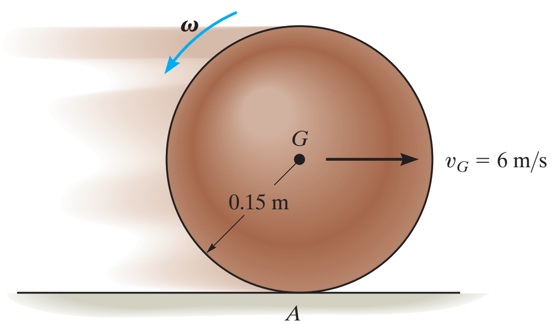
R17–5. The pendulum consists of a 15-kg sphere and a 5-kg slender rod. Compute the reaction at the pin O just after the cord AB is cut.



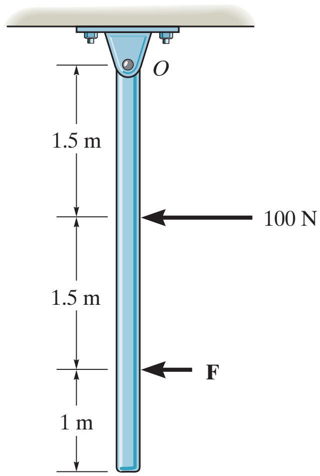
Prob. R17–5

R17–6. Determine the backspin $\boldsymbol{\omega}$ which should be given to the 9-kg ball so that when its center is given an initial horizontal velocity $v_G = 6$ m/s it stops spinning and translating at the same instant. The coefficient of kinetic friction is $\mu_A = 0.3$.



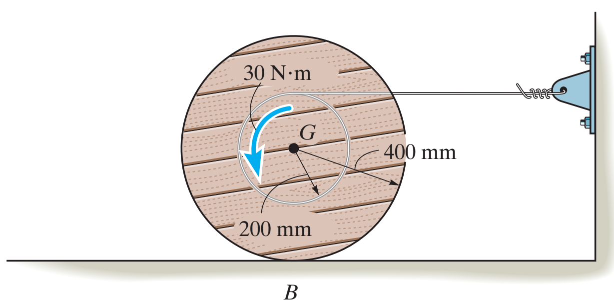
Prob. R17–6

R17–7. At the instant shown, two forces act on the 15-kg slender rod which is pinned at O. Determine the magnitude of force **F** and the initial angular acceleration of the rod so that the horizontal reaction which the *pin exerts on the rod* is 25 N directed to the right.



Prob. R17–7

R17–8. The spool and wire wrapped around its core have a mass of 20 kg and a centroidal radius of gyration $k_G = 250$ mm. If the coefficient of kinetic friction at the ground is $\mu_B = 0.1$, determine the angular acceleration of the spool when the 30-N · m couple moment is applied.



Prob. R17–8

CHAPTER 18

Roller coasters must be able to coast over loops and through turns, and have enough energy to do so safely. Accurate calculation of this energy must account for the size of the cars as they move along the track.

PLANAR KINETICS OF A RIGID BODY: WORK AND ENERGY

Lecture Summary and Quiz, Example, and Problem-solving videos are available where this icon appears.

CHAPTER OBJECTIVES

- To develop formulations for the kinetic energy of a body, and define the various ways a force and couple do work.
- To apply the principle of work and energy to solve rigid-body planar kinetic problems.
- To show how the conservation of energy can be used to solve rigid-body planar kinetic problems.

18.1 KINETIC ENERGY

In this chapter we will apply work and energy methods to solve planar motion problems involving force, velocity, and displacement. But first it will be necessary to develop a means of obtaining the body's kinetic energy when the body is subjected to translation, rotation about a fixed axis, or general plane motion.

To do this we will consider the rigid body shown in Fig. 18–1, which is represented here by a *slab* moving in the inertial x–y reference plane. An arbitrary ith particle of the body, having a mass dm, is located a distance r from the arbitrary point P. If at the *instant* shown the particle has a velocity $\mathbf{v}_i$, then the particle's kinetic energy is $T_i = \frac{1}{2}\, dm\, v_i^2$.

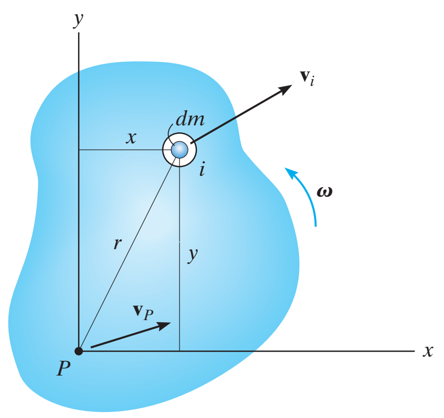


Fig. 18–1

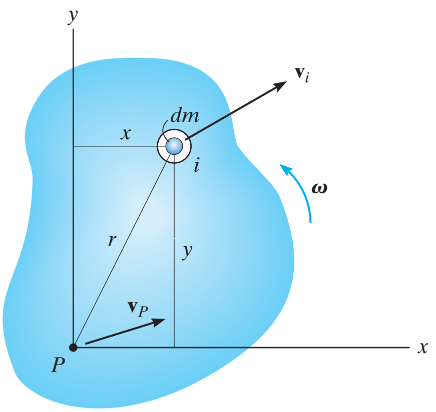


Fig. 18–1 (repeated)

The kinetic energy of the entire body is determined by writing similar expressions for each particle of the body and integrating the results, i.e.,

$$T = \frac{1}{2}\int_m dm\, v_i^2$$

This equation can also be expressed in terms of the velocity of a specific point P rather than the velocity v_i. If the body has an angular velocity $\boldsymbol{\omega}$, then from Fig. 18–1, we have

$$\begin{aligned}\mathbf{v}_i &= \mathbf{v}_P + \mathbf{v}_{i/P}\\ &= (v_P)_x\mathbf{i} + (v_P)_y\mathbf{j} + \omega\mathbf{k} \times (x\mathbf{i} + y\mathbf{j})\\ &= [(v_P)_x - \omega y]\mathbf{i} + [(v_P)_y + \omega x]\mathbf{j}\end{aligned}$$

The square of the magnitude of $\mathbf{v}_i$ is thus

$$\begin{aligned}\mathbf{v}_i \cdot \mathbf{v}_i = v_i^2 &= [(v_P)_x - \omega y]^2 + [(v_P)_y + \omega x]^2\\ &= (v_P)_x^2 - 2(v_P)_x\omega y + \omega^2 y^2 + (v_P)_y^2 + 2(v_P)_y\omega x + \omega^2 x^2\\ &= v_P^2 - 2(v_P)_x\omega y + 2(v_P)_y\omega x + \omega^2 r^2\end{aligned}$$

Substituting this into the equation of kinetic energy yields

$$T = \frac{1}{2}\left(\int_m dm\right)v_P^2 - (v_P)_x\omega\left(\int_m y\, dm\right) + (v_P)_y\omega\left(\int_m x\, dm\right) + \frac{1}{2}\omega^2\left(\int_m r^2\, dm\right)$$

The first integral on the right represents the entire mass m of the body. Since $\bar{y}m = \int y\, dm$ and $\bar{x}m = \int x\, dm$, the second and third integrals locate the body's center of mass G with respect to P. The last integral represents the body's moment of inertia I_P, calculated about the z axis passing through point P. Thus,

$$T = \tfrac{1}{2}mv_P^2 - (v_P)_x\omega\bar{y}m + (v_P)_y\omega\bar{x}m + \tfrac{1}{2}I_P\omega^2 \qquad (18\text{–}1)$$

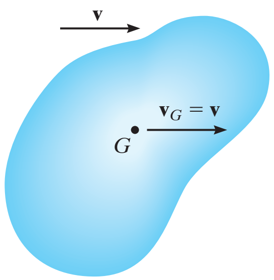
As a special case, if point P coincides with the mass center G of the body, then $\bar{y} = \bar{x} = 0$, and therefore

$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_G\omega^2 \qquad (18\text{–}2)$$

Both terms on the right side are *always positive*, since v_G and ω are squared. The first term represents the translational kinetic energy, referenced from the mass center, and the second term represents the body's rotational kinetic energy about the mass center.

Translation. When a rigid body of mass m is subjected to either rectilinear or curvilinear translation, Fig. 18–2, the kinetic energy due to rotation is zero, since $\boldsymbol{\omega} = \mathbf{0}$. The kinetic energy of the body is therefore

$$T = \tfrac{1}{2} m v_G^2 \qquad (18\text{–}3)$$



Translation

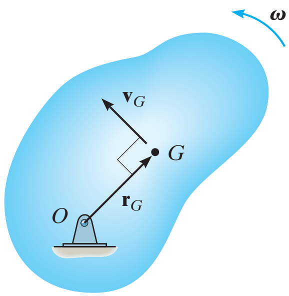
Fig. 18–2

Rotation about a Fixed Axis. When a rigid body rotates about a fixed axis passing through point O, Fig. 18–3, the body has both translational and rotational kinetic energy, so that

$$T = \tfrac{1}{2} m v_G^2 + \tfrac{1}{2} I_G \omega^2 \qquad (18\text{–}4)$$

The body's kinetic energy can also be formulated from its motion about point O. Since $v_G = r_G\omega$, then $T = \frac{1}{2}(I_G + m r_G^2)\omega^2$. By the parallel-axis theorem, the terms inside the parentheses represent the moment of inertia I_O of the body about an axis perpendicular to the plane of motion and passing through point O. Hence,*

$$T = \tfrac{1}{2} I_O \omega^2 \qquad (18\text{–}5)$$



Rotation About a Fixed Axis

Fig. 18–3

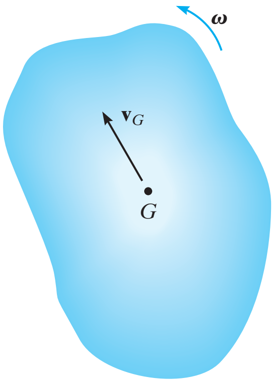
General Plane Motion. When a rigid body is subjected to general plane motion, Fig. 18–4, it has an angular velocity $\boldsymbol{\omega}$ and its mass center has a velocity $\mathbf{v}_G$. Therefore, the kinetic energy is

$$T = \tfrac{1}{2} m v_G^2 + \tfrac{1}{2} I_G \omega^2 \qquad (18\text{–}6)$$

We can also calculate this kinetic energy in terms of the body's motion about its instantaneous center of zero velocity, i.e.,

$$T = \tfrac{1}{2} I_{IC} \omega^2 \qquad (18\text{–}7)$$

where I_{IC} is the moment of inertia of the body about its instantaneous center. The proof is similar to that of Eq. 18–5. (See Prob. 18–3.)



General Plane Motion

Fig. 18–4

18

* The similarity between this derivation and that of $\Sigma M_O = I_O\alpha$ should be noted. Also the same result can be obtained directly from Eq. 18–1 by selecting point P at O, realizing that $v_O = 0$.

System of Bodies. Because energy is a scalar quantity, the total kinetic energy for a system of connected rigid bodies is the sum of the kinetic energies of all its moving parts. Depending on the type of motion, the kinetic energy of each body is found by applying Eq. 18–2 or one of its alternative forms.

18.2 THE WORK OF A FORCE

Several types of forces are often encountered in planar kinetics problems involving a rigid body. The work of each of these forces has been presented in Sec. 14.1 and is listed below as a summary.

Work of a Variable Force. If an external force **F** acts on a body, the work done by the force when the body moves along the path s, Fig. 18–5, is

$$U_F = \int \mathbf{F} \cdot d\mathbf{r} = \int_s F \cos\theta \, ds \qquad (18\text{–}8)$$

Here θ is the angle between the "tails" of the force and the differential displacement. The integration must account for the variation of the force's direction and magnitude.

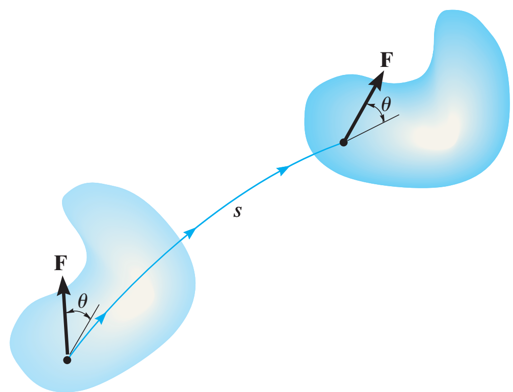


Fig. 18–5

Work of a Constant Force. If an external force $\mathbf{F}_c$ acts on a body, Fig. 18–6, and maintains a constant magnitude F_c and constant direction θ, while the body undergoes a translation s, then the above equation can be integrated, so that the work becomes

$$U_{F_c} = (F_c \cos\theta)s \qquad (18\text{–}9)$$

18

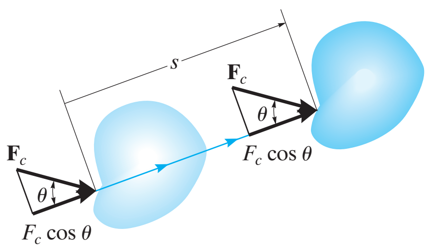


Fig. 18–6

Work of a Weight. The weight of a body does work only when the body's center of mass G undergoes a vertical displacement y. If this displacement is *downward*, Fig. 18–7, the work is positive, since the weight is in the same direction as the displacement.

$$U_W = Wy \tag{18–10}$$

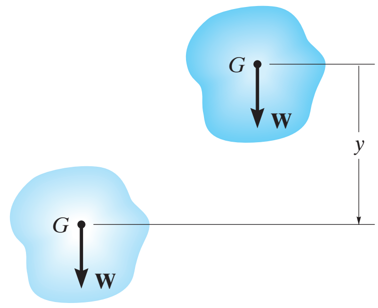


Fig. 18–7

Likewise, if the displacement is *upward* $(-y)$ the work becomes *negative*. In both cases the elevation change is considered to be small so that **W**, which is caused by gravitation, is constant.

Work of a Spring Force. If a linear elastic spring is attached to a body, the spring force $F_s = ks$ acting on the body does work when the spring either stretches or compresses from s_1 to a farther position s_2. In both cases the work will be negative since the displacement of the body is in the opposite direction to the force, Fig. 18–8. The work is

$$U_s = -\left(\tfrac{1}{2}ks_2^2 - \tfrac{1}{2}ks_1^2\right) \tag{18–11}$$

where $|s_2| > |s_1|$.

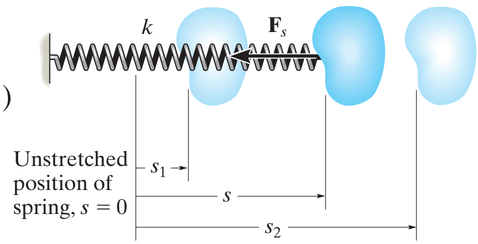


Fig. 18–8

Forces That Do No Work. There are some external forces that do no work when the body is displaced. These forces act either at fixed points on the body, or they have a direction that is perpendicular to their displacement. Examples include the reactions at a pin support about which a body rotates, the normal reaction acting on a body that moves along a fixed surface, and the weight of a body when the center of gravity of the body moves in a horizontal plane, Fig. 18–9. A frictional force $\mathbf{F}_f$ acting on a round body as it *rolls without slipping* over a rough surface also does no work.* This is because, during any instant of time dt, $\mathbf{F}_f$ acts at a point on the body which has zero velocity (instantaneous center, IC) and so the work done by the force on the point is zero. In other words, the point is not displaced in the direction of the force during this instant. Since $\mathbf{F}_f$ contacts successive points for only an instant, the work of $\mathbf{F}_f$ will be zero.

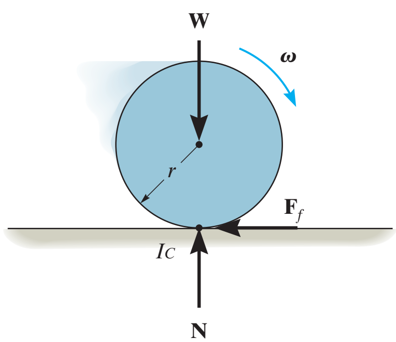


Fig. 18–9

* The work done by a frictional force *when the body slips* is discussed in Sec. 14.3.

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18.3 THE WORK OF A COUPLE MOMENT

Consider the body in Fig. 18–10*a*, which is subjected to a couple moment $M = Fr$. If the body undergoes a differential displacement, then the work done by the couple forces can be found by considering the displacement as the sum of a separate translation plus rotation. When the body translates, the work of each force is produced only by the component of displacement along the line of action of the forces ds_t, Fig. 18–10*b*. Clearly the "positive" work of one force cancels the "negative" work of the other. When the body undergoes a differential rotation $d\theta$ about the arbitrary point O, Fig. 18–10*c*, then each force undergoes a displacement $ds_\theta = (r/2)\,d\theta$ in the direction of the force. Hence, the total work done is

$$\begin{aligned} dU_M &= F\left(\frac{r}{2}d\theta\right) + F\left(\frac{r}{2}d\theta\right) = (Fr)\,d\theta \\ &= M\,d\theta \end{aligned}$$

The work is *positive* when **M** and $d\boldsymbol{\theta}$ have the same sense of direction and negative if these vectors are in the opposite sense.

When the body rotates in the plane through a finite angle θ measured in radians, from θ_1 to θ_2, the work of a couple moment is therefore

$$U_M = \int_{\theta_1}^{\theta_2} M\,d\theta \qquad (18\text{–}12)$$

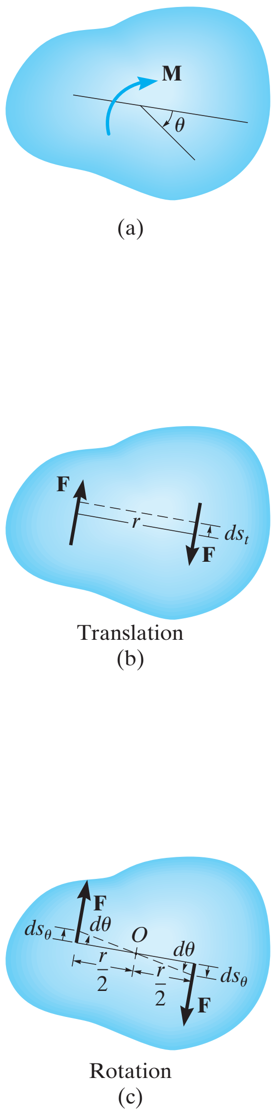


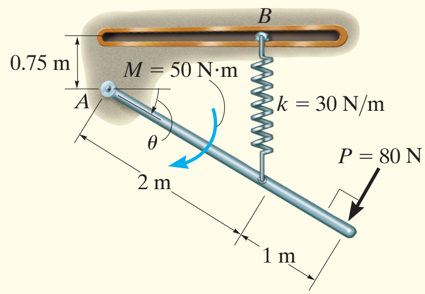
Fig. 18–10

If the couple moment **M** has a constant magnitude, then

$$U_M = M(\theta_2 - \theta_1) \qquad (18\text{–}13)$$

EXAMPLE 18.1

The bar shown in Fig. 18–11*a* has a mass of 10 kg and is subjected to a couple moment of $M = 50 \text{ N} \cdot \text{m}$, and a force of $P = 80$ N, which is always applied perpendicular to the end of the bar. Also, the spring has an unstretched length of 0.5 m and remains in the vertical position due to the roller guide at B. Determine the total work done by all the forces acting on the bar when it has rotated downward from $\theta = 0°$ to $\theta = 90°$.



(a)

SOLUTION

First the free-body diagram of the bar is drawn in order to account for all the forces that act on it, Fig. 18–11*b*.

Weight *W*. Since the weight $10(9.81) \text{ N} = 98.1 \text{ N}$ is displaced downward 1.5 m, the work is

$$U_W = 98.1 \text{ N}(1.5 \text{ m}) = 147.2 \text{ J}$$

Why is the work positive?

Couple Moment *M*. The couple moment rotates through an angle of $\theta = \pi/2$ rad. Hence,

$$U_M = 50 \text{ N} \cdot \text{m}(\pi/2) = 78.5 \text{ J}$$

Spring Force F_s. When $\theta = 0°$, the spring is stretched $(0.75 \text{ m} - 0.5 \text{ m}) = 0.25$ m, and when $\theta = 90°$, the stretch is $(2 \text{ m} + 0.75 \text{ m}) - 0.5 \text{ m} = 2.25$ m. Thus,

$$U_s = -\left[\tfrac{1}{2}(30 \text{ N/m})(2.25 \text{ m})^2 - \tfrac{1}{2}(30 \text{ N/m})(0.25 \text{ m})^2\right] = -75.0 \text{ J}$$

By inspection the spring does negative work on the bar since $\mathbf{F}_s$ acts in the opposite direction to displacement. This checks with the result.

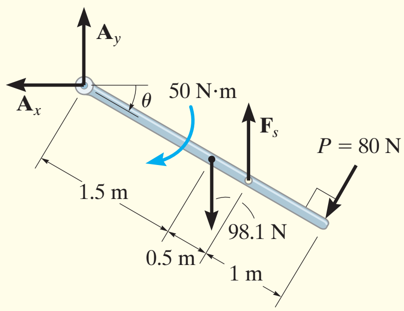
Force *P*. As the bar moves downward, the force is displaced through a distance of $(\pi/2)(3 \text{ m}) = 4.712$ m. The work is positive. Why?

$$U_P = 80 \text{ N}(4.712 \text{ m}) = 377.0 \text{ J}$$

Pin Reactions. Forces $\mathbf{A}_x$ and $\mathbf{A}_y$ do no work since they are not displaced.

Total Work. The work of all the forces when the bar is displaced is thus

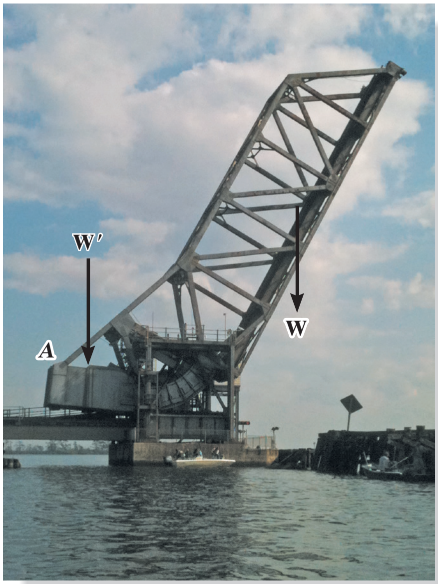
$$U = 147.2 \text{ J} + 78.5 \text{ J} - 75.0 \text{ J} + 377.0 \text{ J} = 528 \text{ J} \qquad \textit{Ans.}$$



(b)

Fig. 18–11

18



The counterweight A on this bascule bridge does positive work as the bridge is lifted, and thereby cancels the negative work done by the weight of the bridge.

18.4 PRINCIPLE OF WORK AND ENERGY

By applying the principle of work and energy, developed in Sec. 14.2, to each of the particles of a rigid body and adding the results algebraically, since energy is a scalar, we obtain a similar result for a rigid body.

$$T_1 + \Sigma U_{1-2} = T_2 \tag{18–14}$$

This equation states that the body's initial translational and rotational kinetic energy, plus the work done by all the external forces and couple moments acting on the body as the body moves from its initial to its final position, is equal to the body's final translational and rotational kinetic energy. The work of the body's internal forces does not have to be considered, since these forces occur in equal but opposite collinear pairs, so that when the body moves, the work of one force cancels that of its counterpart. Furthermore, since the body is rigid, no relative movement between these forces occurs, so that no internal work is done.

When several rigid bodies are pin connected, connected by inextensible cables, or in mesh with one another, Eq. 18–14 can also be applied to the entire system of connected bodies. In all these cases the internal forces, which hold the various members together, do no work and hence are eliminated from the analysis.

The work of the torque or moment developed by the motors is transformed into kinetic energy of rotation of the drum.

PROCEDURE FOR ANALYSIS

The principle of work and energy is used to solve kinetic problems that involve *velocity, force*, and *displacement*, since these terms are involved in the formulation. For application, it is suggested that the following procedure be used.

Kinetic Energy (Kinematic Diagrams).

- The kinetic energy of a body is made up of two parts. Kinetic energy of translation is referenced to the velocity of the mass center, $T = \frac{1}{2}mv_G^2$, and kinetic energy of rotation is determined using the moment of inertia of the body about the mass center, $T = \frac{1}{2}I_G\omega^2$. In the special case of rotation about a fixed axis (or rotation about the *IC*), the kinetic energy can be determined relative to motion about the axis of rotation, $T = \frac{1}{2}I_O\omega^2$.
- *Kinematic diagrams* for velocity may be useful for determining v_G and ω or for establishing a relationship between v_G and ω.

Work (Free-Body Diagram).

- Draw a free-body diagram of the body when it is located at an intermediate point along the path in order to account for all the forces and couple moments which do work on the body as it moves along the path.
- A force does work when it moves through a displacement in the direction of the force.
- Forces that are functions of displacement must be integrated to obtain the work. Graphically, the work is equal to the area under the force–displacement curve.
- The work of a weight is the product of its magnitude and the vertical displacement, $U_W = Wy$. It is positive when the weight moves downwards.
- The work of a spring is of the form $U_s = \frac{1}{2}ks^2$, where k is the spring stiffness and s is the stretch or compression of the spring.
- The work of a couple is the product of the couple moment and the angle in radians through which it rotates, $U_M = M\theta$.
- Since *algebraic addition* of the work terms is required, it is important that the proper sign of each term be specified. Specifically, work is *positive* when the force (couple moment) is in the *same direction* as its displacement (rotation); otherwise, it is negative.

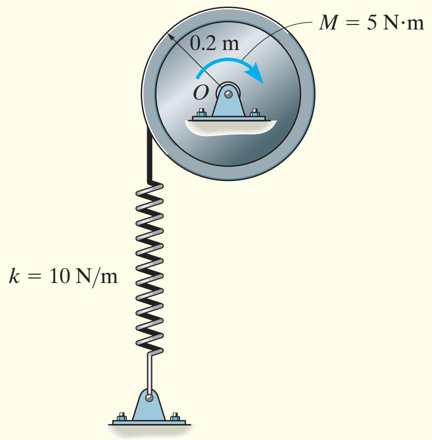
Principle of Work and Energy.

- Apply the principle of work and energy, $T_1 + \Sigma U_{1-2} = T_2$. Since this is a scalar equation, it can be used to solve for only one unknown when it is applied to a single rigid body.

Refer to the companion website for Lecture Summary and Quiz videos.

EXAMPLE 18.2

The 30-kg disk shown in Fig. 18–12*a* is pin supported at its center. Determine the angle through which it must rotate to attain an angular velocity of 2 rad/s starting from rest. It is acted upon by a constant couple moment $M = 5\ \text{N}\cdot\text{m}$. The spring is originally unstretched and its cord wraps around the rim of the disk.



(a)

SOLUTION

Kinetic Energy. Since the disk rotates about a fixed axis, and it is initially at rest, then

$$T_1 = 0$$

$$T_2 = \tfrac{1}{2}I_O\omega_2^2 = \tfrac{1}{2}\left[\tfrac{1}{2}(30\text{ kg})(0.2\text{ m})^2\right](2\text{ rad/s})^2 = 1.2\text{ J}$$

Work (Free-Body Diagram). As shown in Fig. 18–12*b*, the pin reactions $\mathbf{O}_x$ and $\mathbf{O}_y$ and the weight (294.3 N) do no work, since they are not displaced. The *couple moment*, having a constant magnitude, does positive work $U_M = M\theta$ as the disk *rotates* through a clockwise angle of θ rad, and the spring does negative work $U_s = -\tfrac{1}{2}ks^2$.

Principle of Work and Energy.

$$\{T_1\} + \{\Sigma U_{1-2}\} = \{T_2\}$$

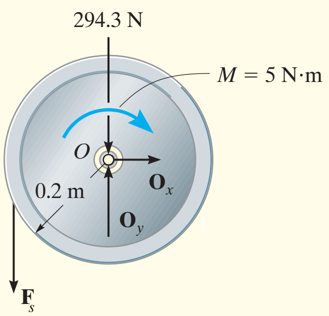
$$\{T_1\} + \left\{M\theta - \tfrac{1}{2}ks^2\right\} = \{T_2\}$$

$$\{0\} + \left\{(5\text{ N}\cdot\text{m})\theta - \frac{1}{2}(10\text{ N/m})[\theta(0.2\text{ m})]^2\right\} = \{1.2\text{ J}\}$$

$$-0.2\theta^2 + 5\theta - 1.2 = 0$$

Solving this quadratic equation for the smallest positive root,

$$\theta = 0.2423\text{ rad} = 0.2423\text{ rad}\left(\frac{180^\circ}{\pi\text{ rad}}\right) = 13.9^\circ \qquad \textit{Ans.}$$



(b)

Fig. 18–12

EXAMPLE 18.3

The wheel shown in Fig. 18–13*a* has a mass of 20 kg and has a radius of gyration $k_G = 0.18$ m about its mass center G. If it is subjected to a clockwise couple moment of $20\ \text{N}\cdot\text{m}$ and rolls from rest without slipping, determine its angular velocity after its center G moves 0.15 m. The spring has a stiffness $k = 150$ N/m and is initially unstretched when the couple moment is applied.

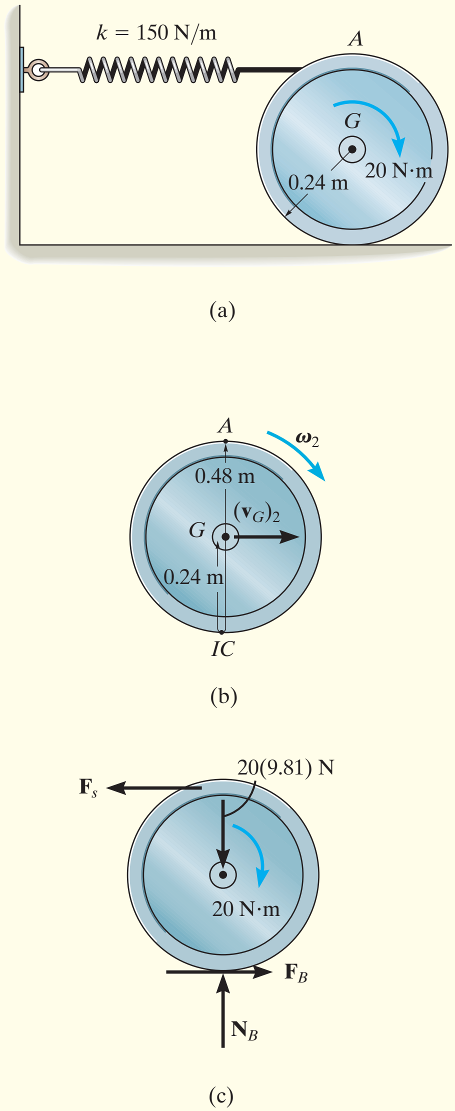


Fig. 18–13

SOLUTION

Kinetic Energy (Kinematic Diagram). Since the wheel is initially at rest,

$$T_1 = 0$$

The kinematic diagram of the wheel when it is in the final position is shown in Fig. 18–13*b*. The final kinetic energy is determined from

$$T_2 = \tfrac{1}{2} I_{IC}\omega_2^2$$

$$= \frac{1}{2}\left[20\ \text{kg}(0.18\ \text{m})^2 + (20\ \text{kg})(0.24\ \text{m})^2\right]\omega_2^2$$

$$T_2 = 0.9\ \omega_2^2$$

Work (Free-Body Diagram). As shown in Fig. 18–13*c*, only the spring force $\mathbf{F}_s$ and the couple moment do work. The normal force does not move along its line of action and the frictional force does *no work*, since the wheel does not slip as it rolls.

The work of $\mathbf{F}_s$ is found using $U_s = -\frac{1}{2}ks^2$. Here the work is negative since $\mathbf{F}_s$ is in the opposite direction to displacement. Since the wheel does not slip when the center G moves 0.15 m, then the wheel rotates $\theta = s_G/r_{G/IC} = 0.15\ \text{m}/0.24\ \text{m} = 0.625$ rad,* Fig. 18–13*b*. Hence, the spring stretches $s = \theta r_{A/IC} = (0.625\ \text{rad})(0.48\ \text{m}) = 0.3$ m.

Principle of Work and Energy.

$$\{T_1\} + \{\Sigma U_{1-2}\} = \{T_2\}$$

$$\{T_1\} + \left\{M\theta - \tfrac{1}{2}ks^2\right\} = \{T_2\}$$

$$\{0\} + \left\{20\ \text{N}\cdot\text{m}(0.625\ \text{rad}) - \frac{1}{2}(150\ \text{N/m})(0.3\ \text{m})^2\right\} = \{0.9\ \omega_2^2\ \text{N}\cdot\text{m}\}$$

$$\omega_2 = 2.53\ \text{rad/s} \curvearrowright$$ *Ans.*

*See Example 16.4.

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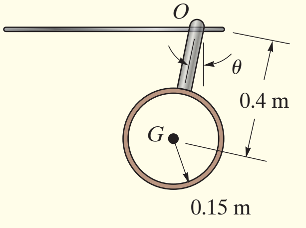
EXAMPLE 18.4

The 700-kg pipe is equally suspended from the two tines of the fork lift shown in the photo. It is undergoing a swinging motion such that when $\theta = 30°$ it is momentarily at rest. Determine the normal and frictional forces acting on each tine which are needed to support the pipe at the instant $\theta = 0°$. Measurements of the pipe and the suspender are shown in Fig. 18–14*a*. Neglect the mass of the suspender and the thickness of the pipe.



(a)

Fig. 18–14

SOLUTION

We must use the equations of motion to find the forces on the tines since these forces do no work. Before doing this, however, we will apply the principle of work and energy to determine the angular velocity of the pipe when $\theta = 0°$.

Kinetic Energy (Kinematic Diagram). Since the pipe is originally at rest, then

$$T_1 = 0$$

The final kinetic energy may be calculated with reference to either the fixed point O or the center of mass G. For the calculation we will consider the pipe to be a thin ring so that $I_G = mr^2$. (See the table on the inside back cover.) If point G is considered, we have

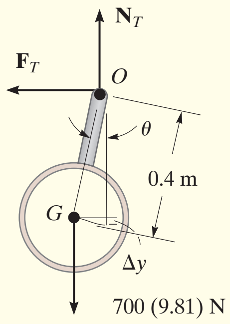
$$\begin{aligned} T_2 &= \tfrac{1}{2}m(v_G)_2^2 + \tfrac{1}{2}I_G\omega_2^2 \\ &= \tfrac{1}{2}(700\text{ kg})[(0.4\text{ m})\omega_2]^2 + \tfrac{1}{2}[700\text{ kg}(0.15\text{ m})^2]\omega_2^2 \\ &= 63.875\omega_2^2 \end{aligned}$$

If point O is considered then the parallel-axis theorem must be used to determine I_O. Hence,

$$\begin{aligned} T_2 = \tfrac{1}{2}I_O\omega_2^2 &= \tfrac{1}{2}[700\text{ kg}(0.15\text{ m})^2 + 700\text{ kg}(0.4\text{ m})^2]\omega_2^2 \\ &= 63.875\omega_2^2 \end{aligned}$$

18

Work (Free-Body Diagram). Fig. 18–14*b*. The normal and frictional forces on the tines do no work since they do not move as the pipe swings. The weight does positive work since the weight moves downward through a vertical distance $y = 0.4\text{ m} - 0.4\cos 30^\circ\text{ m} = 0.05359\text{ m}$.



(b)

Principle of Work and Energy.

$$\{T_1\} + \{\Sigma U_{1-2}\} = \{T_2\}$$

$$\{0\} + \{700(9.81)\text{ N}(0.05359\text{ m})\} = \{63.875\omega_2^2\}$$

$$\omega_2 = 2.400\text{ rad/s}$$

Equations of Motion. Referring to the free-body and kinetic diagrams shown in Fig. 18–14*c*, and using the result for ω_2, we have

$$\overset{+}{\leftarrow} \Sigma F_t = m(a_G)_t; \quad F_T = (700\text{ kg})(a_G)_t$$

$$+\uparrow \Sigma F_n = m(a_G)_n; \quad N_T - 700(9.81)\text{ N} = (700\text{ kg})(2.400\text{ rad/s})^2(0.4\text{ m})$$

$$\circlearrowleft + \Sigma M_O = I_O\alpha; \quad 0 = [(700\text{ kg})(0.15\text{ m})^2 + (700\text{ kg})(0.4\text{ m})^2]\alpha$$

Since $(a_G)_t = (0.4\text{ m})\alpha$, then

$$\alpha = 0 \quad (a_G)_t = 0$$

$$F_T = 0$$

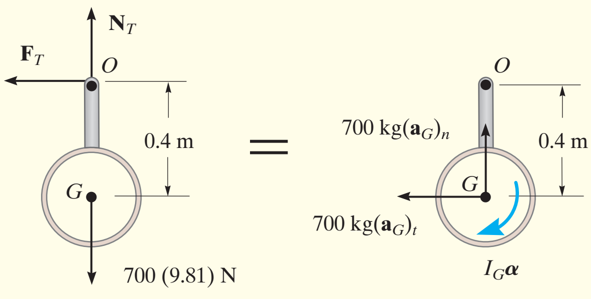
$$N_T = 8.480\text{ kN}$$

There are two tines used to support the load, therefore

$$F'_T = 0 \qquad \textit{Ans.}$$

$$N'_T = \frac{8.480\text{ kN}}{2} = 4.24\text{ kN} \qquad \textit{Ans.}$$

NOTE: Due to the swinging motion the tines are subjected to a *greater* normal force than would be the case if the load were static, in which case $N'_T = 700(9.81)\text{ N}/2 = 3.43\text{ kN}$.



(c)

Fig. 18–14 (cont.)

18

EXAMPLE 18.5

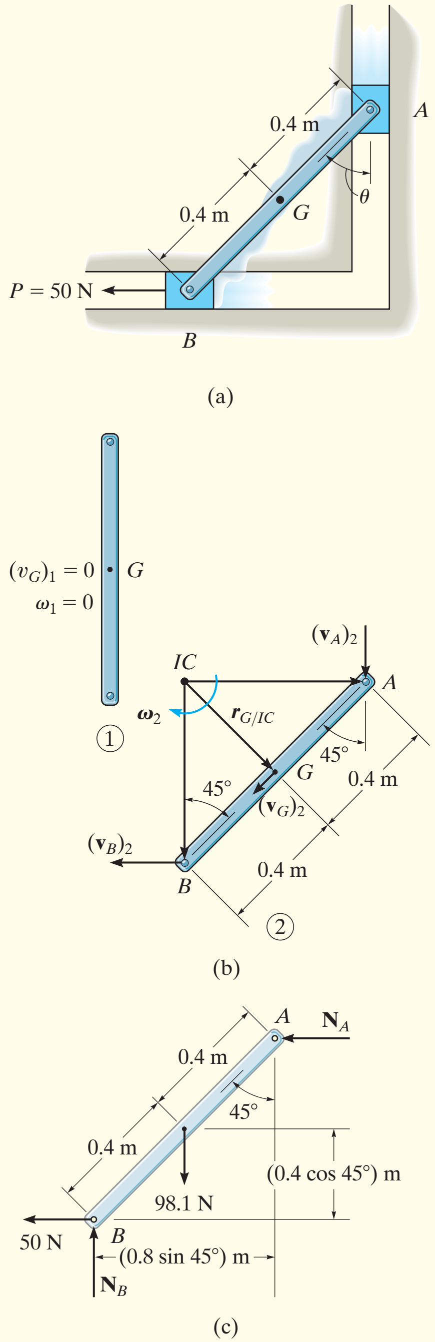


Fig. 18–15

The 10-kg rod shown in Fig. 18–15*a* is constrained so that its ends move along the grooved slots. The rod is initially at rest when $\theta = 0°$. If the slider block at B is subjected to the force $P = 50$ N, determine the angular velocity of the rod at the instant $\theta = 45°$. Neglect friction and the mass of blocks A and B.

SOLUTION

Why can the principle of work and energy be used to solve this problem?

Kinetic Energy (Kinematic Diagrams). Two kinematic diagrams of the rod, when it is in the initial position 1 and final position 2, are shown in Fig. 18–15*b*. When the rod is in position 1, $T_1 = 0$ since $(\mathbf{v}_G)_1 = \boldsymbol{\omega}_1 = \mathbf{0}$. In position 2 the angular velocity is $\boldsymbol{\omega}_2$ and the velocity of the mass center is $(\mathbf{v}_G)_2$. Hence, the kinetic energy is

$$T_2 = \tfrac{1}{2}m(v_G)_2^2 + \tfrac{1}{2}I_G\omega_2^2$$
$$= \tfrac{1}{2}(10\text{ kg})(v_G)_2^2 + \tfrac{1}{2}\left[\tfrac{1}{12}(10\text{ kg})(0.8\text{ m})^2\right]\omega_2^2$$
$$= 5(v_G)_2^2 + 0.2667(\omega_2)^2$$

The two unknowns $(v_G)_2$ and ω_2 can be related from the instantaneous center of zero velocity for the rod. Fig. 18–15*b*. It is seen that as A moves downward with a velocity $(\mathbf{v}_A)_2$, B moves horizontally to the left with a velocity $(\mathbf{v}_B)_2$. Knowing these directions, the IC is located as shown in the figure. Hence,

$$(v_G)_2 = r_{G/IC}\omega_2 = (0.4\tan 45°\text{ m})\omega_2$$
$$= 0.4\omega_2$$

Therefore,

$$T_2 = 0.8\omega_2^2 + 0.2667\omega_2^2 = 1.0667\omega_2^2$$

Of course, we can also determine this result using $T_2 = \tfrac{1}{2}I_{IC}\omega_2^2$.

Work (Free-Body Diagram). Fig. 18–15*c*. The normal forces $\mathbf{N}_A$ and $\mathbf{N}_B$ do no work as the rod is displaced. Why? The 98.1-N weight is displaced a vertical distance of $\Delta y = (0.4 - 0.4\cos 45°)$ m; whereas the 50-N force moves a horizontal distance of $s = (0.8\sin 45°)$ m. Both of these forces do positive work. Why?

Principle of Work and Energy.

$$\{T_1\} + \{\Sigma U_{1-2}\} = \{T_2\}$$
$$\{T_1\} + \{W\,\Delta y + Ps\} = \{T_2\}$$
$$\{0\} + \{98.1\text{ N}(0.4\text{ m} - 0.4\cos 45°\text{ m})$$
$$+ 50\text{ N}(0.8\sin 45°\text{ m})\} = \{1.0667\omega_2^2\text{ J}\}$$

Solving for ω_2 gives

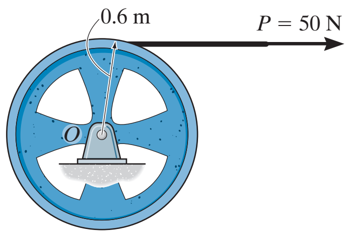
$$\omega_2 = 6.11\text{ rad/s} \circlearrowright \qquad \textit{Ans.}$$

Refer to the companion website for a self quiz of these Example problems.

18

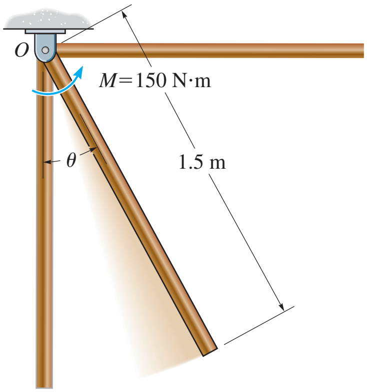
FUNDAMENTAL PROBLEMS

F18–1. The 80-kg wheel has a radius of gyration about its mass center O of $k_O = 400$ mm. Determine its angular velocity after it has rotated 20 revolutions starting from rest.



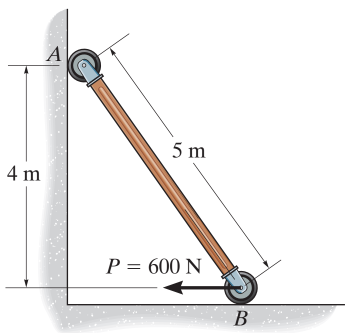
Prob. F18–1

F18–2. The uniform 25-kg slender rod is subjected to a couple moment of $M = 150\ \text{N}\cdot\text{m}$. If the rod is at rest when $\theta = 0°$, determine its angular velocity when $\theta = 90°$.



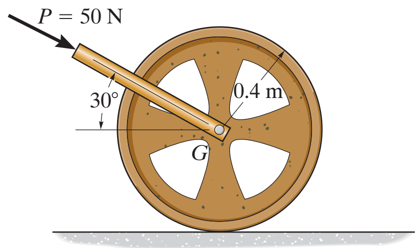
Prob. F18–2

F18–3. The uniform 50-kg slender rod is at rest in the position shown when $P = 600$ N is applied. Determine the angular velocity of the rod when the rod reaches the vertical position.



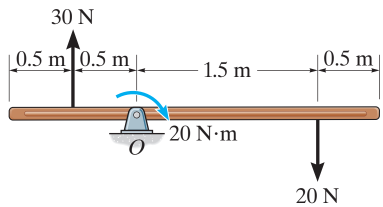
Prob. F18–3

F18–4. The 50-kg wheel is subjected to a force of 50 N. If the wheel starts from rest and rolls without slipping, determine its angular velocity after it has rotated 10 revolutions. The radius of gyration of the wheel about its mass center G is $k_G = 0.3$ m.



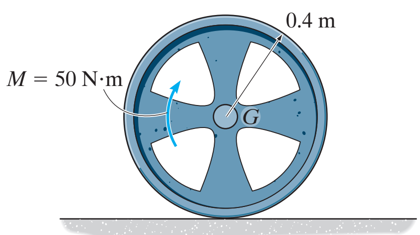
Prob. F18–4

F18–5. If the uniform 30-kg slender rod starts from rest at the position shown, determine its angular velocity after it has rotated 4 revolutions. The forces remain perpendicular to the rod.



Prob. F18–5

F18–6. The 20-kg wheel has a radius of gyration about its center G of $k_G = 300$ mm. When it is subjected to a couple moment of $M = 50\ \text{N}\cdot\text{m}$, it rolls without slipping. Determine the angular velocity of the wheel after its mass center G has traveled through a distance of $s_G = 20$ m, starting from rest.



Prob. F18–6

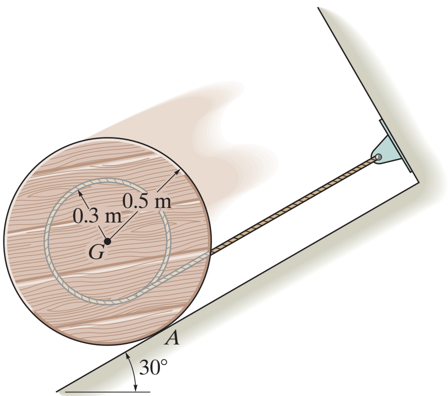
18

PROBLEMS

All solutions must include a free-body diagram.

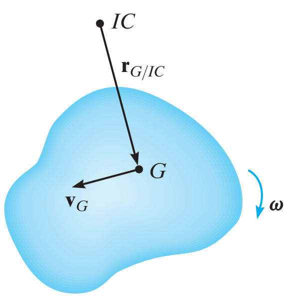
18–1. The spool has a mass of 60 kg and a radius of gyration $k_G = 0.3$ m. If it is released from rest, determine how far its center descends down the smooth plane before it attains an angular velocity of $\omega = 6$ rad/s. Neglect friction and the mass of the cord which is wound around the central core.

18–2. Solve Prob. 18–1 if the coefficient of kinetic friction between the spool and plane at A is $\mu_k = 0.2$.



Probs. 18–1/2

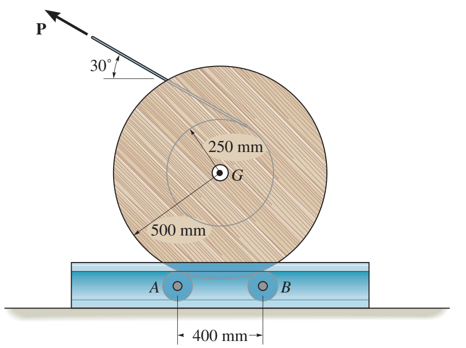
18–3. At a given instant the body of mass m has an angular velocity $\boldsymbol{\omega}$ and its mass center has a velocity $\mathbf{v}_G$. Show that its kinetic energy can be represented as $T = \frac{1}{2}I_{IC}\omega^2$, where I_{IC} is the moment of inertia of the body determined about the instantaneous axis of zero velocity, located a distance $r_{G/IC}$ from the mass center as shown.



Prob. 18–3

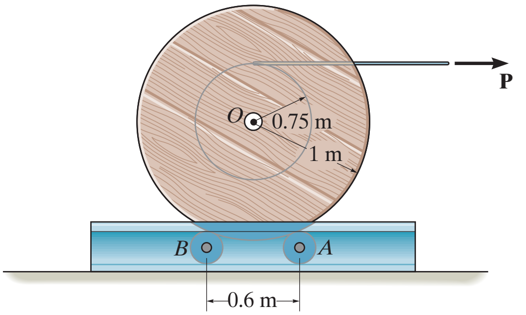
***18–4.** A force of $P = 20$ N is applied to the cable, which causes the 175-kg reel to turn since it is resting on the two rollers A and B of the dispenser. Determine the angular velocity of the reel after it has made two revolutions starting from rest. Neglect the mass of the rollers and the mass of the cable. The radius of gyration of the reel about its center axis is $k_G = 0.42$ m.

18–5. A force of $P = 20$ N is applied to the cable, which causes the 175-kg reel to turn without slipping on the two rollers A and B of the dispenser. Determine the angular velocity of the reel after it has made two revolutions starting from rest. Neglect the mass of the cable. Each roller can be considered as an 18-kg cylinder, having a radius of 0.1 m. The radius of gyration of the reel about its center axis is $k_G = 0.42$ m.



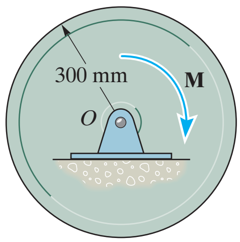
Probs. 18–4/5

18–6. A force of $P = 60$ N is applied to the cable, which causes the 200-kg reel to turn since it is resting on the two rollers A and B of the dispenser. Determine the angular velocity of the reel after it has made two revolutions starting from rest. Neglect the mass of the rollers and the mass of the cable. Assume the radius of gyration of the reel about its center axis remains constant at $k_O = 0.6$ m.



Prob. 18–6

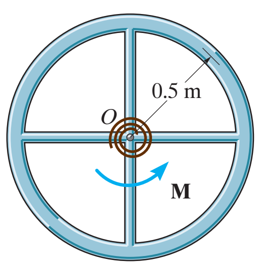
18–7. The disk, which has a mass of 20 kg, is subjected to the couple moment of $M = (2\theta + 4)\ \text{N}\cdot\text{m}$, where θ is in radians. If it starts from rest, determine its angular velocity when it has made two revolutions.



Prob. 18–7

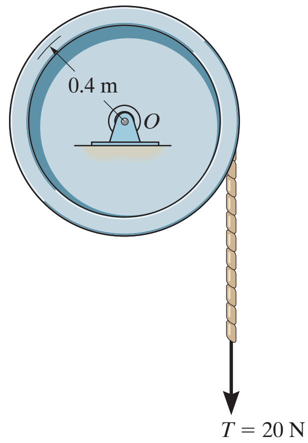
***18–8.** The wheel is made from a 5-kg thin ring and two 2-kg slender rods. If the torsional spring attached to the wheel's center has a stiffness $k = 2\ \text{N}\cdot\text{m/rad}$, and the wheel is rotated until the torque $M = 25\ \text{N}\cdot\text{m}$ is developed, determine the maximum angular velocity of the wheel if it is released from rest.

18–9. The wheel is made from a 5-kg thin ring and two 2-kg slender rods. If the torsional spring attached to the wheel's center has a stiffness $k = 2\ \text{N}\cdot\text{m/rad}$, so that the torque on the center of the wheel is $M = (2\theta)\ \text{N}\cdot\text{m}$, where θ is in radians, determine the maximum angular velocity of the wheel if it is rotated two revolutions and then released from rest.



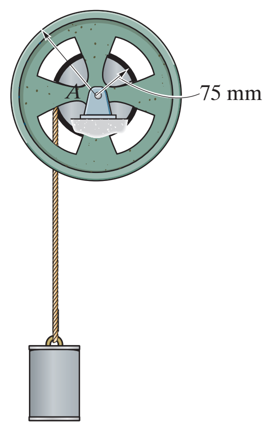
Probs. 18–8/9

18–10. The force of $T = 20$ N is applied to the cord of negligible mass. Determine the angular velocity of the 20-kg wheel when it has rotated 4 revolutions starting from rest. The wheel has a radius of gyration of $k_O = 0.3$ m.



Prob. 18–10

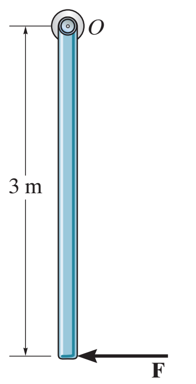
18–11. Determine the velocity of the 50-kg cylinder after it has descended a distance of 2 m. Initially, the system is at rest. The reel has a mass of 25 kg and a radius of gyration about its center of mass A of $k_A = 125$ mm.



Prob. 18–11

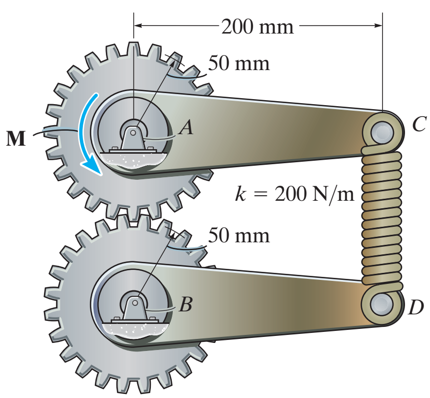
***18–12.** The 10-kg uniform slender rod is suspended at rest from the pin at O when the force of $F = 150$ N is applied to its end. Determine the angular velocity of the rod when it has rotated 90° clockwise from the position shown. The force is always perpendicular to the rod.

18–13. The 10-kg uniform slender rod is suspended at rest from the pin at O when the force of $F = 150$ N is applied to its end. Determine the angular velocity of the rod when it has rotated 180° clockwise from the position shown. The force is always perpendicular to the rod.



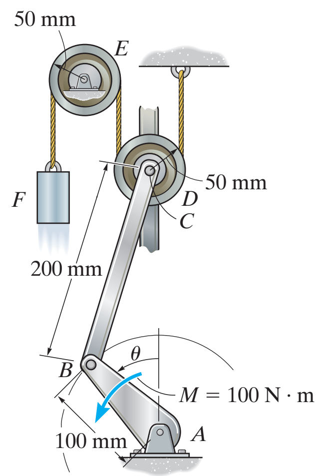
Probs. 18–12/13

18–14. Each gear has mass of 2 kg and a radius of gyration about its pinned mass center A or B, respectively, of $k_g = 40$ mm. Each link has a mass of 2 kg and a radius of gyration about its pinned end, A or B, of $k_l = 50$ mm. If originally the spring is unstretched when the couple moment $M = 20\ \text{N}\cdot\text{m}$ is applied to link AC, determine the angular velocity of the links at the instant link AC rotates $\theta = 45°$. Each gear and link are connected together and rotate about the fixed pins A and B.



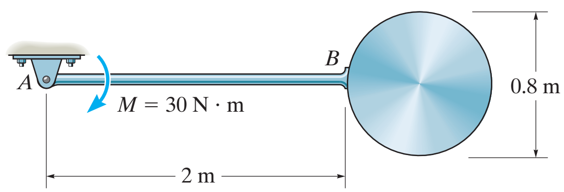
Prob. 18–14

18–15. The system is released from rest at $\theta = 0°$ when a constant couple moment $M = 100\ \text{N}\cdot\text{m}$ is applied. If the mass of the cable and links AB and BC can be neglected, and each pulley can be treated as a disk having a mass of 6 kg, determine the speed of the 10-kg block at the instant link AB has rotated $\theta = 90°$. Note that point C moves along the vertical guide. Also, the cable does not slip on the pulleys.



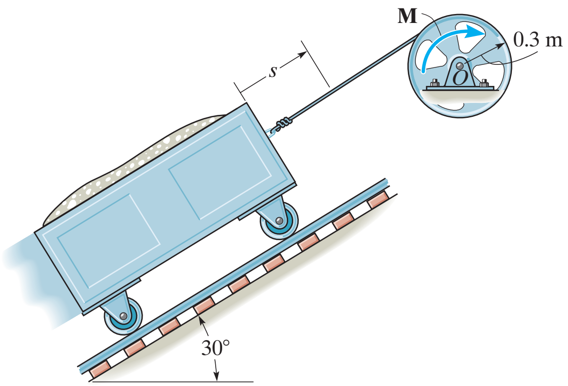
Prob. 18–15

***18–16.** The pendulum consists of a 10-kg uniform disk and a 3-kg uniform slender rod. If it is released from rest in the position shown, determine its angular velocity when it rotates clockwise 90°.



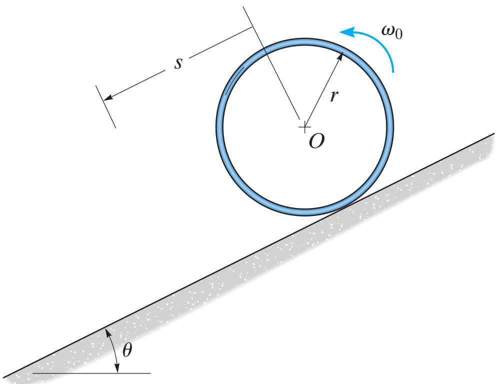
Prob. 18–16

18–17. The wheel has a mass of 100 kg and a radius of gyration of $k_O = 0.2$ m. A motor supplies a torque $M = (40\theta + 900)$ N · m, where θ is in radians, about the drive shaft at O. Determine the speed of the loading car, which has a mass of 300 kg, after it travels $s = 4$ m. Initially the car is at rest when $s = 0$ and $\theta = 0°$. Neglect the mass of the attached cable and the mass of the car's wheels.



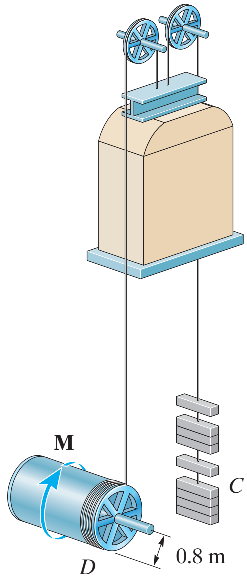
Prob. 18–17

18–18. The center O of the thin ring of mass m is given an angular velocity of ω_0. If the ring rolls without slipping, determine its angular velocity after it has traveled a distance of s down the plane. Neglect its thickness.



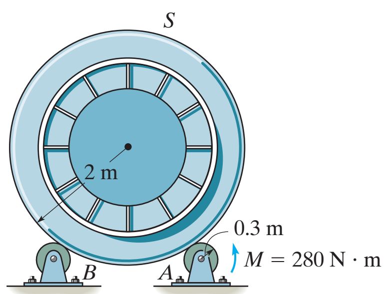
Prob. 18–18

18–19. A motor supplies a constant torque $M = 6$ kN · m to the winding drum that operates the elevator. If the elevator has a mass of 900 kg, the counterweight C has a mass of 200 kg, and the winding drum has a mass of 600 kg and radius of gyration about its axis of $k = 0.6$ m, determine the speed of the elevator after it rises 5 m starting from rest. Neglect the mass of the pulleys.



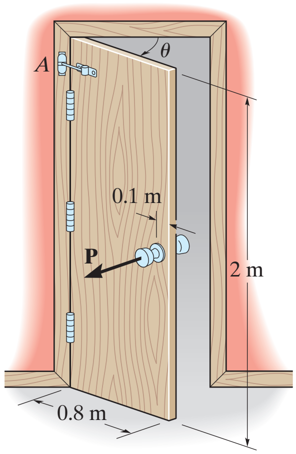
Prob. 18–19

***18–20.** The rotary screen S is used to wash limestone. When empty it has a mass of 800 kg and a radius of gyration of $k_G = 1.75$ m. Rotation is achieved by applying a torque of $M = 280$ N · m about the drive wheel at A. If no slipping occurs at A and the supporting wheel at B is free to roll, determine the angular velocity of the screen after it has rotated 5 revolutions. Neglect the mass of A and B.



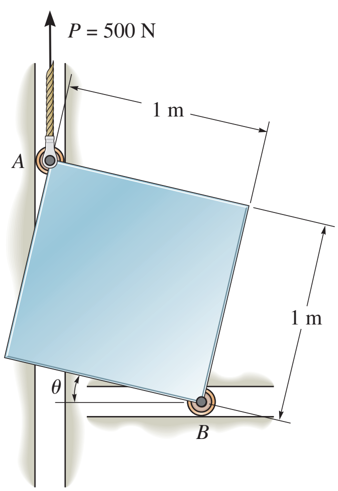
Prob. 18–20

18–21. The uniform door has a mass of 20 kg and can be treated as a thin plate having the dimensions shown. If it is connected to a torsional spring at A, which has stiffness of $k = 80\ \text{N}\cdot\text{m/rad}$, determine the required initial twist of the spring in radians so that the door has an angular velocity of 12 rad/s when it closes at $\theta = 0°$ after being opened at $\theta = 90°$ and released from rest. *Hint:* For a torsional spring $M = k\theta$, where k is the stiffness and θ is the angle of twist.



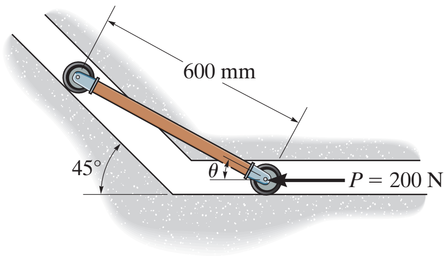
Prob. 18–21

18–22. If corner A of the 60-kg plate is subjected to a vertical force of $P = 500$ N, and the plate is released from rest when $\theta = 0°$, determine the angular velocity of the plate when $\theta = 45°$.



Prob. 18–22

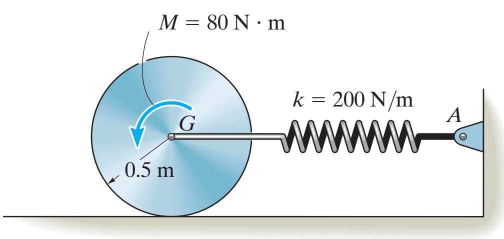
18–23. If $P = 200$ N and the 15-kg uniform slender rod starts from rest at $\theta = 0°$, determine the rod's angular velocity at the instant just before $\theta = 45°$.



Prob. 18–23

***18–24.** The 30-kg disk is originally at rest, and the spring is unstretched. A couple moment of $M = 80\ \text{N}\cdot\text{m}$ is then applied to the disk as shown. Determine its angular velocity when its mass center G has moved 0.5 m along the plane. The disk rolls without slipping.

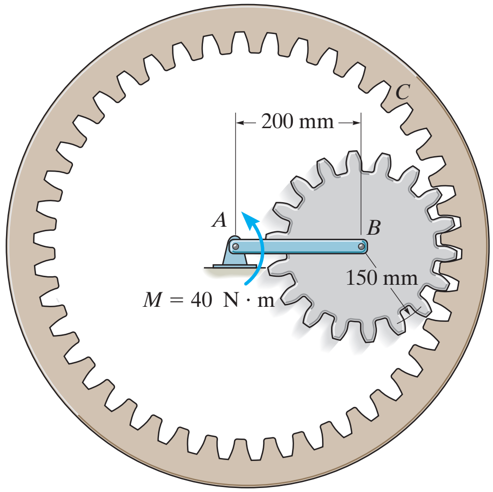
18–25. The 30-kg disk is originally at rest, and the spring is unstretched. A couple moment $M = 80\ \text{N}\cdot\text{m}$ is then applied to the disk as shown. Determine how far the center of mass of the disk travels along the plane before it momentarily stops. The disk rolls without slipping.



Probs. 18–24/25

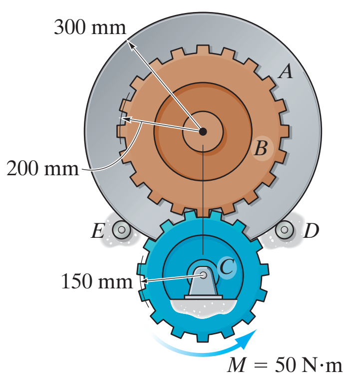
18

18–26. The link AB is subjected to a couple moment of $M = 40\ \text{N}\cdot\text{m}$. If the ring gear C is fixed, determine the angular velocity of the 15-kg inner gear when the link has made two revolutions starting from rest. Neglect the mass of the link and assume the inner gear is a disk. Motion occurs in the vertical plane.



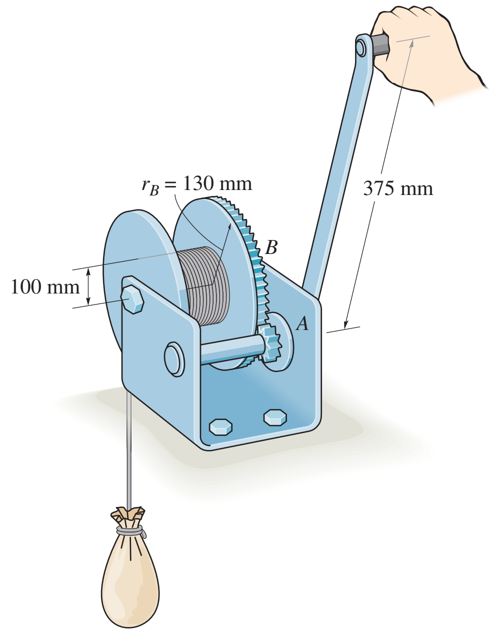
Prob. 18–26

18–27. Gear B is rigidly attached to drum A and is supported by two small rollers at E and D. Gear B is in mesh with gear C and is subjected to a torque of $M = 50\ \text{N}\cdot\text{m}$. Determine the angular velocity of the drum after C has rotated 10 revolutions, starting from rest. Gear B and the drum have 100 kg and a radius of gyration about their rotating axis of 250 mm. Gear C has a mass of 30 kg and a radius of gyration about its rotating axis of 125 mm.



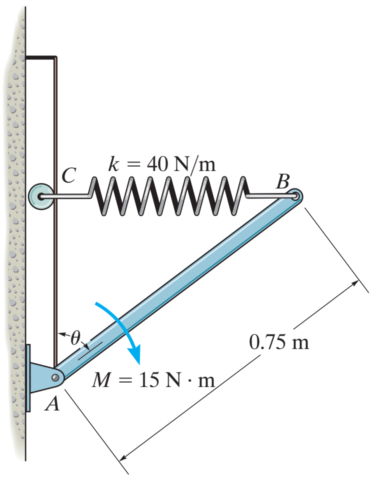
Prob. 18–27

***18–28.** The hand winch is used to lift the 50-kg load. Determine the work required to rotate the handle five revolutions, starting and ending at rest. The gear at A has a radius of 20 mm.



Prob. 18–28

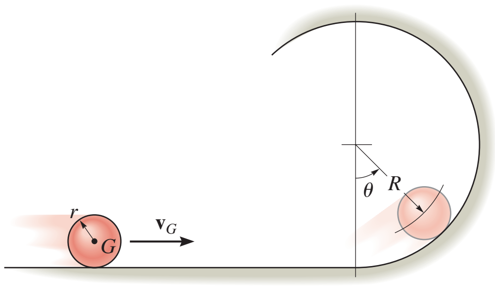
18–29. The 10-kg rod AB is pin connected at A and subjected to a couple moment of $M = 15\ \text{N}\cdot\text{m}$. If the rod is released from rest when the spring is unstretched at $\theta = 30°$, determine the rod's angular velocity at the instant $\theta = 60°$. As the rod rotates, the spring always remains horizontal, because of the roller support at C.



Prob. 18–29

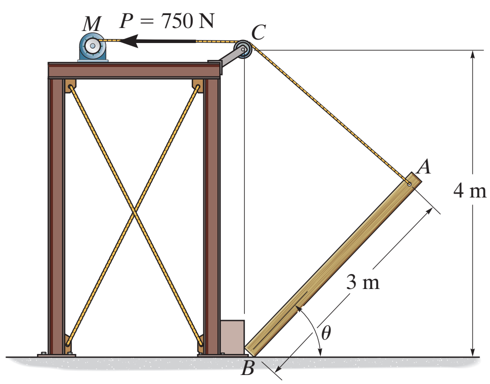
18–30. A ball of mass m and radius r is cast onto the horizontal surface such that it rolls without slipping. Determine its angular velocity at the instant $\theta = 90°$, if it has an initial speed of v_G as shown.

18–31. A ball of mass m and radius r is cast onto the horizontal surface such that it rolls without slipping. Determine the minimum speed v_G of its mass center G so that it rolls completely around the loop of radius $R + r$ without leaving the track.



Probs. 18–30/31

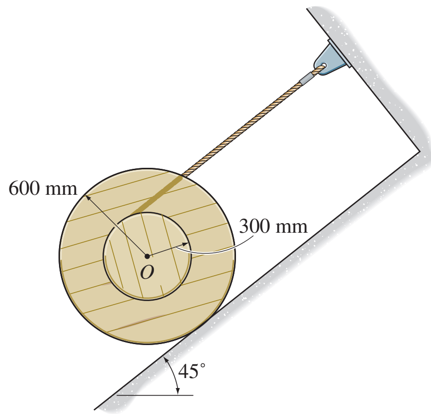
***18–32.** Motor M exerts a constant force of $P = 750$ N on the rope. If the 100-kg post is at rest when $\theta = 0°$, determine the angular velocity of the post at the instant $\theta = 60°$. Neglect the mass of the pulley and its size, and consider the post as a slender rod.



Prob. 18–32

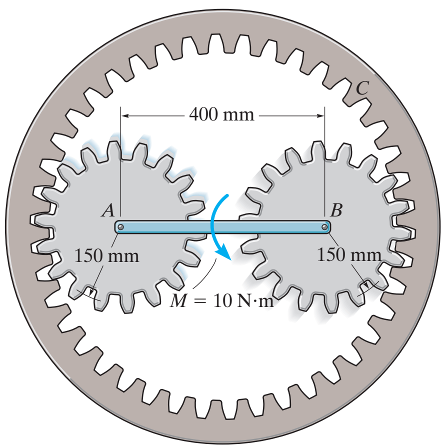
18–33. The spool has a mass of 100 kg and a radius of gyration of 400 mm about its center of mass O. If it is released from rest, determine its angular velocity after its center O has moved down the plane a distance of 2 m. The contact surface between the spool and the inclined plane is smooth.

18–34. The spool has a mass of 100 kg and a radius of gyration of 400 mm about its center of mass O. If it is released from rest, determine its angular velocity after its center O has moved down the plane a distance of 2 m. The coefficient of kinetic friction between the spool and the inclined plane is $\mu_k = 0.15$.



Probs. 18–33/34

18–35. The two 2-kg gears A and B are attached to the ends of a 3-kg slender bar. The gears roll within the fixed ring gear C, which lies in the horizontal plane. If a 10-N · m torque is applied to the center of the bar as shown, determine the number of revolutions the bar must rotate starting from rest in order for it to have an angular velocity of $\omega_{AB} = 20$ rad/s. For the calculation, assume the gears can be approximated by thin disks. What is the result if the gears lie in the vertical plane?



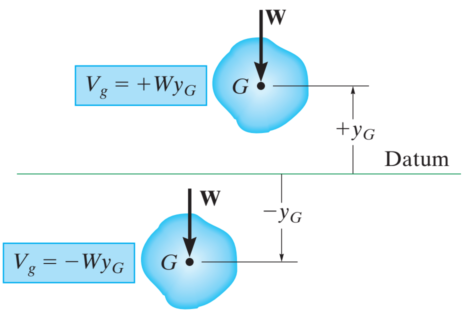
Prob. 18–35

18.5 CONSERVATION OF ENERGY

When a force system acting on a rigid body consists only of *conservative forces*, then the conservation of energy theorem can be used to solve a problem that otherwise would be solved using the principle of work and energy. This theorem is often easier to apply since the work of a conservative force is independent of the path and depends only on the initial and final positions of the body. It was shown in Sec. 14.5 that the work of a conservative force can be expressed as the difference in the body's potential energy measured from an arbitrarily selected reference or datum.

Gravitational Potential Energy. Since the total weight of a body can be considered concentrated at its center of gravity, the gravitational potential energy of the body is determined by knowing the height of the body's center of gravity above or below a horizontal datum.

$$V_g = W y_G \tag{18–15}$$



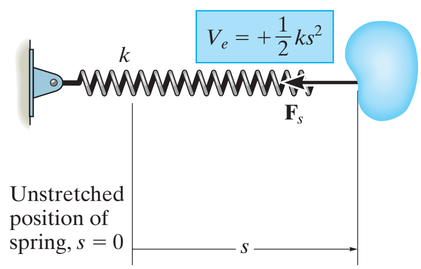
Gravitational potential energy

Fig. 18–16

Here the potential energy is positive when G is above the datum, since the weight has the ability to do positive work when the body moves back to the datum, Fig. 18–16. Likewise, if G is located below the datum $(-y_G)$, the gravitational potential energy is negative, since the weight does negative work when the body returns to the datum.

Elastic Potential Energy. The force developed by an elastic spring is also a conservative force. The elastic potential energy which a spring imparts to an attached body when the spring is stretched or compressed from an initial undeformed position $(s = 0)$ to a final position s, Fig. 18–17, is

$$V_e = \tfrac{1}{2}ks^2 \tag{18–16}$$



Elastic potential energy

Fig. 18–17

In the deformed position, the spring force acting on the body always has the ability for doing positive work when the spring returns back to its original undeformed position (see Sec. 14.5).

Conservation of Energy. In general, if a body is subjected to both gravitational and elastic forces, the total potential energy can be expressed as a potential function represented as the algebraic sum

$$V = V_g + V_e \tag{18–17}$$

Realizing that the work of conservative forces can be written as a difference in their potential energies, i.e., $(\Sigma U_{1-2})_{\text{cons}} = V_1 - V_2$, Eq. 14–16, we can write the principle of work and energy for a rigid body as

$$T_1 + V_1 + (\Sigma U_{1-2})_{\text{noncons}} = T_2 + V_2 \tag{18–18}$$

Here $(\Sigma U_{1-2})_{\text{noncons}}$ represents the work of the nonconservative forces such as friction. If this term is zero, then

$$T_1 + V_1 = T_2 + V_2 \tag{18–19}$$

This equation is referred to as the ***conservation of energy***. It states that the sum of the potential and kinetic energies of the body remains constant when the body moves from one position to another. It also applies to a *system* of smooth, pin-connected rigid bodies, bodies connected by inextensible cords, and bodies in mesh with other bodies. In all these cases the forces acting at the points of contact are eliminated from the analysis, since they occur in equal but opposite collinear pairs and each pair of forces moves through an equal distance when the system undergoes a displacement.

It is important to remember that only problems involving conservative force systems can be solved by using Eq. 18–19. As stated in Sec. 14.5, dry friction or fluid drag-resistant forces, which depend on velocity or acceleration, are nonconservative. The work of such forces is transformed into thermal energy used to heat up the surfaces of contact, and consequently this energy is dissipated into the surroundings and may not be recovered. Therefore, problems involving frictional forces can be solved by using either the principle of work and energy written in the form of Eq. 18–18, if it applies, or the equations of motion.

The torsional springs located at the top of the garage door wind up as the door is lowered. When the door is raised, the potential energy stored in the springs is then transferred into gravitational potential energy of the door's weight, thereby making it easy to open.

PROCEDURE FOR ANALYSIS

The conservation of energy equation is used to solve problems involving *velocity, displacement*, and *conservative force systems*. For application it is suggested that the following procedure be used.

Potential Energy.

- Draw two diagrams showing the body or system located at its initial and final positions on the path.
- If the center of gravity, G, is subjected to a vertical displacement, establish a fixed horizontal datum from which to measure the body's gravitational potential energy V_g.
- Data pertaining to the elevation y_G of the body's center of gravity from the datum and the extension or compression of any connecting springs can be determined from the problem geometry and listed on the two diagrams.
- The potential energy is determined from $V = V_g + V_e$. Here $V_g = Wy_G$, which can be positive or negative, and $V_e = \frac{1}{2}ks^2$, which is always positive.

Kinetic Energy.

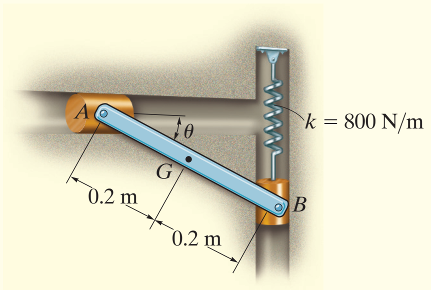
- The kinetic energy of the body consists of two parts, namely, translational kinetic energy referenced from the mass center, $T = \frac{1}{2}mv_G^2$, and rotational kinetic energy about the mass center, $T = \frac{1}{2}I_G\omega^2$.
- Kinematic diagrams for velocity may be useful for establishing a relationship between v_G and ω.

Conservation of Energy.

- Apply the conservation of energy equation $T_1 + V_1 = T_2 + V_2$.

Refer to the companion website for Lecture Summary and Quiz videos.

EXAMPLE 18.6



(a)

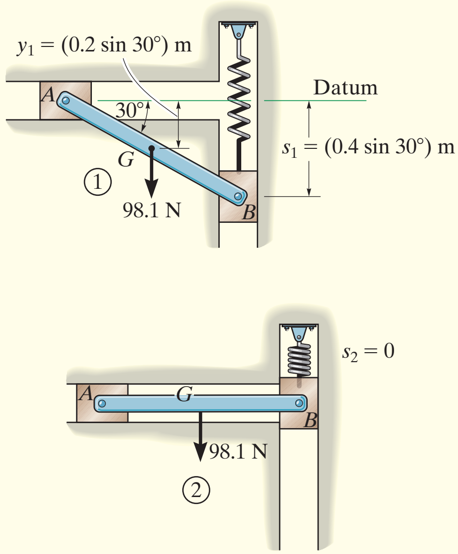
The 10-kg rod AB shown in Fig. 18–18a is confined so that its ends move in the horizontal and vertical slots. The spring has a stiffness of $k = 800$ N/m and is unstretched when $\theta = 0°$. Determine the angular velocity of AB when $\theta = 0°$, if the rod is released from rest when $\theta = 30°$. Neglect the mass of the slider blocks.

SOLUTION

Here the system consists of the rod and spring.

Potential Energy. The two diagrams of the rod, when it is located at its initial and final positions, are shown in Fig. 18–18b. The datum, used to measure the gravitational potential energy, is placed in line with the rod when $\theta = 0°$.

When the rod is in position 1, the center of gravity G is located below the datum so its gravitational potential energy is negative. Positive elastic potential energy is stored in the spring, since it is stretched a distance of $s_1 = (0.4 \sin 30°)$ m. Thus,



(b)

$$V_1 = -Wy_1 + \tfrac{1}{2}ks_1^2$$

$$= -(98.1\text{ N})(0.2 \sin 30°\text{ m}) + \tfrac{1}{2}(800\text{ N/m})(0.4 \sin 30°\text{ m})^2 = 6.19\text{ J}$$

When the rod is in position 2, the potential energy of the rod is zero, since the center of gravity G is located at the datum, and the spring is unstretched, $s_2 = 0$. Thus,

$$V_2 = 0$$

Kinetic Energy. The rod is released from rest from position 1, thus $(\mathbf{v}_G)_1 = \boldsymbol{\omega}_1 = \mathbf{0}$, and so

$$T_1 = 0$$

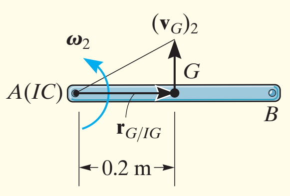
In position 2, the angular velocity is $\boldsymbol{\omega}_2$ and the rod's mass center has a velocity of $(\mathbf{v}_G)_2$. Thus,

$$T_2 = \tfrac{1}{2}m(v_G)_2^2 + \tfrac{1}{2}I_G\omega_2^2$$

$$= \tfrac{1}{2}(10\text{ kg})(v_G)_2^2 + \tfrac{1}{2}\left[\tfrac{1}{12}(10\text{ kg})(0.4\text{ m})^2\right]\omega_2^2$$

Using kinematics, $(\mathbf{v}_G)_2$ can be related to $\boldsymbol{\omega}_2$ as shown in Fig. 18–18c. At the instant considered, the instantaneous center of zero velocity (IC) for the rod is at point A; hence, $(v_G)_2 = (r_{G/IC})\omega_2 = (0.2\text{ m})\omega_2$. Substituting into the above expression and simplifying (or using $\tfrac{1}{2}I_{IC}\omega_2^2$), we get

$$T_2 = 0.2667\omega_2^2$$



(c)

Fig. 18–18

Conservation of Energy.

$$\{T_1\} + \{V_1\} = \{T_2\} + \{V_2\}$$

$$\{0\} + \{6.19\text{ J}\} = \{0.2667\omega_2^2\} + \{0\}$$

$$\omega_2 = 4.82\text{ rad/s} \circlearrowleft \qquad \textit{Ans.}$$

18

EXAMPLE 18.7

The wheel shown in Fig. 18–19*a* has a mass of 15 kg and a radius of gyration of $k_G = 0.18$ m. It is attached to a spring which has a stiffness $k = 30$ N/m and an unstretched length of 0.3 m. If the wheel is released from rest in the position shown and rolls without slipping, determine its angular velocity at the instant G moves 0.9 m to the left.

SOLUTION

Here the system consists of the wheel and spring.

Potential Energy. Two diagrams of the wheel, when it is at the initial and final positions, are shown in Fig. 18–19*b*. A gravitational datum is not needed here since the weight is not displaced vertically. From the problem geometry the spring is stretched $s_1 = \left(\sqrt{0.9^2 + 1.2^2} - 0.3\right) = 1.2$ m in the initial position, and stretched $s_2 = (1.2 - 0.3) = 0.9$ m in the final position. Hence, the positive spring potential energy is

$$V_1 = \tfrac{1}{2}ks_1^2 = \tfrac{1}{2}(30\ \text{N/m})(1.2\ \text{m})^2 = 21.6\ \text{N}\cdot\text{m}$$

$$V_2 = \tfrac{1}{2}ks_2^2 = \tfrac{1}{2}(30\ \text{N/m})(0.9\ \text{m})^2 = 12.15\ \text{N}\cdot\text{m}$$

Kinetic Energy. The wheel is released from rest and so $(\mathbf{v}_G)_1 = \mathbf{0}$, $\boldsymbol{\omega}_1 = \mathbf{0}$. Therefore,

$$T_1 = 0$$

Since the instantaneous center of zero velocity is at the ground, Fig. 18–19*c*, we have

$$T_2 = \frac{1}{2}I_{IC}\omega_2^2$$

$$= \frac{1}{2}\left[(15\ \text{kg})(0.18\ \text{m})^2 + (15\ \text{kg})(0.225\ \text{m})^2\right]\omega_2^2$$

$$= 0.6227\omega_2^2$$

Conservation of Energy.

$$\{T_1\} + \{V_1\} = \{T_2\} + \{V_2\}$$

$$\{0\} + \{21.6\ \text{N}\cdot\text{m}\} = \{0.6227\omega_2^2\} + \{12.15\ \text{N}\cdot\text{m}\}$$

$$\omega_2 = 3.90\ \text{rad/s} \circlearrowleft \qquad \textit{Ans.}$$

NOTE: If the principle of work and energy were used to solve this problem, then the work of the spring would have to be determined by considering both the change in magnitude and direction of the spring force.

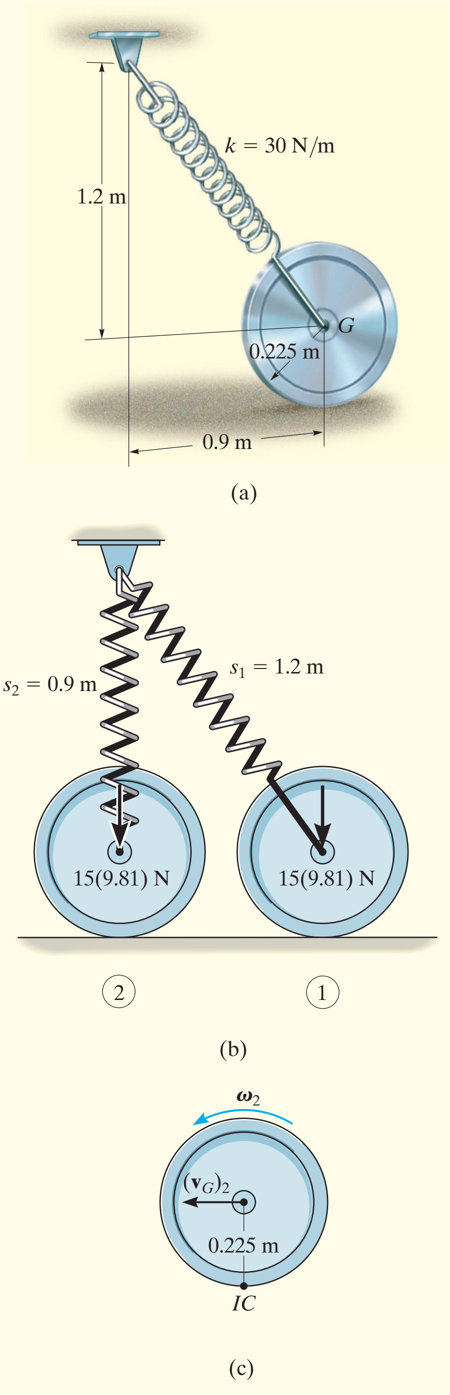
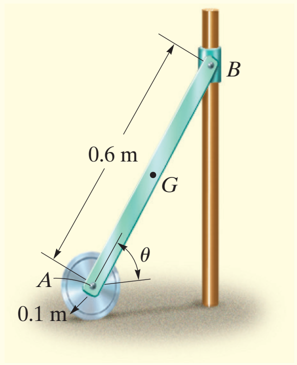


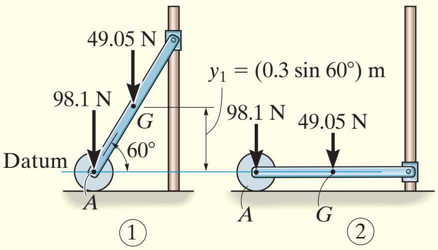
Fig. 18–19

18

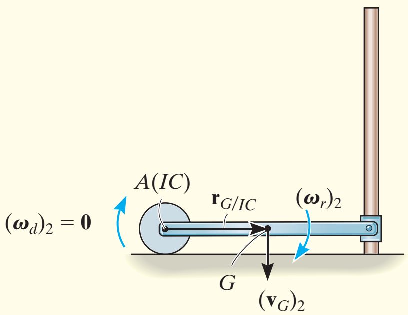
EXAMPLE 18.8



(a)



(b)



(c)

Fig. 18–20

The 10-kg disk shown in Fig. 18–20*a* is attached to a uniform 5-kg rod *AB*. If the assembly is released from rest when $\theta = 60°$, determine the angular velocity of the rod when $\theta = 0°$. Assume that the disk rolls without slipping. Neglect friction along the guide and the mass of the collar at *B*.

SOLUTION

The system consists of the disk and rod.

Potential Energy. Two diagrams for the rod and disk, when they are located at their initial and final positions, are shown in Fig. 18–20*b*. For convenience the datum passes through point *A*.

When the system is in position 1, only the rod's weight has positive potential energy. Thus,

$$V_1 = W_r y_1 = (49.05 \text{ N})(0.3 \sin 60° \text{ m}) = 12.74 \text{ J}$$

When the system is in position 2, both the weight of the rod and the weight of the disk have zero potential energy. Why? Thus,

$$V_2 = 0$$

Kinetic Energy. Since the entire system is at rest at the initial position,

$$T_1 = 0$$

In the final position the rod has an angular velocity $(\boldsymbol{\omega}_r)_2$ and its mass center has a velocity $(\mathbf{v}_G)_2$, Fig. 18–20*c*. Since the rod is *fully extended* in this position, the disk is momentarily at rest, so $(\boldsymbol{\omega}_d)_2 = \mathbf{0}$ and $(\mathbf{v}_A)_2 = \mathbf{0}$. For the rod $(\mathbf{v}_G)_2$ can be related to $(\boldsymbol{\omega}_r)_2$ from the instantaneous center of zero velocity, which is located at point *A*, Fig. 18–20*c*. Hence, $(v_G)_2 = r_{G/IC}(\omega_r)_2$ or $(v_G)_2 = 0.3(\omega_r)_2$. Thus,

$$T_2 = \frac{1}{2}m_r(v_G)_2^2 + \frac{1}{2}I_G(\omega_r)_2^2 + \frac{1}{2}m_d(v_A)_2^2 + \frac{1}{2}I_A(\omega_d)_2^2$$

$$= \frac{1}{2}(5 \text{ kg})[(0.3 \text{ m})(\omega_r)_2]^2 + \frac{1}{2}\left[\frac{1}{12}(5 \text{ kg})(0.6 \text{ m})^2\right](\omega_r)_2^2 + 0 + 0$$

$$= 0.3(\omega_r)_2^2$$

Conservation of Energy.

$$\{T_1\} + \{V_1\} = \{T_2\} + \{V_2\}$$

$$\{0\} + \{12.74 \text{ J}\} = \{0.3(\omega_r)_2^2\} + \{0\}$$

$$(\omega_r)_2 = 6.52 \text{ rad/s} \curvearrowright \qquad \textit{Ans.}$$

NOTE: We can also determine the final kinetic energy of the rod using $T_2 = \frac{1}{2}I_{IC}\omega_2^2$.

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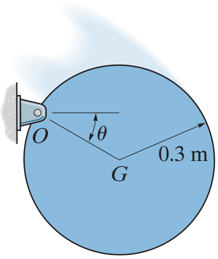
Refer to the companion website for a self quiz of these Example problems.

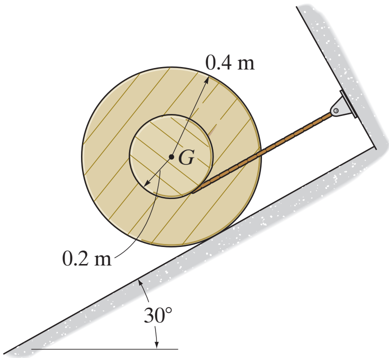
FUNDAMENTAL PROBLEMS

F18–7. If the 30-kg disk is released from rest when $\theta = 0°$, determine its angular velocity when $\theta = 90°$.



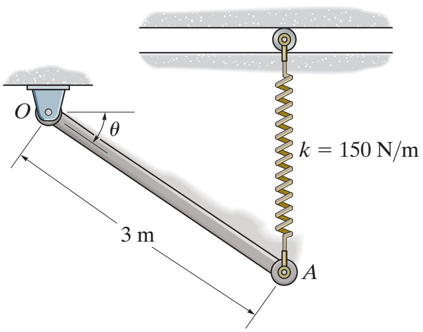
Prob. F18–7

F18–8. The 50-kg reel has a radius of gyration about its center G of $k_G = 300$ mm. If it is released from rest, determine its angular velocity when its center G has traveled 6 m down the smooth inclined plane.



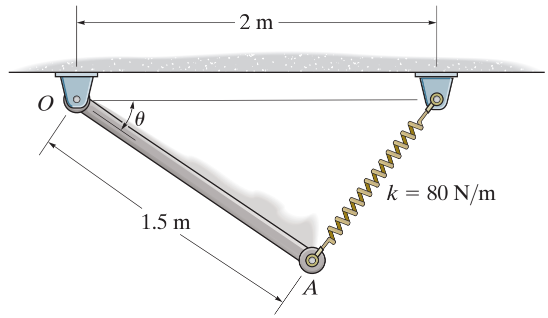
Prob. F18–8

F18–9. The 60-kg rod OA is released from rest when $\theta = 0°$. Determine its angular velocity when $\theta = 45°$. The spring remains vertical during the motion and is unstretched when $\theta = 0°$.



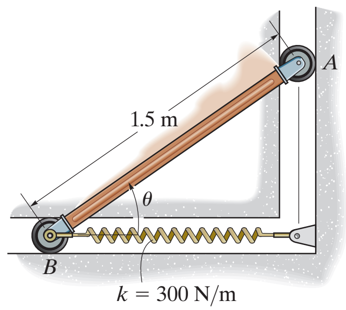
Prob. F18–9

F18–10. The 30-kg rod is released from rest when $\theta = 0°$. Determine the angular velocity of the rod when $\theta = 90°$. The spring is unstretched when $\theta = 0°$.



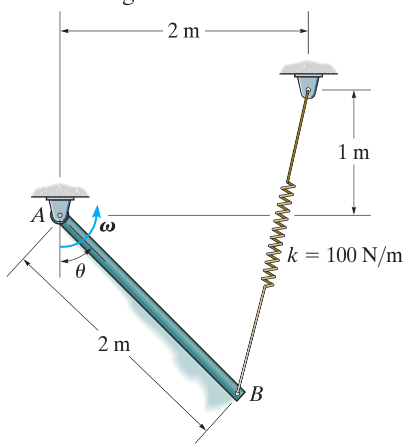
Prob. F18–10

F18–11. The 30-kg rod is released from rest when $\theta = 45°$. Determine the angular velocity of the rod when $\theta = 0°$. The spring is unstretched when $\theta = 45°$.



Prob. F18–11

F18–12. The 20-kg rod is released from rest when $\theta = 0°$. Determine its angular velocity when $\theta = 90°$. The spring has an unstretched length of 0.5 m.

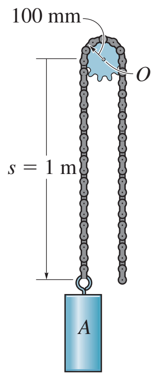


Prob. F18–12

PROBLEMS

***18–36.** A chain that has a negligible mass is draped over the sprocket which has a mass of 2 kg and a radius of gyration of $k_O = 50$ mm. If the 4-kg block *A* is released from rest in the position $s = 1$ m, determine the angular velocity of the sprocket at the instant $s = 2$ m.

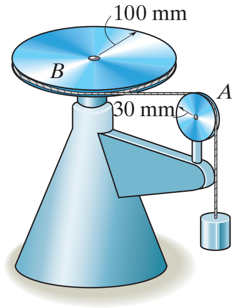
18–37. Solve Prob. 18–36 if the chain has a mass of 0.8 kg/m. For the calculation, neglect the portion of the chain that wraps over the sprocket.



Probs. 18–36/37

18–38. The assembly consists of a 3-kg pulley *A* and 10-kg pulley *B*. If a 2-kg block is suspended from the cord, determine the block's speed after it descends 0.5 m starting from rest. Neglect the mass of the cord and treat the pulleys as thin disks. No slipping occurs.

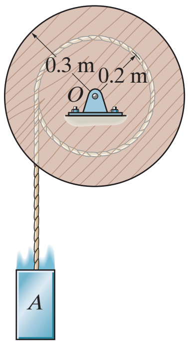
18–39. The assembly consists of a 3-kg pulley *A* and 10-kg pulley *B*. If a 2-kg block is suspended from the cord, determine the distance the block must descend, starting from rest, in order to cause *B* to have an angular velocity of 6 rad/s. Neglect the mass of the cord and treat the pulleys as thin disks. No slipping occurs.



Probs. 18–38/39

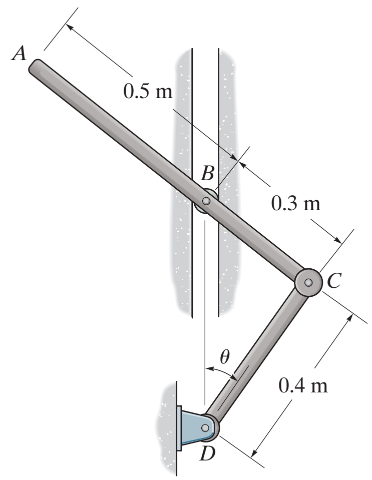
***18–40.** The spool has a mass of 50 kg and a radius of gyration of $k_O = 0.280$ m. If the 20-kg block *A* is released from rest, determine the distance the block must fall in order for the spool to have an angular velocity $\omega = 5$ rad/s. Also, what is the tension in the cord while the block is in motion? Neglect the mass of the cord.

18–41. The spool has a mass of 50 kg and a radius of gyration of $k_O = 0.280$ m. If the 20-kg block *A* is released from rest, determine the velocity of the block when it descends 0.5 m.



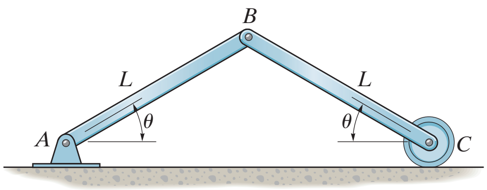
Probs. 18–40/41

18–42. The 6-kg rod *ABC* is connected to the 3-kg rod *CD*. If the system is released from rest when $\theta = 0°$, determine the angular velocity of rod *ABC* at the instant it becomes horizontal.



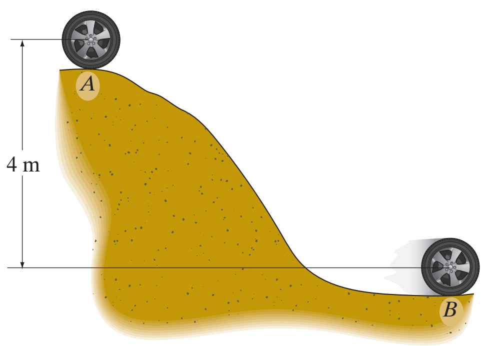
Prob. 18–42

18–43. The two bars are released from rest at the position θ. Determine their angular velocities at the instant they become horizontal. Neglect the mass of the roller at C. Each bar has a mass m and length L.



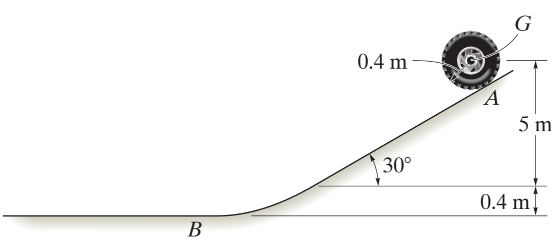
Prob. 18–43

***18–44.** The tire has a mass of 8 kg, a radius of 0.3 m, and a radius of gyration about its center of gravity G of $k_G = 0.25$ m. If it is released from rest at the top of the hill, determine the velocity of its center at the bottom of the hill if it rolls without slipping.



Prob. 18–44

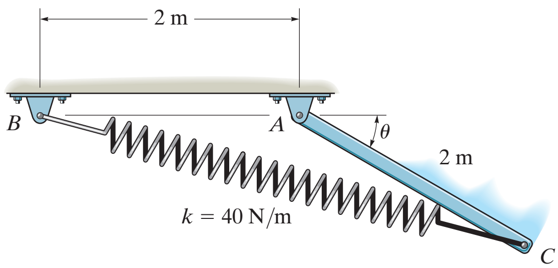
18–45. An automobile tire has a mass of 7 kg and radius of gyration of $k_G = 0.3$ m. If it is released from rest at A on the incline, determine its angular velocity when it reaches the horizontal plane. The tire rolls without slipping.



Prob. 18–45

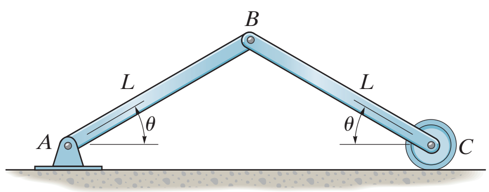
18–46. The 12-kg slender rod is attached to a spring, which has an unstretched length of 2 m. If the rod is released from rest when $\theta = 30°$, determine its angular velocity at the instant $\theta = 90°$.

18–47. The 12-kg slender rod is attached to a spring, which has an unstretched length of 2 m. If the rod is released from rest when $\theta = 30°$, determine the angular velocity of the rod the instant the spring becomes unstretched.



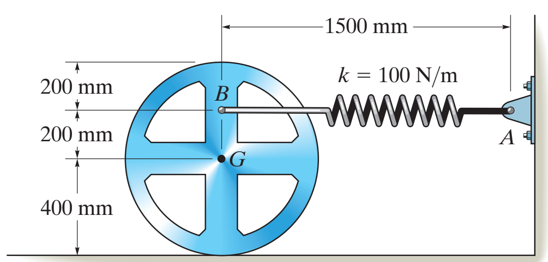
Probs. 18–46/47

***18–48.** The two bars are released from rest at the position $\theta = 90°$. Determine their angular velocities at the instant they become horizontal. Neglect the mass of the roller at C. Each bar has a mass m and length L.



Prob. 18–48

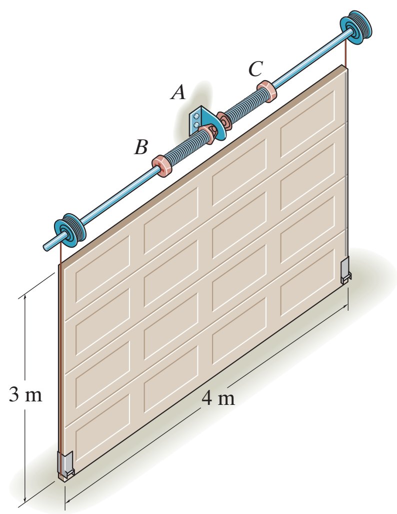
18–49. The 40-kg wheel has a radius of gyration about its center of gravity G of $k_G = 250$ mm. If it rolls without slipping, determine its angular velocity when it has rotated clockwise 90° from the position shown. The spring AB has a stiffness $k = 100$ N/m and an unstretched length of 500 mm. The wheel is released from rest.



Prob. 18–49

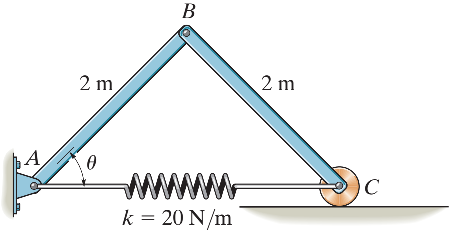
18

18–50. The uniform garage door has a mass of 150 kg and is guided along smooth tracks at its ends. Lifting is done using the two springs, which are attached to the anchor bracket at A and to the counterbalance shaft at B and C. As the door is raised, the springs begin to unwind from the shaft, thereby assisting the lift. If each spring provides a torsional moment of $M = (0.7\theta)$ N·m, where θ is in radians, determine the angle θ_0 at which both the left-wound and right-wound spring should be attached at A so that the door is completely balanced by the springs, i.e., when the door is in the vertical position and is given a slight force upward, the springs will lift the door along the side tracks to the horizontal plane with no final angular velocity. *Note*: The elastic potential energy of a torsional spring is $V_e = \frac{1}{2}k\theta^2$, where $M = k\theta$ and in this case $k = 0.7$ N·m/rad.



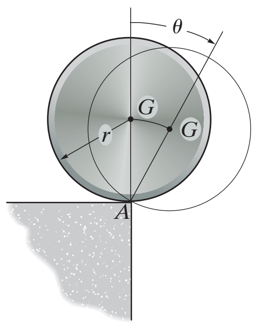
Prob. 18–50

18–51. The two 12-kg slender rods are pin connected and released from rest at the position $\theta = 60°$. If the spring has an unstretched length of 1.5 m, determine the angular velocity of rod BC, when the system is at the position $\theta = 30°$.



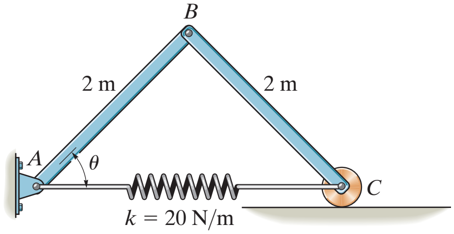
Prob. 18–51

***18–52.** A large roll of paper having a mass of 20 kg and a radius $r = 150$ mm is resting over the edge of a corner, such that the end of the paper on the roll is attached to the horizontal surface. If the roll is disturbed slightly from its equilibrium position, determine the angle θ at which it begins to leap off the corner A as it falls. The radius of gyration of the roll is $k_G = 75$ mm.



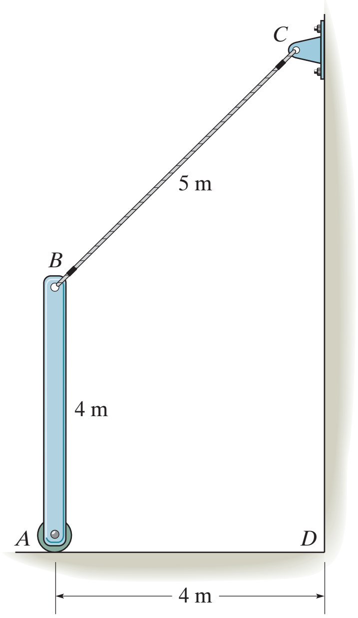
Prob. 18–52

18–53. The two 12-kg slender rods are pin connected and released from rest at the position $\theta = 60°$. If the spring has an unstretched length of 1.5 m, determine the angular velocity of rod BC when the system is at the position $\theta = 0°$. Neglect the mass of the roller at C.



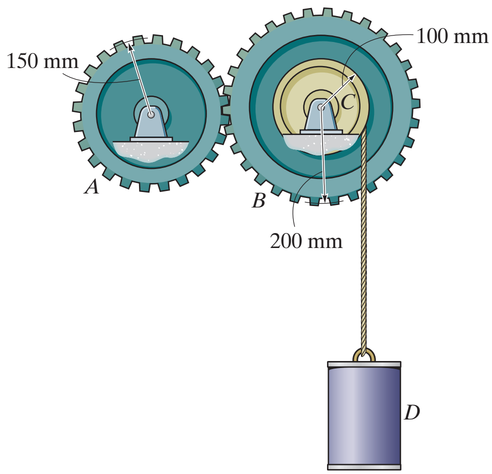
Prob. 18–53

18–54. The slender 15-kg bar is initially at rest and standing in the vertical position when the bottom end A is displaced slightly to the right. If the track in which it moves is smooth, determine the speed at which end A strikes the corner D. The bar is constrained to move in the vertical plane. Neglect the mass of the cord BC.



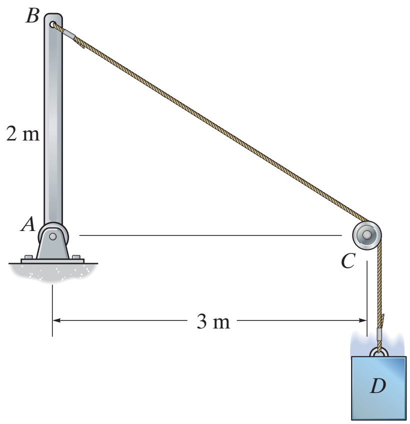
Prob. 18–54

18–55. Determine the speed of the 50-kg cylinder after it has descended a distance of 2 m, starting from rest. Gear A has a mass of 10 kg and a radius of gyration of 125 mm about its center of mass. Gear B and drum C have a combined mass of 30 kg and a radius of gyration about their center of mass of 150 mm.



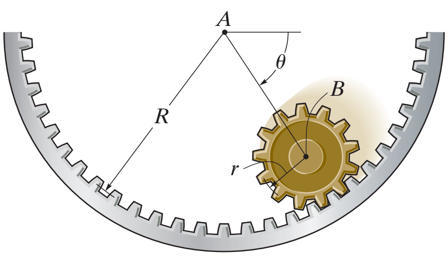
Prob. 18–55

***18–56.** The slender rod has a mass of 10 kg and is released from rest when it is in the vertical position. Determine its angular velocity at the instant it becomes horizontal. The attached block at D has a mass of 4 kg. The mass of the cable and pulley at C can be neglected.



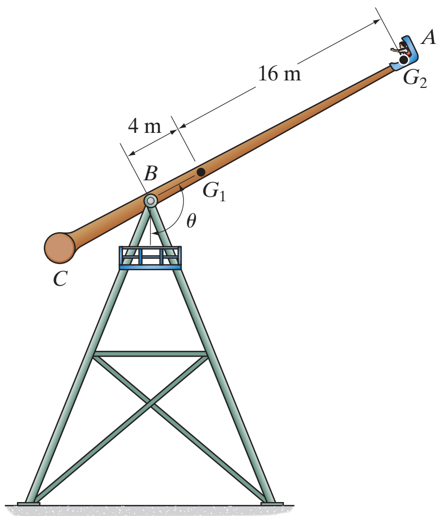
Prob. 18–56

18–57. The small gear has a mass m and may be treated as a uniform disk. If it is released from rest at $\theta = 0°$, and rolls along the fixed circular gear rack, determine the angular velocity of the radial line AB at the instant $\theta = 90°$.



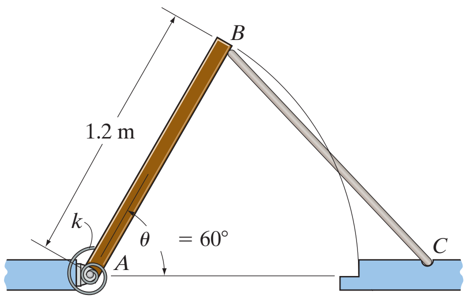
Prob. 18–57

18–58. The arm and seat of the amusement-park ride have a mass of 1.5 Mg, with the center of mass located at point G_1. The passenger seated at A has a mass of 125 kg, with the center of mass located at G_2. If the arm is raised to a position where $\theta = 150°$ and released from rest, determine the speed of the passenger at the instant $\theta = 0°$. The arm has a radius of gyration of $k_{G1} = 12$ m about its center of mass G_1. Neglect the size of the passenger.



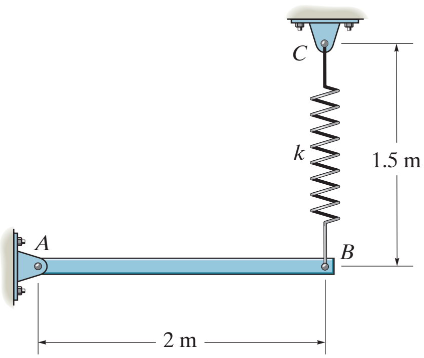
Prob. 18–58

18–59. The uniform rectangular door panel has a mass of 25 kg and is held in equilibrium above the horizontal at the position $\theta = 60°$ by rod BC. Determine the required stiffness of the torsional spring at A, so that the door's angular velocity becomes zero when the door reaches the closed position ($\theta = 0°$) once the supporting rod BC is removed. The spring is undeformed when $\theta = 60°$.



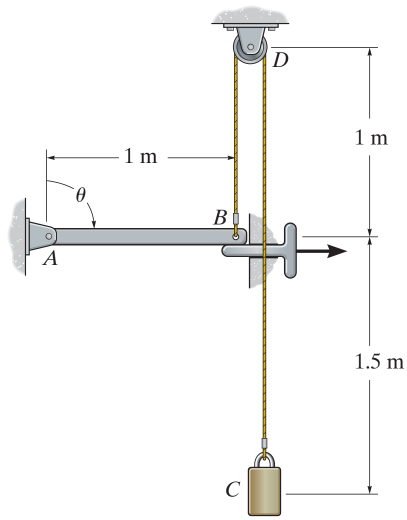
Prob. 18–59

***18–60.** The slender 6-kg bar AB is horizontal and at rest and the spring is unstretched. Determine the stiffness k of the spring so that the motion of the bar is momentarily stopped when it has rotated clockwise 90° after being released.



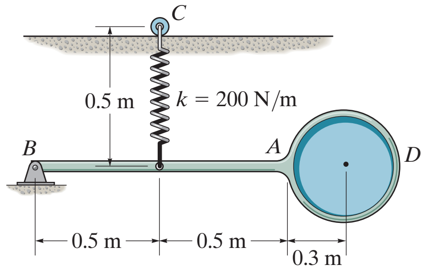
Prob. 18–60

18–61. The uniform bar AB has a mass of 12 kg and is pin connected at A. If the support at B is removed ($\theta = 90°$), determine the velocity of the 5-kg block C at the instant the bar rotates downward to $\theta = 150°$. Neglect the size and mass of the pulley at D.



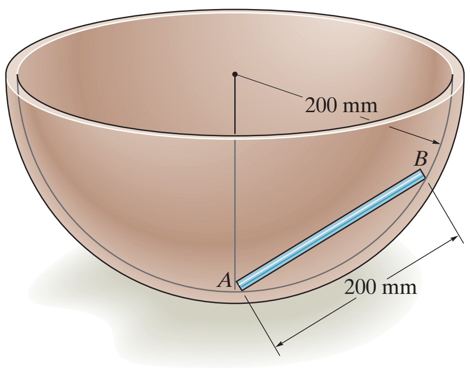
Prob. 18–61

18–62. The pendulum consists of a 6-kg slender rod fixed to a 15-kg disk. If the spring has an unstretched length of 0.2 m, determine the angular velocity of the pendulum when it is released from rest and rotates clockwise 90° from the position shown. The roller at C allows the spring to always remain vertical.



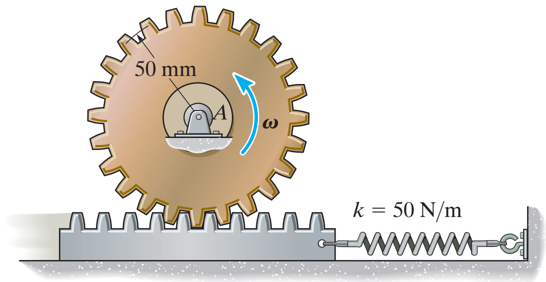
Prob. 18–62

18–63. The 500-g rod AB rests along the smooth inner surface of a hemispherical bowl. If the rod is released from rest from the position shown, determine its angular velocity at the instant it swings downward and becomes horizontal.



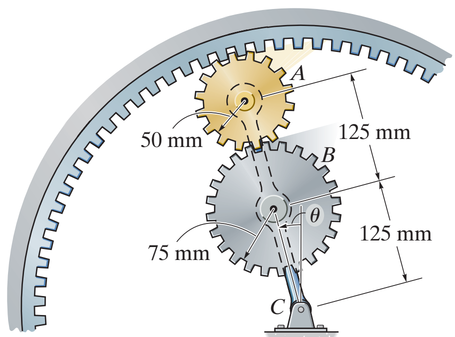
Prob. 18–63

***18–64.** The gear is pinned at A and has a mass of 0.3 kg and a radius of gyration of $k_A = 25$ mm. The gear rack has a mass of 0.5 kg. If the spring is stretched 50 mm and released from rest, determine the angular velocity of the gear at the instant the spring compresses 20 mm. Neglect friction.



Prob. 18–64

18–65. Gear A has a mass of 0.5 kg and a radius of gyration of $k_A = 40$ mm, and gear B has a mass of 0.8 kg and a radius of gyration of $k_B = 55$ mm. The link is pinned at C and has a mass of 0.35 kg. If the link can be treated as a slender rod, determine the angular velocity of the rod after the assembly is released from rest when $\theta = 0°$ and falls to $\theta = 90°$.



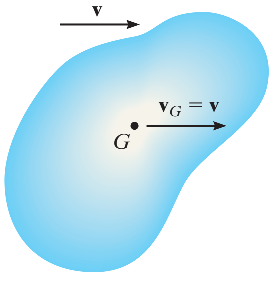
Prob. 18–65

18

CHAPTER REVIEW

Kinetic Energy

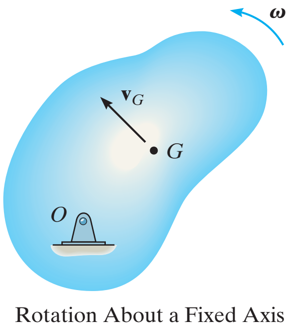
The kinetic energy of a rigid body that undergoes planar motion can be calculated using the motion of its mass center or its motion about a fixed axis or instantaneous axis of rotation. It includes a scalar sum of its translational and rotational kinetic energies.



Translation

Translation

$$T = \tfrac{1}{2} m v_G^2$$



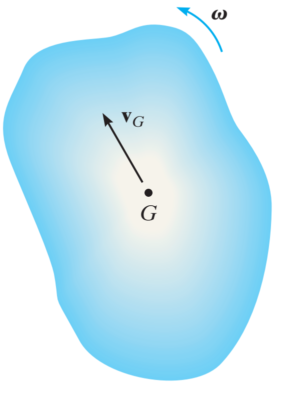
Rotation About a Fixed Axis

Rotation About a Fixed Axis

$$T = \tfrac{1}{2} m v_G^2 + \tfrac{1}{2} I_G \omega^2$$

or

$$T = \tfrac{1}{2} I_O \omega^2$$



General Plane Motion

General Plane Motion

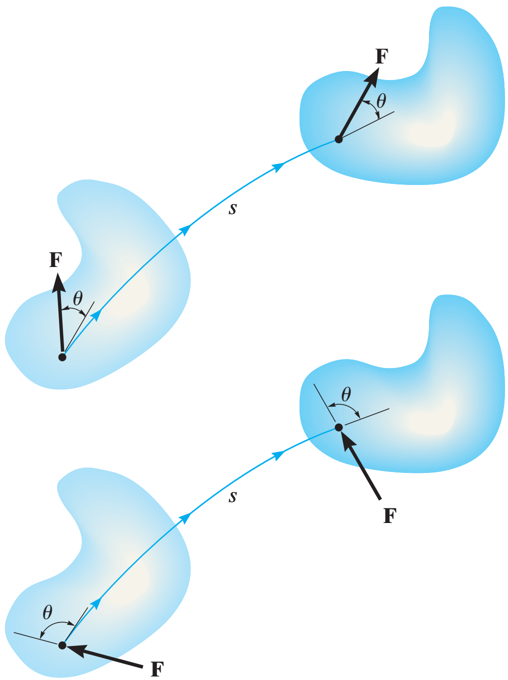
$$T = \tfrac{1}{2} m v_G^2 + \tfrac{1}{2} I_G \omega^2$$

or

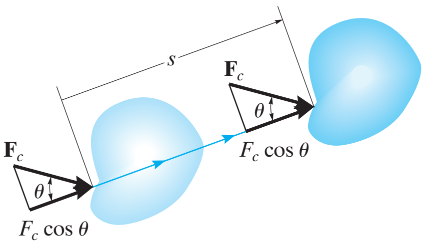
$$T = \tfrac{1}{2} I_{IC} \omega^2$$

Work of a Force and a Couple Moment

A force does work when it undergoes a displacement *ds* in the direction of the force, and a couple moment does work when it undergoes a rotation.

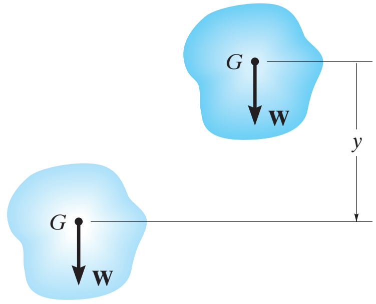


$$U_F = \int F \cos\theta \, ds$$



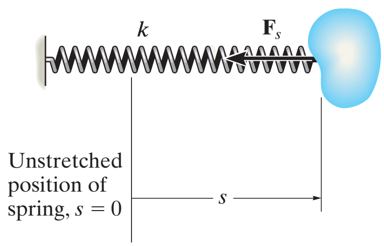
$$U_{F_C} = (F_c \cos\theta)s$$

Constant Force



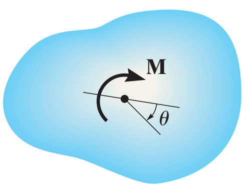
$$U_W = Wy$$

Weight



$$U = -\frac{1}{2} k s^2$$

Spring



$$U_M = \int_{\theta_1}^{\theta_2} M \, d\theta$$

$$U_M = M(\theta_2 - \theta_1)$$

Constant Magnitude

Principle of Work and Energy

Problems that involve velocity, force, and displacement can be solved using the principle of work and energy. For application, a free-body diagram should be drawn in order to account for the work of all of the forces and couple moments that act on the body as it moves along the path.

$$T_1 + \Sigma U_{1-2} = T_2$$

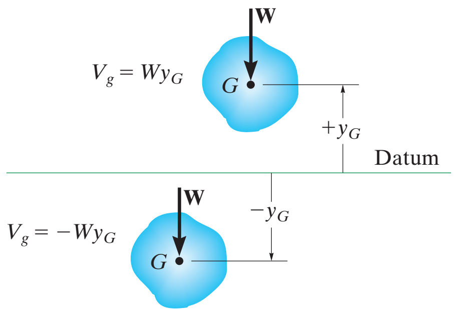
Conservation of Energy

If a rigid body is subjected only to conservative forces, then the conservation-of-energy equation can be used to solve the problem. This equation requires that the sum of the potential and kinetic energies of the body remain the same at any two points along the path.

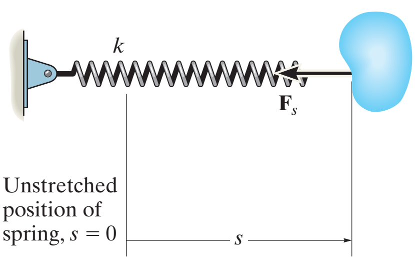
$$T_1 + V_1 = T_2 + V_2$$

where $V = V_g + V_e$

The potential energy is the sum of the body's gravitational and elastic potential energies. The gravitational potential energy will be positive if the body's center of gravity is located above a datum. If it is below the datum, then it will be negative. The elastic potential energy is always positive, regardless if the spring is stretched or compressed.



Gravitational potential energy

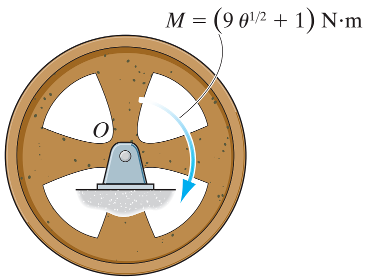


$$V_e = \frac{1}{2}ks^2$$

Elastic potential energy

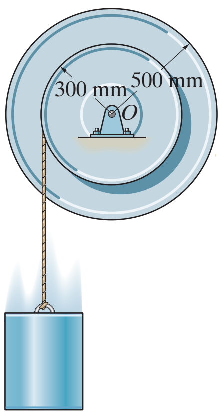
REVIEW PROBLEMS

R18–1. The 50-kg flywheel has a radius of gyration of $k_O = 200$ mm about its center of mass. If it is subjected to a torque of $M = (9\theta^{1/2} + 1)$ N · m, where θ is in radians, determine its angular velocity when it has rotated 5 revolutions, starting from rest.



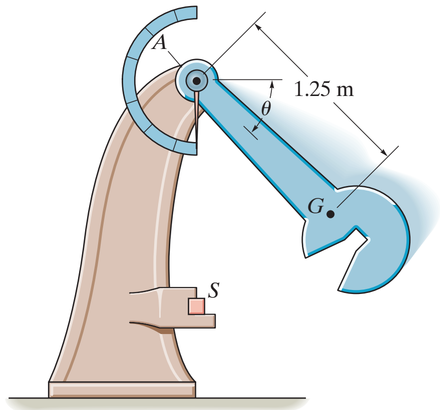
Prob. R18–1

R18–2. The spool has a mass of 40 kg and a radius of gyration of $k_O = 0.3$ m. If the 10-kg block is released from rest, determine the distance the block must fall in order for the spool to have an angular velocity $\omega = 15$ rad/s. Also, what is the tension in the cord while the block is in motion? Neglect the mass of the cord.



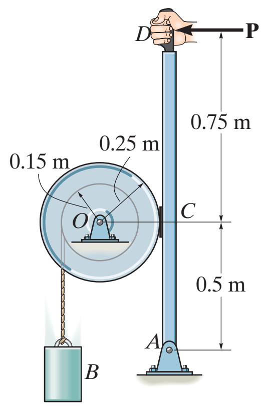
Prob. R18–2

R18–3. The pendulum of the Charpy impact machine has a mass of 50 kg and a radius of gyration of $k_A = 1.75$ m. If it is released from rest when $\theta = 0°$, determine its angular velocity just before it strikes the specimen S, $\theta = 90°$.



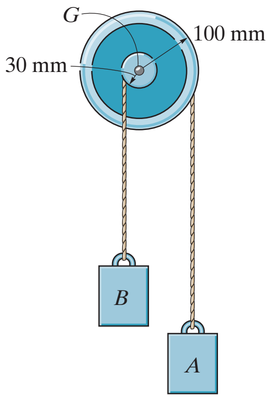
Prob. R18–3

R18–4. The drum has a mass of 50 kg and a radius of gyration about the pin at O of $k_O = 0.23$ m. Starting from rest, the suspended 15-kg block B is allowed to fall 3 m without applying the brake ACD. Determine the speed of the block at this instant. If the coefficient of kinetic friction at the brake pad C is $\mu_k = 0.5$, determine the force P that must be applied at the brake handle which will then stop the block after it descends *another* 3 m. Neglect the thickness of the handle.



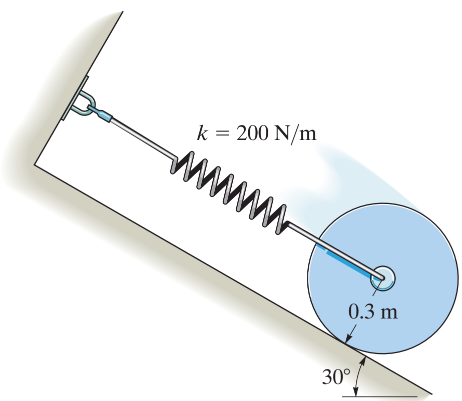
Prob. R18–4

R18–5. The compound disk pulley consists of a hub and attached outer rim. If it has a mass of 3 kg and a radius of gyration of $k_G = 45$ mm, determine the speed of block A after A descends 0.2 m from rest. Blocks A and B each have a mass of 2 kg. Neglect the mass of the cords.



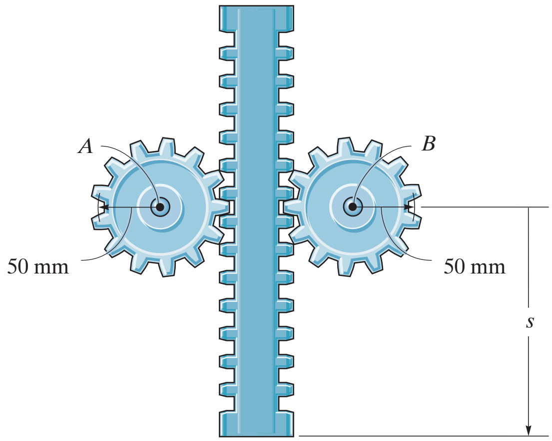
Prob. R18–5

R18–6. At the instant the spring becomes undeformed, the center of the 40-kg disk has a speed of 4 m/s. From this point determine the distance d the disk moves down the plane before momentarily stopping. The disk rolls without slipping.



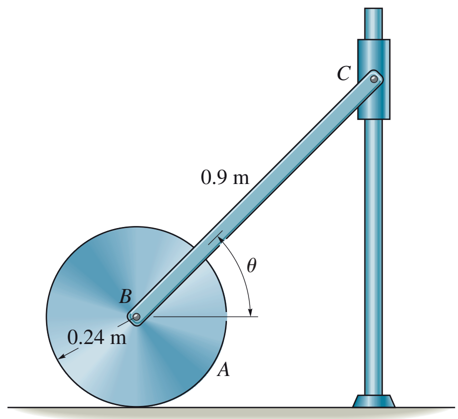
Prob. R18–6

R18–7. The gear rack has a mass of 6 kg, and the gears each have a mass of 4 kg and a radius of gyration of $k = 30$ mm at their centers. If the rack is originally moving downward at 2 m/s, when $s = 0$, determine the speed of the rack when $s = 600$ mm. The gears are free to turn about their centers A and B.



Prob. R18–7

R18–8. The system consists of a 10-kg disk A, 2-kg slender rod BC, and a 0.5-kg smooth collar C. If the disk rolls without slipping, determine the velocity of the collar at the instant the rod becomes horizontal, i.e., $\theta = 0°$. The system is released from rest when $\theta = 45°$.



Prob. R18–8

CHAPTER 19

The astronaut maneuvers in space by applying short impulses using the jetpack. The results of the motion can be determined using impulse and momentum methods.

PLANAR KINETICS OF A RIGID BODY: IMPULSE AND MOMENTUM

Lecture Summary and Quiz, Example, and Problem-solving videos are available where this icon appears.

CHAPTER OBJECTIVES

- To develop formulations for the linear and angular momentum of a body.
- To apply the principles of linear and angular impulse and momentum to solve rigid-body planar kinetic problems.
- To discuss application of the conservation of momentum.
- To analyze the mechanics of eccentric impact.

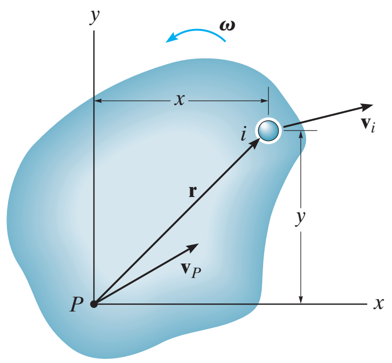
19.1 LINEAR AND ANGULAR MOMENTUM

In this chapter we will use the principles of linear and angular impulse and momentum to solve problems involving force, velocity, and time as it relates to the planar motion of a rigid body. Before doing this, we will first formalize the methods for obtaining a body's linear and angular momentum, assuming the body is symmetric with respect to an inertial x–y reference plane.

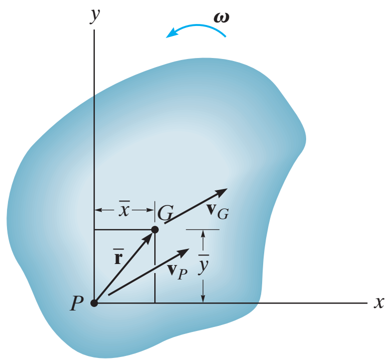
Linear Momentum. The linear momentum of a rigid body is determined by summing vectorially the linear momenta of all the particles of the body, i.e., $\mathbf{L} = \Sigma m_i \mathbf{v}_i$. Since $\Sigma m_i \mathbf{v}_i = m\mathbf{v}_G$ (see Sec. 15.2) we can then reference the linear momentum from the mass center, that is,

$$\boxed{\mathbf{L} = m\mathbf{v}_G} \qquad (19\text{–}1)$$

This equation states that the body's linear momentum is a vector quantity having a magnitude mv_G, which is commonly measured in units of $\text{kg} \cdot \text{m/s}$, and a direction defined by $\mathbf{v}_G$, the velocity of the body's mass center.



(a)



(b)

Fig. 19–1

Angular Momentum. Consider the body in Fig. 19–1*a*, which is subjected to general plane motion. At the instant shown, the arbitrary ith particle of the body has a mass m_i and a velocity $\mathbf{v}_i$. The angular momentum of this particle about the point P is equal to the "moment" of the particle's linear momentum about P, that is,

$$(\mathbf{H}_P)_i = \mathbf{r} \times m_i \mathbf{v}_i$$

This momentum can also be expressed in terms of the velocity of point P rather than the velocity $\mathbf{v}_i$. If the body has an angular velocity $\boldsymbol{\omega}$, then from Fig. 19–1*a*, we have

$$\mathbf{v}_i = \mathbf{v}_P + \mathbf{v}_{i/P} = \mathbf{v}_P + \boldsymbol{\omega} \times \mathbf{r}$$

Letting $m_i \rightarrow dm_i$, expressing $\mathbf{v}_i$ in terms of $\mathbf{v}_P$ and using Cartesian vectors, we have

$$(H_P)_i\mathbf{k} = m_i(x\mathbf{i} + y\mathbf{j}) \times [(v_P)_x\mathbf{i} + (v_P)_y\mathbf{j} + \omega\mathbf{k} \times (x\mathbf{i} + y\mathbf{j})]$$
$$(H_P)_i = -dm_i y(v_P)_x + dm_i x(v_P)_y + dm_i \omega r^2$$

Integrating over the entire mass m of the body, we obtain

$$H_P = -\left(\int_m y\, dm\right)(v_P)_x + \left(\int_m x\, dm\right)(v_P)_y + \left(\int_m r^2\, dm\right)\omega$$

Here H_P represents the angular momentum of the body about an axis (the z axis) perpendicular to the plane of motion that passes through point P. Since $\bar{y}m = \int y\, dm$ and $\bar{x}m = \int x\, dm$, the integrals for the first and second terms on the right are used to locate the body's center of mass G with respect to P, Fig. 19–1*b*. The last integral represents the body's moment of inertia about point P. Thus,

$$H_P = -\bar{y}m(v_P)_x + \bar{x}m(v_P)_y + I_P\omega \tag{19–2}$$

This equation reduces to a simpler form if P coincides with the mass center G for the body,* in which case $\bar{x} = \bar{y} = 0$. Hence,

$$\boxed{H_G = I_G\omega} \tag{19–3}$$

* It also reduces to a simpler form, $H_P = I_P\omega$, if point P is a *fixed point* (see Eq. 19–9) or if the velocity of P is directed along the line PG.

Here the angular momentum of the body about G is equal to the product of the moment of inertia of the body about an axis passing through G and the body's angular velocity. Realize that $\mathbf{H}_G$ is a vector quantity having a magnitude $I_G\omega$, which is commonly measured in units of $\text{kg}\cdot\text{m}^2/\text{s}$, and a direction defined by $\boldsymbol{\omega}$, which is always perpendicular to the plane of motion.

Equation 19–2 can also be rewritten in terms of the x and y components of the velocity of the body's mass center, $(\mathbf{v}_G)_x$ and $(\mathbf{v}_G)_y$, and the body's moment of inertia I_G. Since G is located at coordinates $(\bar{x},\bar{y})$, then by the parallel-axis theorem, $I_P = I_G + m(\bar{x}^2 + \bar{y}^2)$. Substituting into Eq. 19–2 and rearranging terms, we have

$$H_P = \bar{y}m[-(v_P)_x + \bar{y}\omega] + \bar{x}m[(v_P)_y + \bar{x}\omega] + I_G\omega \qquad (19\text{–}4)$$

From the kinematic diagram of Fig. 19–1b, $\mathbf{v}_G$ can be expressed in terms of $\mathbf{v}_P$ as

$$\mathbf{v}_G = \mathbf{v}_P + \boldsymbol{\omega} \times \bar{\mathbf{r}}$$
$$(v_G)_x\mathbf{i} + (v_G)_y\mathbf{j} = (v_P)_x\mathbf{i} + (v_P)_y\mathbf{j} + \omega\mathbf{k} \times (\bar{x}\mathbf{i} + \bar{y}\mathbf{j})$$

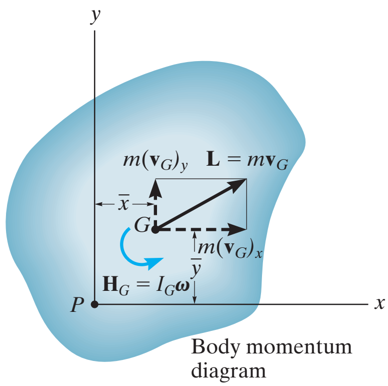
Carrying out the cross product and equating the respective $\mathbf{i}$ and $\mathbf{j}$ components yields the two scalar equations

$$(v_G)_x = (v_P)_x - \bar{y}\omega$$
$$(v_G)_y = (v_P)_y + \bar{x}\omega$$

Substituting these results into Eq. 19–4 yields

$$(\circlearrowleft +)H_P = -\bar{y}m(v_G)_x + \bar{x}m(v_G)_y + I_G\omega \qquad (19\text{–}5)$$

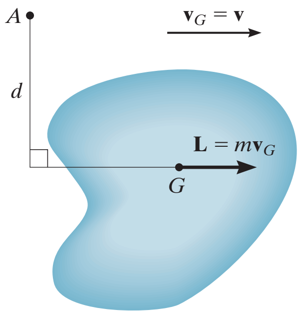
As shown in Fig. 19–1c, *this result indicates that when the angular momentum of the body is calculated about point P, it is equivalent to the moment of the linear momentum* $m\mathbf{v}_G$*, or its components* $m(\mathbf{v}_G)_x$ *and* $m(\mathbf{v}_G)_y$*, about P plus the angular momentum* $I_G\boldsymbol{\omega}$. Using these results, we will now consider three types of motion.



Body momentum diagram

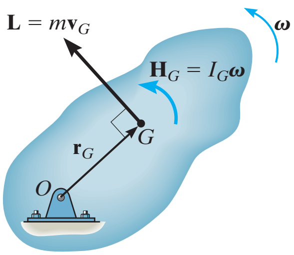
(c)

Fig. 19–1



Translation

(a)



Rotation about a fixed axis

(b)

Fig. 19–2

Translation.

When a rigid body is subjected to either rectilinear or curvilinear translation, Fig. 19–2a, then $\boldsymbol{\omega} = \mathbf{0}$ and its mass center has a velocity of $\mathbf{v}_G = \mathbf{v}$. Hence, the linear momentum, and the angular momentum about G, become

$$\boxed{\begin{aligned} L &= mv_G \\ H_G &= 0 \end{aligned}} \tag{19–6}$$

If the angular momentum is calculated about some other point A, the "moment" of the linear momentum $\mathbf{L}$ must be found about the point. Since d is the "moment arm," as shown in Fig. 19–2a, then in accordance with Eq. 19–5, $H_A = (d)(mv_G)\circlearrowleft$.

Rotation About a Fixed Axis.

When a rigid body is rotating about a fixed axis, Fig. 19–2b, the linear momentum, and the angular momentum about G, are

$$\boxed{\begin{aligned} L &= mv_G \\ H_G &= I_G\omega \end{aligned}} \tag{19–7}$$

It is sometimes convenient to calculate the angular momentum about the pin at O. Noting that $\mathbf{L}$ (or $\mathbf{v}_G$) is always *perpendicular to* $\mathbf{r}_G$, we have

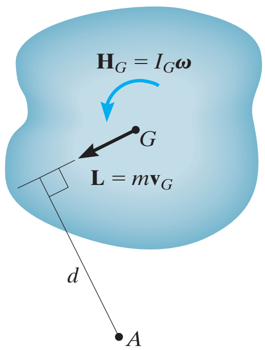
$$(\circlearrowleft +)\; H_O = I_G\omega + r_G(mv_G) \tag{19–8}$$

Since $v_G = r_G\omega$, this equation can be written as $H_O = (I_G + mr_G^2)\omega$. Using the parallel-axis theorem,*

$$\boxed{H_O = I_O\omega} \tag{19–9}$$

For the calculation, then, either Eq. 19–8 or 19–9 can be used.

*The similarity between this derivation and that of Eq. 17–16 ($\Sigma M_O = I_O\alpha$) and Eq. 18–5 $\left(T = \frac{1}{2}I_O\omega^2\right)$ should be noted. Also, this same result can be obtained from Eq. 19–2 by selecting point P at O, realizing that $(v_O)_x = (v_O)_y = 0$.



General plane motion

(c)

Fig. 19–2

General Plane Motion. When a rigid body is subjected to general plane motion, Fig. 19–2*c*, the linear momentum, and the angular momentum about G, become

$$\begin{aligned} L &= mv_G \\ H_G &= I_G\omega \end{aligned} \tag{19–10}$$

If the angular momentum is calculated about point A, Fig. 19–2*c*, it is necessary to include the moment of $\mathbf{L}$ and $\mathbf{H}_G$ about this point. In this case,

$$(\circlearrowright+) \quad H_A = I_G\omega + (d)(mv_G)$$

Here d is the moment arm, as shown in the figure.

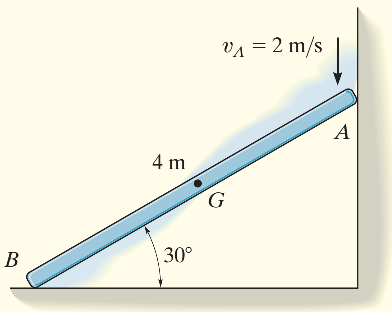
As a special case, if point A is the instantaneous center of zero velocity, then, like Eq. 19–9, we can write the above equation in simplified form as

$$H_{IC} = I_{IC}\omega \tag{19–11}$$

where I_{IC} is the moment of inertia of the body about the IC. (See Prob. 19–1.)

EXAMPLE 19.1

At a given instant the 5-kg slender bar has the motion shown in Fig. 19–3*a*. Determine its angular momentum about point *G* and about the *IC* at this instant.



(a)

SOLUTION

Bar. The bar undergoes *general plane motion*. The *IC* is established in Fig. 19–3*b*, so that

$$\omega = \frac{2\text{ m/s}}{4\text{ m}\cos 30^\circ} = 0.5774\text{ rad/s}$$

$$v_G = (0.5774\text{ rad/s})(2\text{ m}) = 1.155\text{ m/s}$$

Thus,

$$(\circlearrowright +)H_G = I_G\omega = \left[\tfrac{1}{12}(5\text{ kg})(4\text{ m})^2\right](0.5774\text{ rad/s}) = 3.85\text{ kg}\cdot\text{m}^2/\text{s}\circlearrowright$$

Ans.

The angular momentum about the *IC* is

$$(\circlearrowright +)H_{IC} = I_G\omega + d(mv_G)$$

$$= \left[\tfrac{1}{12}(5\text{ kg})(4\text{ m})^2\right](0.5774\text{ rad/s}) + (2\text{ m})(5\text{ kg})(1.155\text{ m/s})$$

$$= 15.4\text{ kg}\cdot\text{m}^2/\text{s}\circlearrowright$$

Ans.

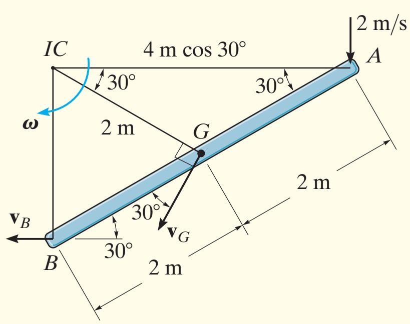
We can also use

$$(\circlearrowright +)H_{IC} = I_{IC}\omega$$

$$= \left[\tfrac{1}{12}(5\text{ kg})(4\text{ m})^2 + (5\text{ kg})(2\text{ m})^2\right](0.5774\text{ rad/s})$$

$$= 15.4\text{ kg}\cdot\text{m}^2/\text{s}\circlearrowright$$

Ans.



(b)

Fig. 19–3

19.2 PRINCIPLE OF IMPULSE AND MOMENTUM

Like the case for particle motion, the principle of impulse and momentum for a rigid body can be developed by combining the equation of motion with kinematics.

Principle of Linear Impulse and Momentum. The equation of translational motion for a rigid body can be written as $\Sigma\mathbf{F} = m\mathbf{a}_G = m(d\mathbf{v}_G/dt)$. Since the mass of the body is constant,

$$\Sigma\mathbf{F} = \frac{d}{dt}(m\mathbf{v}_G)$$

Multiplying both sides by dt and integrating from $t = t_1$, $\mathbf{v}_G = (\mathbf{v}_G)_1$ to $t = t_2$, $\mathbf{v}_G = (\mathbf{v}_G)_2$ yields

$$\Sigma\int_{t_1}^{t_2}\mathbf{F}\,dt = m(\mathbf{v}_G)_2 - m(\mathbf{v}_G)_1$$

This equation is referred to as the ***principle of linear impulse and momentum***. It states that the sum of all the impulses created by the external force system which acts on the body during the time interval t_1 to t_2 is equal to the change in the linear momentum of the body during this time interval.

Principle of Angular Impulse and Momentum. If the body has general plane motion then $\Sigma M_G = I_G\alpha = I_G(d\omega/dt)$. Since the moment of inertia is constant,

$$\Sigma M_G = \frac{d}{dt}(I_G\omega)$$

Multiplying both sides by dt and integrating from $t = t_1$, $\omega = \omega_1$ to $t = t_2$, $\omega = \omega_2$ gives

$$\Sigma\int_{t_1}^{t_2} M_G\,dt = I_G\omega_2 - I_G\omega_1 \qquad (19\text{–}12)$$

In a similar manner, for rotation about a fixed axis passing through point O, Eq. 17–16 ($\Sigma M_O = I_O\,(d\omega/dt)$) when integrated becomes

$$\Sigma\int_{t_1}^{t_2} M_O\,dt = I_O\omega_2 - I_O\omega_1 \qquad (19\text{–}13)$$

Equations 19–12 and 19–13 are referred to as the ***principle of angular impulse and momentum***. Both equations state that the sum of the angular impulses acting on the body during the time interval t_1 to t_2 is equal to the change in the body's angular momentum during this time interval.

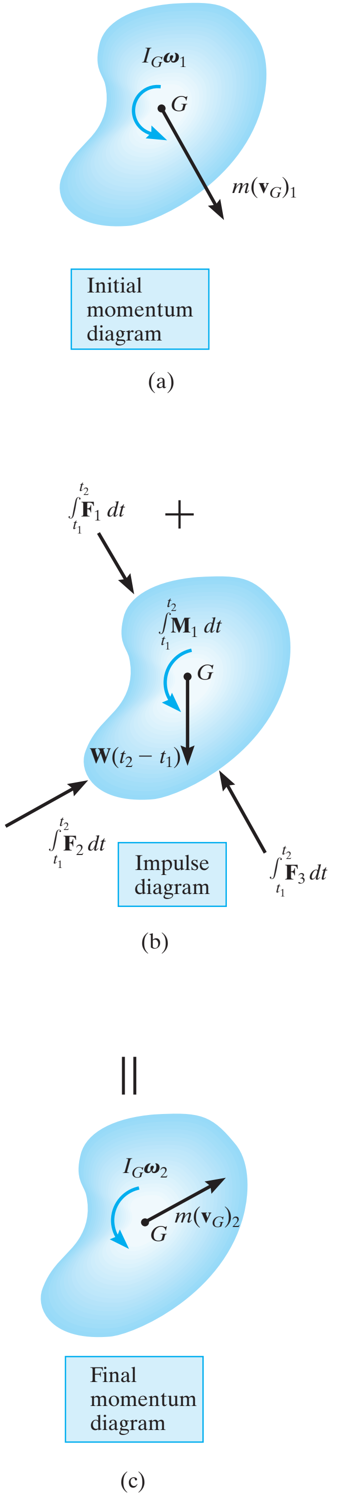


Fig. 19–4

To summarize these concepts, if motion occurs in the x–y plane, the following three scalar equations can be written to describe the planar motion of the body.

$$
\begin{aligned}
m(v_{Gx})_1 + \Sigma\int_{t_1}^{t_2} F_x\,dt &= m(v_{Gx})_2\\
m(v_{Gy})_1 + \Sigma\int_{t_1}^{t_2} F_y\,dt &= m(v_{Gy})_2\\
I_G\omega_1 + \Sigma\int_{t_1}^{t_2} M_G\,dt &= I_G\omega_2
\end{aligned}
\qquad (19\text{–}14)
$$

The terms in these equations can be shown graphically on the impulse and momentum diagrams for the body, Fig. 19–4. The linear momentum $m\mathbf{v}_G$ is applied at the body's mass center, Figs. 19–4a and 19–4c; whereas the angular momentum $I_G\boldsymbol{\omega}$ is a free vector, and therefore, like a couple moment, it can be applied at any point on the body. When the impulse diagram is constructed, Fig. 19–4b, the forces **F** and moment **M** vary with time, and are indicated by the integrals. However, if **F** and **M** are constant, then integration of the impulses yields $\mathbf{F}(t_2 - t_1)$ and $\mathbf{M}(t_2 - t_1)$. Such is the case for the body's weight **W**, Fig. 19–4b.

Equations 19–14 can also be applied to an entire system of connected bodies, rather than to each body separately. This eliminates the need to include interaction impulses which occur at the connections since they are internal to the system and cancel out. The resultant equations may be written in symbolic form as

$$
\begin{aligned}
\left(\Sigma\begin{matrix}\text{syst. linear}\\\text{momentum}\end{matrix}\right)_{x1} + \left(\Sigma\begin{matrix}\text{syst. linear}\\\text{impulse}\end{matrix}\right)_{x(1-2)} &= \left(\Sigma\begin{matrix}\text{syst. linear}\\\text{momentum}\end{matrix}\right)_{x2}\\
\left(\Sigma\begin{matrix}\text{syst. linear}\\\text{momentum}\end{matrix}\right)_{y1} + \left(\Sigma\begin{matrix}\text{syst. linear}\\\text{impulse}\end{matrix}\right)_{y(1-2)} &= \left(\Sigma\begin{matrix}\text{syst. linear}\\\text{momentum}\end{matrix}\right)_{y2}\\
\left(\Sigma\begin{matrix}\text{syst. angular}\\\text{momentum}\end{matrix}\right)_{O1} + \left(\Sigma\begin{matrix}\text{syst. angular}\\\text{impulse}\end{matrix}\right)_{O(1-2)} &= \left(\Sigma\begin{matrix}\text{syst. angular}\\\text{momentum}\end{matrix}\right)_{O2}
\end{aligned}
$$

(19–15)

As indicated by the third equation, the system's angular momentum and angular impulse must be calculated with respect to the same reference point O for all the bodies of the system.

PROCEDURE FOR ANALYSIS

Impulse and momentum principles are used to solve kinetic problems that involve *velocity, force*, and *time* since these terms are involved in the formulation.

Free-Body Diagram.

- Establish the x, y, z inertial frame of reference and draw the free-body diagram in order to account for all the forces and couple moments that produce impulses on the body.
- The direction and sense of the initial and final velocity of the body's mass center, $\mathbf{v}_G$, and the body's angular velocity $\boldsymbol{\omega}$ should be established. If any of these motions is unknown, assume that the sense of its components is in the direction of the positive inertial coordinates.
- Calculate the moment of inertia I_G or I_O.
- As an alternative procedure, draw the impulse and momentum diagrams for the body or system of bodies as shown in Fig. 19–4. These diagrams are particularly helpful in order to visualize the "moment" terms used in the principle of angular impulse and momentum, if application is about a fixed point O, the IC, or another point other than the body's mass center G.

Principle of Impulse and Momentum.

- Apply the three scalar equations of impulse and momentum.
- The angular momentum of a rigid body rotating about a fixed axis is the moment of $m\mathbf{v}_G$ plus $I_G\boldsymbol{\omega}$ about the axis. This is equal to $H_O = I_O\omega$, where I_O is the moment of inertia of the body about the axis.
- Forces that are functions of time must be integrated to obtain the impulse.
- The principle of angular impulse and momentum is often used to eliminate unknown impulsive forces that are parallel to or pass through a common axis, since the moment of these forces is zero about this axis.

Kinematics.

- If more than three equations are needed for a complete solution, it may be possible to relate the velocity of the body's mass center to the body's angular velocity using kinematics.

Refer to the companion website for Lecture Summary and Quiz videos.

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EXAMPLE 19.2

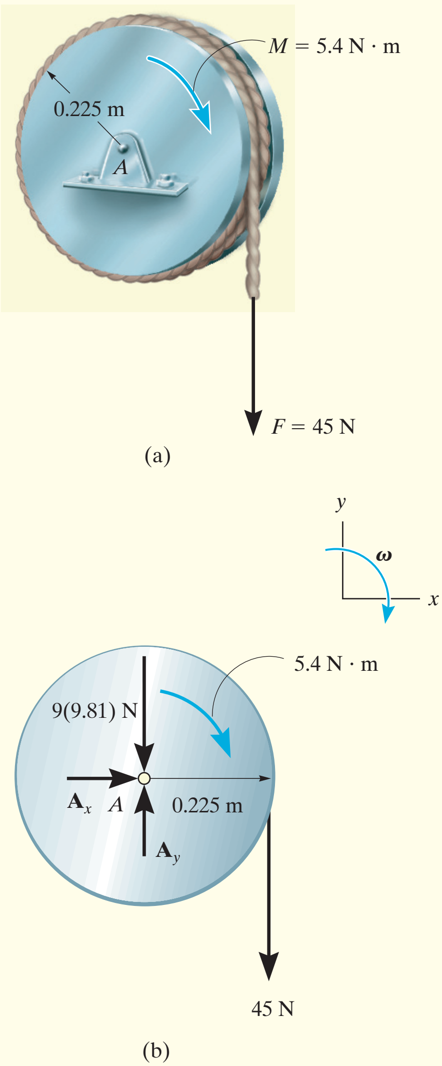


Fig. 19–5

The 9-kg disk shown in Fig. 19–5*a* is acted upon by a couple moment of 5.4 N · m and a force of 45 N which is applied to the cord wrapped around its periphery. Determine the angular velocity of the disk two seconds after starting from rest. Also, what are the force components of reaction at the pin?

SOLUTION

Since angular velocity, force, and time are involved in the problem, we will apply the principle of impulse and momentum to the solution.

Free-Body Diagram. Fig. 19–5*b*. The disk's mass center does not move; however, the loading causes the disk to rotate clockwise.

The moment of inertia of the disk about its fixed axis of rotation is

$$I_A = \frac{1}{2}mr^2 = \frac{1}{2}(9\text{ kg})\,(0.225\text{ m})^2 = 0.228\text{ kg}\cdot\text{m}^2$$

Principle of Impulse and Momentum.

$$(\overset{+}{\rightarrow}) \qquad m(v_{Ax})_1 + \Sigma\int_{t_1}^{t_2} F_x\,dt = m(v_{Ax})_2$$

$$0 + A_x(2\text{ s}) = 0$$

$$(+\uparrow) \qquad m(v_{Ay})_1 + \Sigma\int_{t_1}^{t_2} F_y\,dt = m(v_{Ay})_2$$

$$0 + A_y(2\text{ s}) - [9(9.81)\text{ N}](2\text{ s}) - 45\text{ N}(2\text{ s}) = 0$$

$$(\curvearrowright +) \qquad I_A\omega_1 + \Sigma\int_{t_1}^{t_2} M_A\,dt = I_A\omega_2$$

$$0 + 5.4\text{ N}\cdot\text{m}(2\text{ s}) + [45\text{ N}(2\text{ s})](0.225\text{ m}) = 0.228\omega_2$$

Solving these equations yields

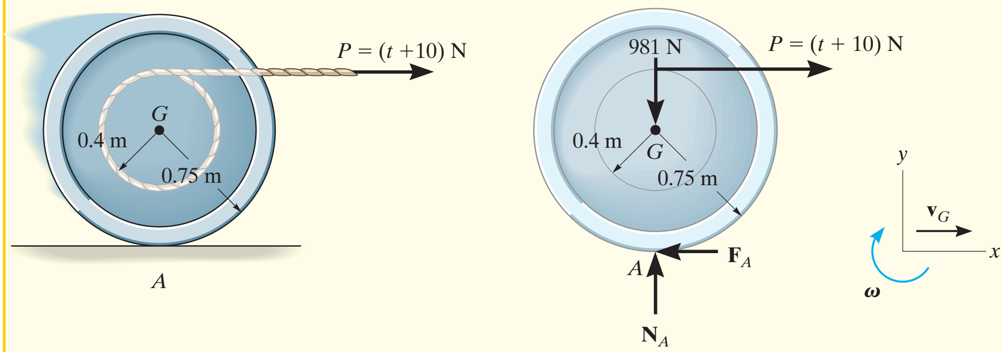
$$A_x = 0 \qquad \textit{Ans.}$$

$$A_y = 133\text{ N} \qquad \textit{Ans.}$$

$$\omega_2 = 136\text{ rad/s}\,\circlearrowright \qquad \textit{Ans.}$$

EXAMPLE 19.3

The 100-kg spool shown in Fig. 19–6*a* has a radius of gyration $k_G = 0.35$ m. A cable is wrapped around the central hub of the spool, and a horizontal force having a variable magnitude of $P = (t + 10)$ N is applied, where t is in seconds. If the spool is initially at rest, determine its angular velocity in 5 s. Assume that the spool rolls without slipping at A.



(a) (b)

Fig. 19–6

SOLUTION

Free-Body Diagram. From the free-body diagram, Fig. 19–6*b*, the *variable* force **P** will cause the friction force $\mathbf{F}_A$ to be variable, and thus the impulses created by both **P** and $\mathbf{F}_A$ must be determined by integration. Force **P** causes the mass center to have a velocity $\mathbf{v}_G$ to the right, and so the spool has a clockwise angular velocity $\boldsymbol{\omega}$.

Principle of Impulse and Momentum. A direct solution for $\boldsymbol{\omega}$ can be obtained by applying the principle of angular impulse and momentum about point A, the IC, in order to eliminate the unknown friction impulse.

$$(\curvearrowright +) \qquad I_A\omega_1 + \Sigma\int M_A\,dt = I_A\omega_2$$

$$0 + \left[\int_0^{5\,\text{s}} (t + 10)\text{ N}\,dt\right](0.75\text{ m} + 0.4\text{ m}) = [100\text{ kg}\,(0.35\text{ m})^2 + (100\text{ kg})(0.75\text{ m})^2]\omega_2$$

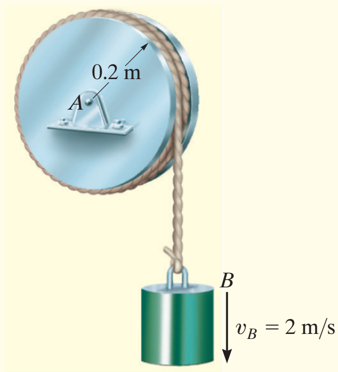
$$62.5(1.15) = 68.5\omega_2$$

$$\omega_2 = 1.05\text{ rad/s} \curvearrowright \qquad \textit{Ans.}$$

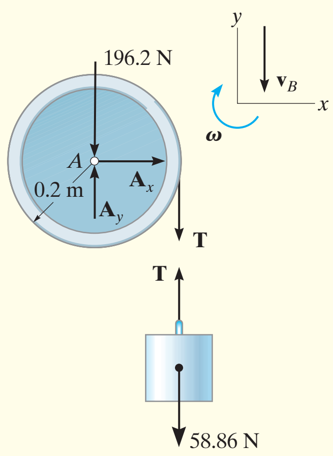
NOTE: Try solving this problem by applying the principle of impulse and momentum about G and using the principle of linear impulse and momentum in the x direction.

EXAMPLE 19.4

The cylinder B, shown in Fig. 19–7a has a mass of 6 kg. It is attached to a cord which is wrapped around the periphery of a 20-kg disk that has a moment of inertia $I_A = 0.40\ \text{kg}\cdot\text{m}^2$. If the cylinder is initially moving downward with a speed of 2 m/s, determine its speed in 3 s. Neglect the mass of the cord in the calculation.



(a)



(b)

Fig. 19–7

SOLUTION I

Free-Body Diagram. The free-body diagrams of the cylinder and disk are shown in Fig. 19–7b. All the forces are constant since the weight of the cylinder causes the motion. The downward motion of the cylinder, $\mathbf{v}_B$, causes $\boldsymbol{\omega}$ of the disk to be clockwise.

Principle of Impulse and Momentum. We can eliminate $\mathbf{A}_x$ and $\mathbf{A}_y$ from the analysis by applying the principle of angular impulse and momentum about point A. Hence

Disk

$$(\curvearrowright +) \qquad I_A\omega_1 + \Sigma \int M_A\, dt = I_A\omega_2$$

$$0.40\ \text{kg}\cdot\text{m}^2(\omega_1) + T(3\ \text{s})(0.2\ \text{m}) = (0.40\ \text{kg}\cdot\text{m}^2)\omega_2$$

Cylinder

$$(+\uparrow) \qquad m_B(v_B)_1 + \Sigma \int F_y\, dt = m_B(v_B)_2$$

$$-6\ \text{kg}(2\ \text{m/s}) + T(3\ \text{s}) - 58.86\ \text{N}(3\ \text{s}) = -6\ \text{kg}(v_B)_2$$

Kinematics. Since $\omega = v_B/r$, then $\omega_1 = (2\ \text{m/s})/(0.2\ \text{m}) = 10\ \text{rad/s}$ and $\omega_2 = (v_B)_2/0.2\ \text{m} = 5(v_B)_2$. Substituting and solving the equations simultaneously for $(v_B)_2$ yields

$$(v_B)_2 = 13.0\ \text{m/s} \downarrow \qquad \textit{Ans.}$$

Refer to the companion website for a self quiz of these Example problems.

SOLUTION II

Impulse and Momentum Diagrams. We can obtain $(v_B)_2$ directly by considering the *system* consisting of the cylinder, the cord, and the disk. The impulse and momentum diagrams have been drawn to clarify application of the principle of angular impulse and momentum about point A, Fig. 19–7*c*.

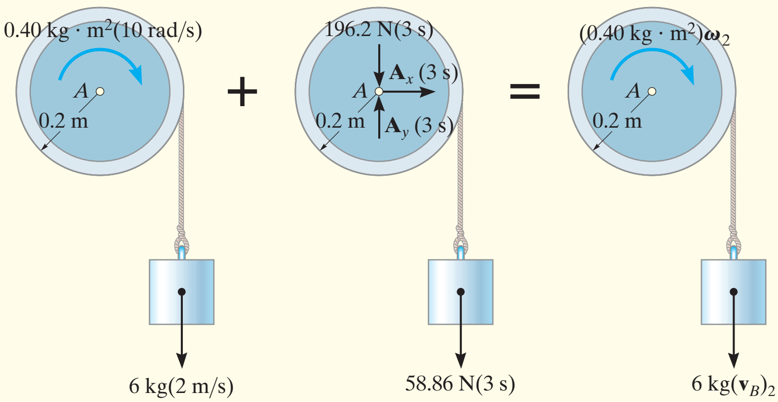
Principle of Angular Impulse and Momentum. Realizing that $\omega_1 = 10 \text{ rad/s}$ and $\omega_2 = 5(v_B)_2$, we have

$$(\circlearrowright +) \left(\Sigma \begin{matrix}\text{syst. angular}\\ \text{momentum}\end{matrix}\right)_{A1} + \left(\Sigma \begin{matrix}\text{syst. angular}\\ \text{impulse}\end{matrix}\right)_{A(1-2)} = \left(\Sigma \begin{matrix}\text{syst. angular}\\ \text{momentum}\end{matrix}\right)_{A2}$$

$$(6 \text{ kg})(2 \text{ m/s})(0.2 \text{ m}) + (0.40 \text{ kg}\cdot\text{m}^2)(10 \text{ rad/s}) + (58.86 \text{ N})(3 \text{ s})(0.2 \text{ m})$$

$$= (6 \text{ kg})(v_B)_2(0.2 \text{ m}) + (0.40 \text{ kg}\cdot\text{m}^2)[5(v_B)_2]$$

$$(v_B)_2 = 13.0 \text{ m/s} \downarrow \qquad \textit{Ans.}$$



(c)

Fig. 19–7

EXAMPLE 19.5

A Charpy impact test is used to test a material to determine its energy absorption characteristics during impact. The test is performed using the pendulum shown in Fig. 19–8*a*, which has a mass m, mass center at G, and a radius of gyration k_G about G. Determine the distance r_P from the pin at A to the point P where the impact with the specimen S should occur so that the horizontal force at the pin A is essentially zero during the impact. For the calculation, assume the specimen absorbs all the pendulum's kinetic energy during the time it falls and thereby stops the pendulum from swinging when $\theta = 0°$.

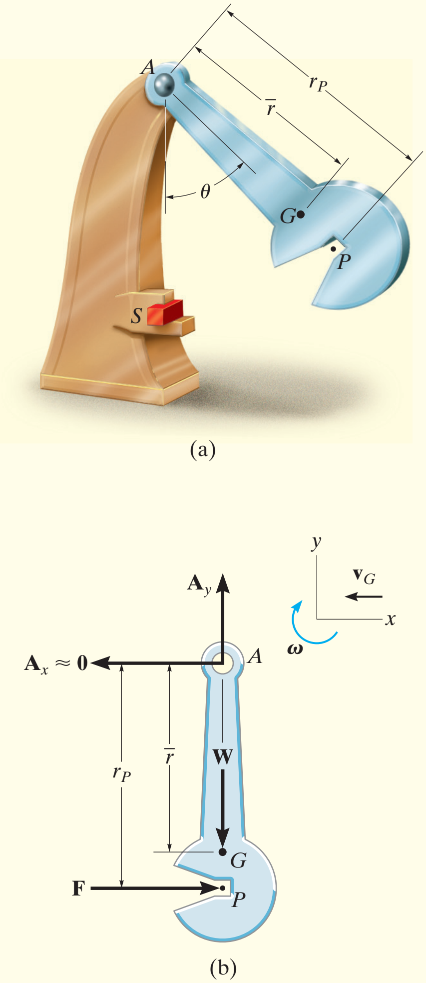


Fig. 19–8

SOLUTION

Free-Body Diagram. As shown on the free-body diagram, Fig. 19–8*b*, the conditions of the problem require the horizontal force at A to be zero. Just before impact, the pendulum has a clockwise angular velocity $\boldsymbol{\omega}_1$, and the mass center of the pendulum is moving to the left at $(v_G)_1 = \bar{r}\omega_1$.

Principle of Impulse and Momentum. We will apply the principle of angular impulse and momentum about point A. Thus,

$$I_A\omega_1 + \Sigma \int M_A \, dt = I_A\omega_2$$

$$(\circlearrowright +) \qquad I_A\omega_1 - \left(\int F\,dt\right) r_P = 0$$

$$m(v_G)_1 + \Sigma \int F\,dt = m(v_G)_2$$

$$(\overset{+}{\leftarrow}) \qquad -m(\bar{r}\omega_1) + \int F\,dt = 0$$

Eliminating the impulse $\int F\,dt$ and substituting $I_A = mk_G^2 + m\bar{r}^2$ yields

$$[mk_G^2 + m\bar{r}^2]\omega_1 - m(\bar{r}\omega_1)r_P = 0$$

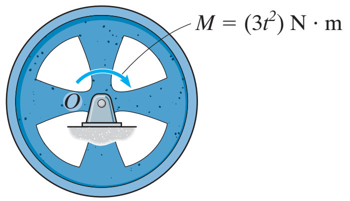
Factoring out $m\omega_1$ and solving for r_P, we obtain

$$r_P = \bar{r} + \frac{k_G^2}{\bar{r}} \qquad \textit{Ans.}$$

NOTE: Point P, so defined, is called the ***center of percussion***. By placing the striking point at P, the force developed at the pin will be minimized. Many sports rackets, clubs, etc., are designed so that collision with the object being struck occurs at the center of percussion. As a consequence, no "sting" or little sensation occurs in the hand of the player. (Also see Probs. 17–65 and 19–2.)

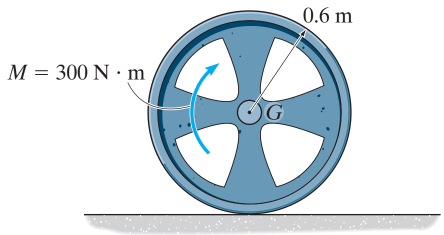
FUNDAMENTAL PROBLEMS

F19–1. The 60-kg wheel has a radius of gyration about its center O of $k_O = 300$ mm. If it is subjected to a couple moment of $M = (3t^2)$ N·m, where t is in seconds, determine the angular velocity of the wheel when $t = 4$ s, starting from rest.



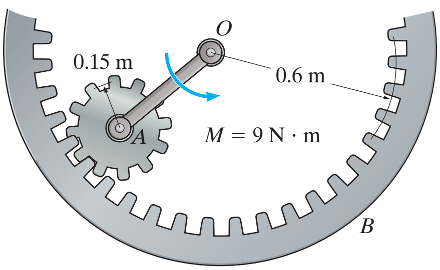
Prob. F19–1

F19–2. The 300-kg wheel has a radius of gyration about its mass center G of $k_G = 400$ mm. If the wheel is subjected to a couple moment of $M = 300$ N·m and no slipping occurs, determine its angular velocity 6 s after it starts from rest. Also, determine the friction force that the ground applies to the wheel.



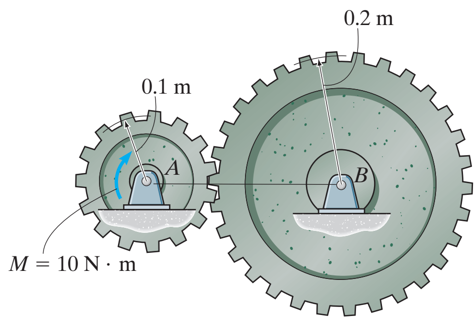
Prob. F19–2

F19–3. If rod OA of negligible mass is subjected to the couple moment $M = 9$ N·m, determine the angular velocity of the 10-kg inner gear $t = 5$ s after it starts from rest. The gear has a radius of gyration about its mass center of $k_A = 100$ mm, and it rolls on the fixed outer gear, B. Motion occurs in the horizontal plane.



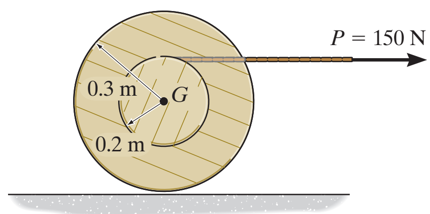
Prob. F19–3

F19–4. Gears A and B of mass 10 kg and 50 kg have radii of gyration about their respective mass centers of $k_A = 80$ mm and $k_B = 150$ mm. If gear A is subjected to the couple moment $M = 10$ N·m when it is at rest, determine the angular velocity of gear B when $t = 5$ s.



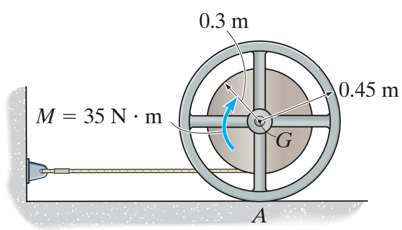
Prob. F19–4

F19–5. The 50-kg spool is subjected to a horizontal force of $P = 150$ N. If the spool rolls without slipping, determine its angular velocity 3 s after it starts from rest. The radius of gyration of the spool about its center of mass is $k_G = 175$ mm.



Prob. F19–5

F19–6. The reel has a mass of 70 kg and a radius of gyration about its center of gravity of $k_G = 0.375$ m. If it is subjected to a torque of $M = 35$ N·m, and starts from rest when the torque is applied, determine its angular velocity in 3 seconds. The coefficient of kinetic friction between the reel and the horizontal plane is $\mu_k = 0.15$.

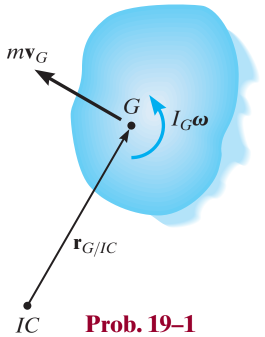


Prob. F19–6

PROBLEMS

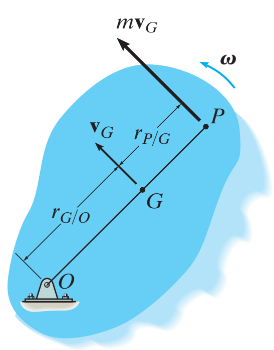
All solutions must include a free-body diagram.

19–1. At a given instant, the body has a linear momentum $\mathbf{L} = m\mathbf{v}_G$ and an angular momentum $\mathbf{H}_G = I_G\boldsymbol{\omega}$ calculated about its mass center. Show that the angular momentum of the body calculated about the instantaneous center of zero velocity IC can be expressed as $\mathbf{H}_{IC} = I_{IC}\boldsymbol{\omega}$, where I_{IC} represents the body's moment of inertia calculated about the instantaneous axis of zero velocity. As shown, the IC is located at a distance $r_{G/IC}$ away from the mass center G.



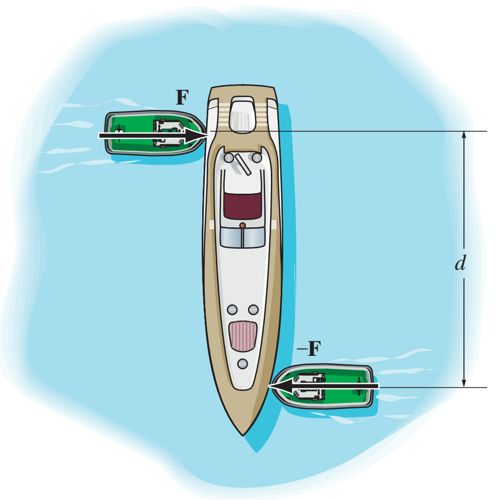
Prob. 19–1

19–2. The rigid body (slab) has a mass m and rotates with an angular velocity $\boldsymbol{\omega}$ about an axis passing through the fixed point O. Show that the momenta of all the particles composing the body can be represented by a single vector having a magnitude mv_G and acting through point P, called the *center of percussion*, which lies at a distance $r_{P/G} = k_G^2/r_{G/O}$ from the mass center G. Here k_G is the radius of gyration of the body, calculated about an axis perpendicular to the plane of motion and passing through G.



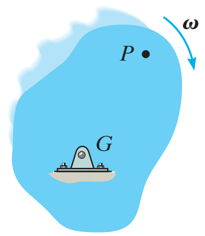
Prob. 19–2

19–3. The two tugboats each exert a constant force $\mathbf{F}$ on the boat such that these forces are always directed perpendicular to the boat's centerline. If the boat has a mass m and a radius of gyration k_G about its center of mass G, determine the angular velocity of the boat as a function of time t. The boat is originally at rest.



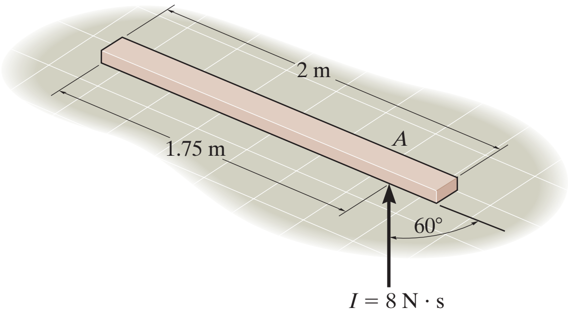
Prob. 19–3

***19–4.** Show that if a slab is rotating about a fixed axis perpendicular to the slab and passing through its mass center G, the angular momentum is the same when calculated about any other point P.



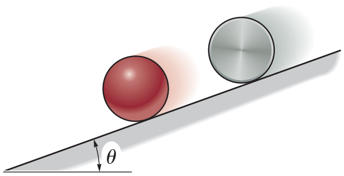
Prob. 19–4

19–5. The 4-kg slender rod rests on a smooth floor. If it is kicked so as to receive a horizontal impulse $I = 8\ \text{N}\cdot\text{s}$ at point A as shown, determine its angular velocity and the speed of its mass center.



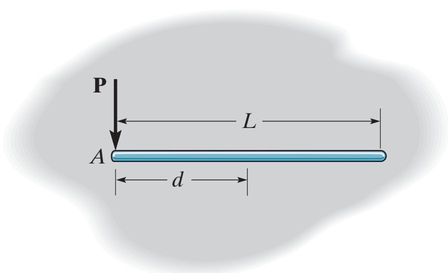
Prob. 19–5

19–6. A sphere and cylinder are released from rest on the ramp at $t = 0$. If each has a mass m and a radius r, determine their angular velocities at time t. Assume no slipping occurs.



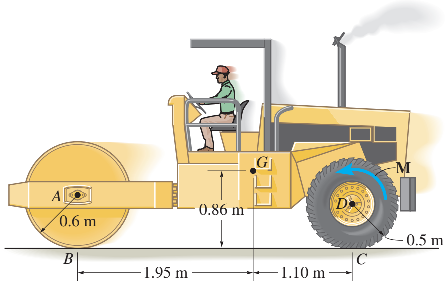
Prob. 19–6

19–7. The rod of length L and mass m lies on a smooth horizontal surface and is subjected to a force $\mathbf{P}$ at its end A as shown. Determine the location d of the point about which the rod begins to turn, i.e, the point that has zero velocity.



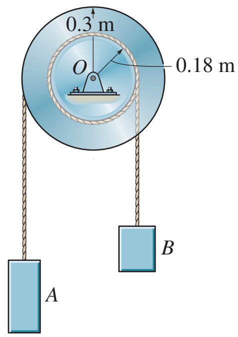
Prob. 19–7

***19–8.** The frame of the roller has a mass of 5.5 Mg and a center of mass at G. The roller has a mass of 2 Mg and a radius of gyration about its mass center of $k_A = 0.45$ m. If a torque of $M = 600\ \text{N}\cdot\text{m}$ is applied to the rear wheels, determine the speed of the compactor in $t = 4$ s, starting from rest. No slipping occurs. Neglect the mass of the driving wheels.



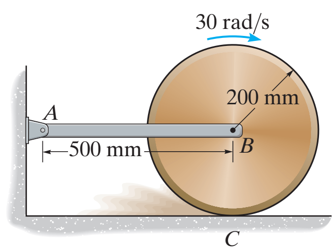
Prob. 19–8

19–9. The spool has a mass of 30 kg and a radius of gyration $k_O = 0.25$ m. Block A has a mass of 25 kg, and block B has a mass of 10 kg. If they are released from rest, determine the time required for block A to attain a speed of 2 m/s. Neglect the mass of the ropes.



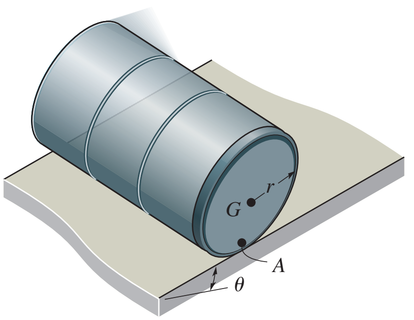
Prob. 19–9

19–10. The 50-kg cylinder has an angular velocity of 30 rad/s when it is brought into contact with the surface at C. If the coefficient of kinetic friction is $\mu_C = 0.2$, determine how long it takes for the cylinder to stop spinning. What force is developed in the link AB during this time? The axis of the cylinder is connected to *two* symmetrical links. (Only AB is shown.) Neglect the weight of the links.



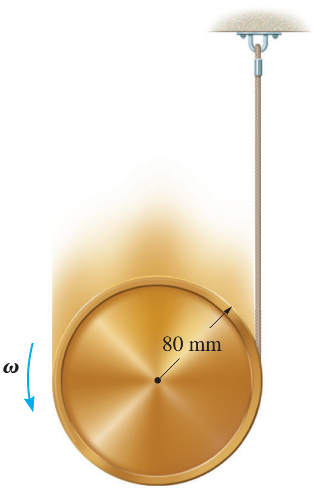
Prob. 19–10

19–11. The drum of mass m, radius r, and radius of gyration k_G rolls along an inclined plane for which the coefficient of static friction is μ_s. If the drum is released from rest, determine the maximum angle θ for the incline so that it rolls without slipping.



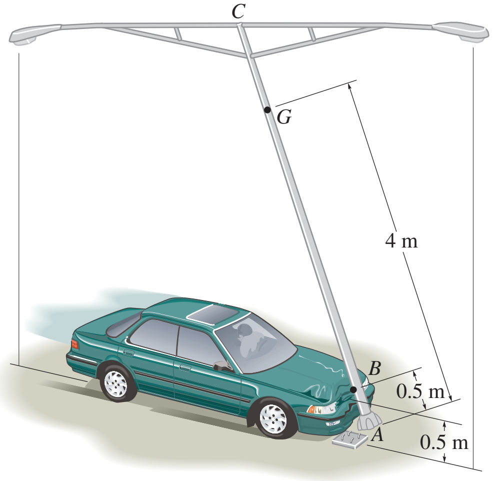
Prob. 19–11

***19–12.** A wire of negligible mass is wrapped around the outer surface of the 2-kg disk. If the disk is released from rest, determine its angular velocity in 3 s.



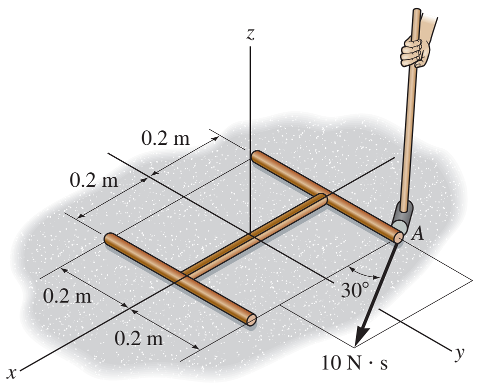
Prob. 19–12

19–13. The car strikes the side of a light pole, which is designed to break away from its base with negligible resistance. From a video taken of the collision it is observed that the pole was given an angular velocity of 60 rad/s when AC was vertical. The pole has a mass of 175 kg, a center of mass at G, and a radius of gyration about an axis perpendicular to the plane of the pole assembly and passing through G of $k_G = 2.25$ m. Determine the horizontal impulse which the car exerts on the pole at the instant AC is essentially vertical.



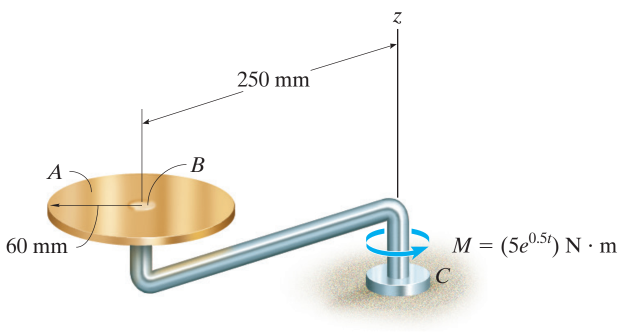
Prob. 19–13

19–14. The smooth rod assembly shown is at rest when it is struck by a hammer at A with an impulse of 10 N · s. Determine the angular velocity of the assembly and the magnitude of velocity of its mass center immediately after it has been struck. The rods have a mass per unit length of 6 kg/m.



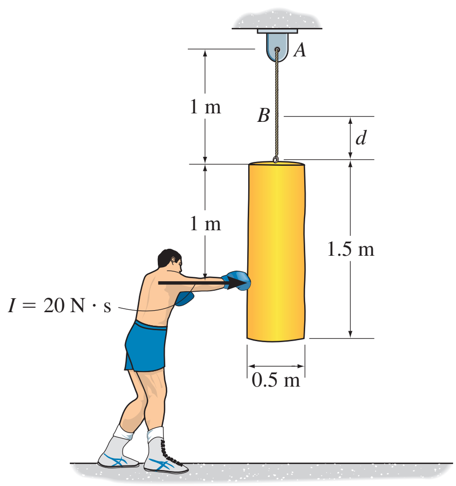
Prob. 19–14

19–15. A 4-kg disk A is mounted on arm BC, which has a negligible mass. If a torque of $M = (5e^{0.5t})$ N · m, where t is in seconds, is applied to the arm at C, determine the angular velocity of BC in 2 s starting from rest. Solve the problem assuming that (a) the disk is set in a smooth bearing at B so that it moves with curvilinear translation, (b) the disk is fixed to the shaft BC, and (c) the disk is given an initial freely spinning angular velocity of $\boldsymbol{\omega}_D = \{-80\mathbf{k}\}$ rad/s prior to application of the torque.



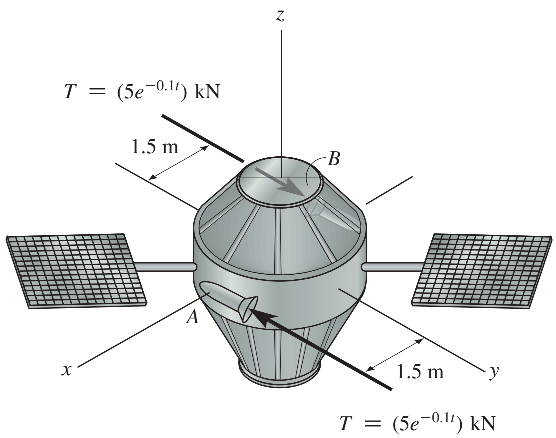
Prob. 19–15

***19–16.** If the boxer hits the 75-kg punching bag with an impulse of $I = 20$ N · s, determine the angular velocity of the bag immediately after it has been hit. Also, find the location d of point B, about which the bag appears to rotate. Treat the bag as a uniform cylinder.



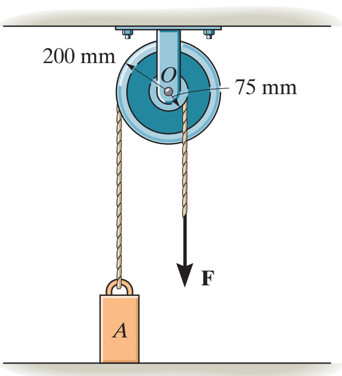
Prob. 19–16

19–17. The 200-kg satellite has a radius of gyration about the centroidal z axis of $k_z = 1.25$ m. Initially it is rotating with a constant angular velocity of $\boldsymbol{\omega}_0 = \{1500\mathbf{k}\}$ rev/min. If the two jets A and B, both in the x–y plane, are fired simultaneously and produce a thrust of $T = (5e^{-0.1t})$ kN, where t is in seconds, determine the angular velocity of the satellite, five seconds after firing.



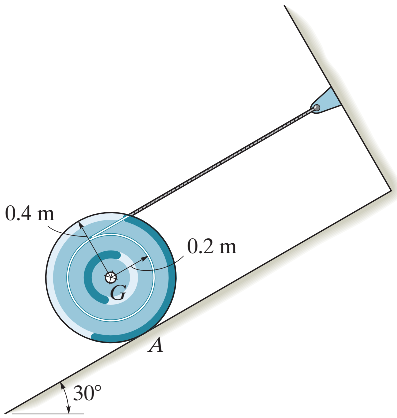
Prob. 19–17

19–18. The double pulley consists of two wheels which are attached to one another and turn at the same rate. The pulley has a mass of 15 kg and a radius of gyration $k_O = 110$ mm. If the block at A has a mass of 40 kg, determine the speed of the block in 3 s after a constant force $F = 2$ kN is applied to the rope wrapped around the inner hub of the pulley. The block is originally at rest. Neglect the mass of the rope.



Prob. 19–18

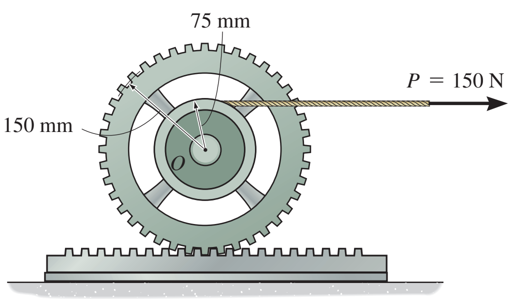
19–19. The 100-kg spool is resting on the inclined surface for which the coefficient of kinetic friction is $\mu_k = 0.1$. Determine the angular velocity of the spool when $t = 4$ s after it is released from rest. The radius of gyration about the mass center is $k_G = 0.25$ m.



Prob. 19–19

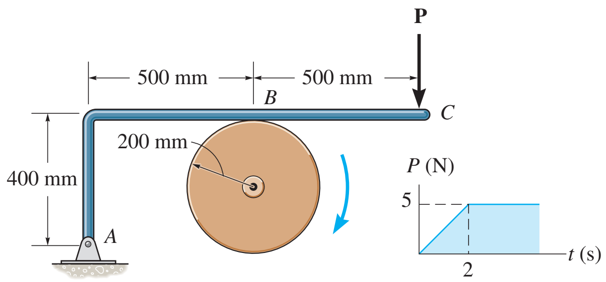
****19–20.** If the cord is subjected to a horizontal force of $P = 150$ N, and the gear rack is fixed to the horizontal plane, determine the angular velocity of the gear in 4 s, starting from rest. The mass of the gear is 50 kg, and it has a radius of gyration about its center of mass O of $k_O = 125$ mm.

19–21. If the cord is subjected to a horizontal force of $P = 150$ N, and the gear is supported by a fixed pin at O, determine the angular velocity of the gear and the velocity of the 20-kg gear rack in 4 s, starting from rest. The mass of the gear is 50 kg and it has a radius of gyration of $k_O = 125$ mm. Assume that the contact surface between the gear rack and the horizontal plane is smooth.



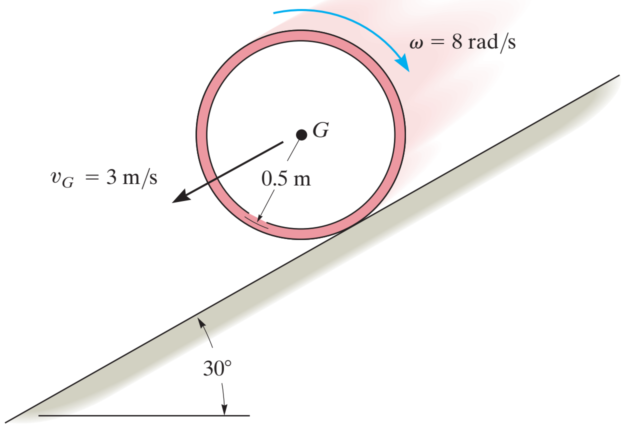
Probs. 19–20/21

19–22. The 12-kg disk has an angular velocity of $\omega = 20$ rad/s. If the brake ABC is applied such that the magnitude of force $\mathbf{P}$ varies with time as shown, determine the time needed to stop the disk. The coefficient of kinetic friction at B is $\mu_k = 0.4$. Neglect the thickness of the brake.



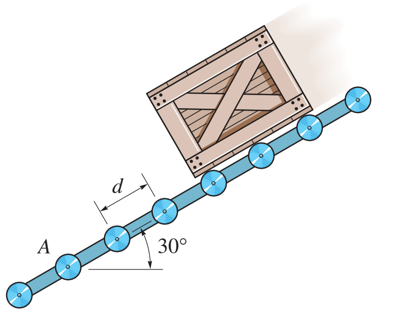
Prob. 19–22

19–23. The hoop (thin ring) has a mass of 5 kg and is released down the inclined plane such that it has a backspin $\omega = 8$ rad/s and its center has a velocity $v_G = 3$ m/s as shown. If the coefficient of kinetic friction between the hoop and the plane is $\mu_k = 0.6$, determine how long the hoop rolls before it stops slipping.



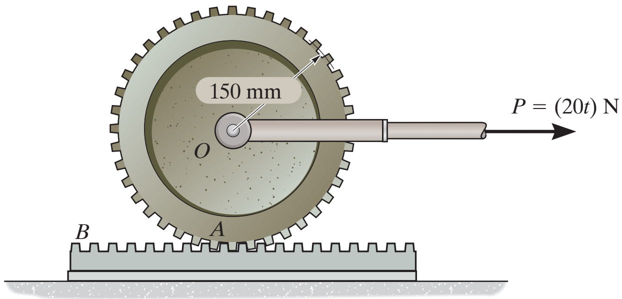
Prob. 19–23

*19–24. The crate has a mass m_c. Determine the constant speed v_0 it acquires as it moves down the conveyor. The rollers each have a radius of r, mass m, and are spaced d apart. Note that friction causes each roller to rotate when the crate comes in contact with it.



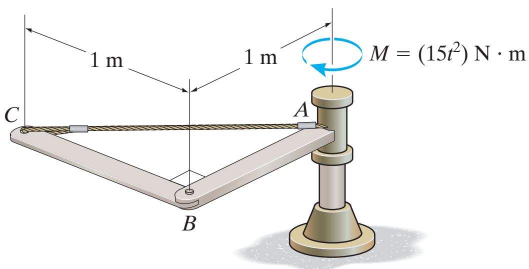
Prob. 19–24

19–25. The 30-kg gear is subjected to a force of $P = (20t)$ N, where t is in seconds. Determine the angular velocity of the gear at $t = 4$ s, starting from rest. Gear rack B is fixed to the horizontal plane, and the gear's radius of gyration about its mass center O is $k_O = 125$ mm.



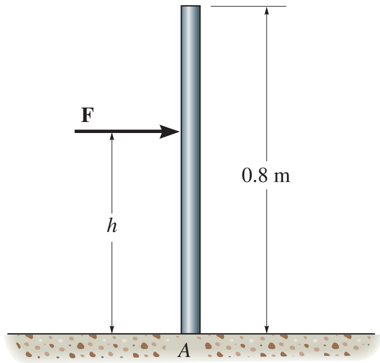
Prob. 19–25

19–26. If the shaft is subjected to a torque of $M = (15t^2)$ N·m, where t is in seconds, determine the angular velocity of the assembly when $t = 3$ s, starting from rest. Rods AB and BC each have a mass of 9 kg.



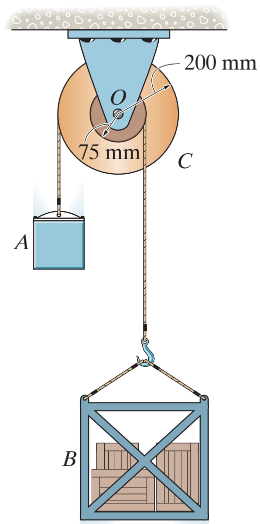
Prob. 19–26

19–27. A thin rod having a mass of 4 kg is balanced vertically as shown. Determine the height h at which it can be struck with a horizontal force **F** and not slip on the floor. This requires that the frictional force at A be essentially zero.



Prob. 19–27

*19–28. The double pulley consists of two wheels which are attached to one another and turn at the same rate. The pulley has a mass of 15 kg and a radius of gyration of $k_O = 110$ mm. If the block at A has a mass of 40 kg and the container at B has a mass of 85 kg, including its contents, determine the speed of the container when $t = 3$ s after it is released from rest.



Prob. 19–28

19.3 CONSERVATION OF MOMENTUM

Conservation of Linear Momentum. If the sum of all the *linear impulses* acting on a system of connected rigid bodies is *zero* in a specific direction, then the linear momentum of the system is constant, or conserved in this direction, that is,

$$\left(\sum \begin{matrix}\text{syst. linear}\\ \text{momentum}\end{matrix}\right)_1 = \left(\sum \begin{matrix}\text{syst. linear}\\ \text{momentum}\end{matrix}\right)_2 \qquad (19\text{–}16)$$

This equation is referred to as the ***conservation of linear momentum***.

Without introducing appreciable errors in the calculations, it may also be possible to apply Eq. 19–16 in a specified direction for which the linear impulses are small or *nonimpulsive*. Specifically, nonimpulsive forces occur when small forces act over very short periods of time. Typical examples include the force of a slightly deformed spring, the initial contact force with soft ground, and in some cases the weight of the body.

Conservation of Angular Momentum. The angular momentum of a system of connected rigid bodies is conserved about the system's center of mass G, or a fixed point O, when the sum of all the angular impulses about these points is zero or appreciably small (nonimpulsive). The third of Eqs. 19–15 then becomes

$$\left(\sum \begin{matrix}\text{syst. angular}\\ \text{momentum}\end{matrix}\right)_{O1} = \left(\sum \begin{matrix}\text{syst. angular}\\ \text{momentum}\end{matrix}\right)_{O2} \qquad (19\text{–}17)$$

This equation is referred to as the ***conservation of angular momentum***.

In the case of a single rigid body, Eq. 19–17 applied to point G becomes $(I_G\boldsymbol{\omega})_1 = (I_G\boldsymbol{\omega})_2$. For example, consider a swimmer who executes a somersault after jumping off a diving board. Angular momentum is conserved about his center of mass G since his resultant weight passes through this point. By tucking his arms and legs in close to his chest, he decreases his body's moment of inertia and thus increases his angular velocity ($I_G\boldsymbol{\omega}$ must be constant). If he straightens out just before entering the water, his body's moment of inertia is increased, and so his angular velocity decreases. Since his weight creates a linear impulse during the motion, this example also illustrates how the angular momentum of a body can be conserved and yet the linear momentum is not. Such cases occur whenever the external forces creating the linear impulse pass through either the center of mass of the body or a fixed axis of rotation.

PROCEDURE FOR ANALYSIS

The conservation of linear or angular momentum should be applied using the following procedure.

Free-Body Diagram.

- Establish the x, y inertial frame of reference and draw the free-body diagram for the body or system of bodies during the time of impact. From this diagram classify each of the applied forces as being either "impulsive" or "nonimpulsive."

- By inspection of the free-body diagram, the *conservation of linear momentum* applies in a given direction when *no* external impulsive forces act on the body or system in that direction; whereas the *conservation of angular momentum* applies about a fixed point O or at the mass center G of a body or system of bodies when all the external impulsive forces acting on the body or system create zero moment (or zero angular impulse) about O or G.

- As an alternative procedure, draw the impulse and momentum diagrams for the body or system of bodies. These diagrams are particularly helpful in order to visualize the "moment" terms used in the conservation of angular momentum equation, if it is decided that angular momenta are to be calculated about a point other than the body's mass center G.

Conservation of Momentum.

- Apply the conservation of linear or angular momentum in the appropriate directions.

Kinematics.

- If the motion appears to be complicated, kinematic (velocity) diagrams may be helpful in obtaining the necessary kinematic relations.

EXAMPLE 19.6

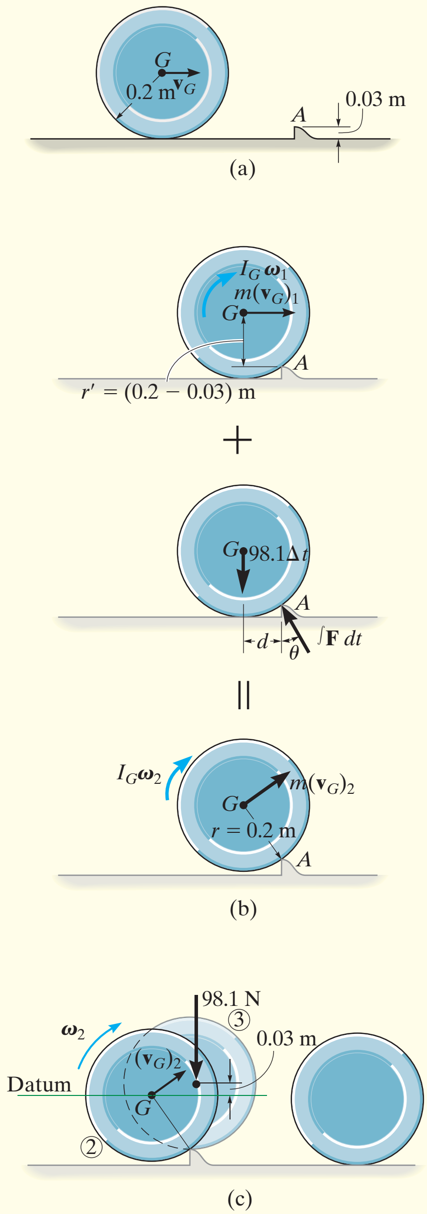


Fig. 19–9

The 10-kg wheel shown in Fig. 19–9*a* has a moment of inertia $I_G = 0.156 \text{ kg}\cdot\text{m}^2$. Assuming that the wheel does not slip or rebound, determine the minimum velocity $\mathbf{v}_G$ it must have to just roll over the obstruction at A.

SOLUTION

Impulse and Momentum Diagrams. Since no slipping or rebounding occurs, the wheel essentially *pivots* about point A during contact. This condition is shown in Fig. 19–9*b*, which indicates, respectively, the momentum of the wheel *just before impact*, the impulses given to the wheel *during impact*, and the momentum of the wheel *just after impact*. Only two impulses (forces) act on the wheel. By comparison, the force at A is much greater than that of the weight, and since the time of impact is very short, the weight can be considered nonimpulsive. The impulsive force $\mathbf{F}$ at A has both an unknown magnitude and an unknown direction θ. We can eliminate this force from the analysis by noting that angular momentum about A is essentially *conserved* since $(98.1\Delta t)d \approx 0$.

Conservation of Angular Momentum. With reference to Fig. 19–9*b*,

$$(\circlearrowleft+) \qquad (H_A)_1 = (H_A)_2$$

$$r'm(v_G)_1 + I_G\omega_1 = rm(v_G)_2 + I_G\omega_2$$

$$(0.2 \text{ m} - 0.03 \text{ m})(10 \text{ kg})(v_G)_1 + (0.156 \text{ kg}\cdot\text{m}^2)(\omega_1) =$$
$$(0.2 \text{ m})(10 \text{ kg})(v_G)_2 + (0.156 \text{ kg}\cdot\text{m}^2)(\omega_2)$$

Kinematics. Since no slipping occurs, in general $\omega = v_G/r = v_G/0.2 \text{ m} = 5v_G$. Substituting this into the above equation and simplifying yields

$$(v_G)_2 = 0.8921(v_G)_1 \qquad (1)$$

Conservation of Energy.* In order to roll over the obstruction, the wheel must pass position 3 shown in Fig. 19–9*c*. Hence, if $(v_G)_2$ [or $(v_G)_1$] is to be a minimum, it is necessary that the kinetic energy of the wheel at position 2 be equal to the potential energy at position 3. Placing the datum through the center of gravity, as shown in the figure, and applying the conservation of energy equation, we have

$$\{T_2\} + \{V_2\} = \{T_3\} + \{V_3\}$$

$$\left\{\tfrac{1}{2}(10 \text{ kg})(v_G)_2^2 + \tfrac{1}{2}(0.156 \text{ kg}\cdot\text{m}^2)\omega_2^2\right\} + \{0\} =$$
$$\{0\} + \{(98.1 \text{ N})(0.03 \text{ m})\}$$

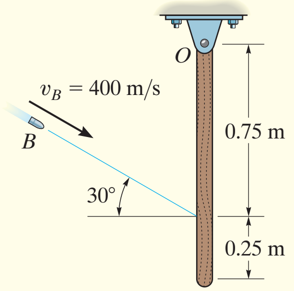
Substituting $\omega_2 = 5(v_G)_2$ and Eq. 1 into this equation, and solving,

$$(v_G)_1 = 0.729 \text{ m/s} \rightarrow \qquad \textit{Ans.}$$

* This principle *does not apply during impact*, since energy is *lost* during the collision. However, just after impact, as in Fig. 19–9*c*, it can be used.

EXAMPLE 19.7

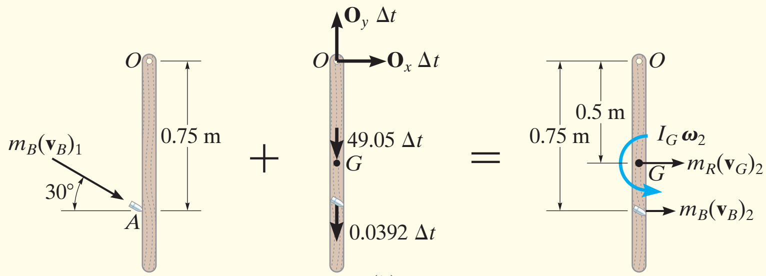
The 5-kg slender rod shown in Fig. 19–10*a* is pinned at O and is initially at rest. If a 4-g bullet is fired into the rod with a velocity of 400 m/s, as shown in the figure, determine the angular velocity of the rod just after the bullet becomes embedded in it.



(a)

SOLUTION

Impulse and Momentum Diagrams. The impulse which the bullet exerts on the rod can be eliminated from the analysis by considering the bullet and rod as a single system. To clarify the principles involved, the impulse and momentum diagrams are shown in Fig. 19–10*b*. The momentum diagrams are drawn *just before and just after impact.* As shown on the impulse diagram, the external impulses are caused by the reactions at O and the weights of the bullet and rod. Since the time of impact, Δt, is very short, the rod moves only a slight amount, and so the "moments" of the weight impulses about point O are essentially zero. Therefore angular momentum is conserved about this point.



(b)

Conservation of Angular Momentum. From Fig. 19–10*b*, we have

$$(\circlearrowright+) \qquad \Sigma(H_O)_1 = \Sigma(H_O)_2$$

$$m_B(v_B)_1 \cos 30°(0.75 \text{ m}) = m_B(v_B)_2(0.75 \text{ m}) + m_R(v_G)_2(0.5 \text{ m}) + I_G\omega_2$$

$$(0.004 \text{ kg})(400 \cos 30° \text{ m/s})(0.75 \text{ m}) =$$

$$(0.004 \text{ kg})(v_B)_2(0.75 \text{ m}) + (5 \text{ kg})(v_G)_2(0.5 \text{ m}) + \left[\tfrac{1}{12}(5 \text{ kg})(1 \text{ m})^2\right]\omega_2$$

or

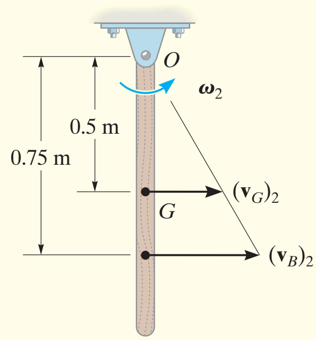
$$1.039 = 0.003(v_B)_2 + 2.50(v_G)_2 + 0.4167\omega_2$$

Kinematics. Since the rod is pinned at O, from Fig. 19–10*c* we have

$$(v_G)_2 = (0.5 \text{ m})\omega_2 \quad (v_B)_2 = (0.75 \text{ m})\omega_2$$

Substituting into the above equation and solving yields

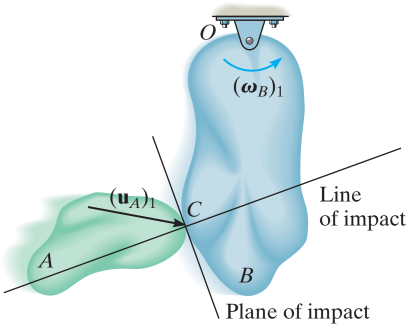
$$\omega_2 = 0.623 \text{ rad/s} \circlearrowright \qquad \textit{Ans.}$$



(c)

Fig. 19–10

*19.4 ECCENTRIC IMPACT



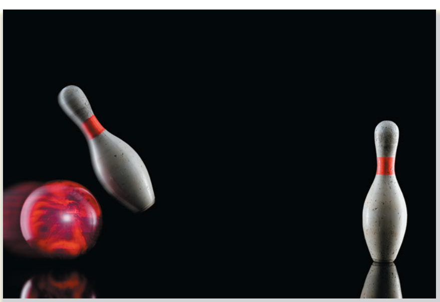
(a)

Fig. 19–11

The concepts involving central and oblique impact of particles were presented in Sec. 15.4. We will now expand this treatment and discuss the eccentric impact of two bodies. ***Eccentric impact*** occurs when the line connecting the mass centers of the two bodies does not coincide with the line of impact.* This type of impact often occurs when one or both of the bodies are constrained to rotate about a fixed axis. For example, consider the collision at C between the two bodies A and B, shown in Fig. 19–11*a*. It is assumed that just before collision B is rotating counterclockwise with an angular velocity $(\boldsymbol{\omega}_B)_1$, and the velocity of the contact point C located on A is $(\mathbf{u}_A)_1$. Kinematic diagrams for both bodies just before collision are shown in Fig. 19–11*b*. Provided the bodies are smooth, the impulsive forces they exert on each other are directed along the line of impact. Hence, the component of velocity of point C on body B, which is directed along the line of impact, is $(v_B)_1 = (\omega_B)_1 r$, Fig. 19–11*b*. Likewise, on body A the component of velocity $(\mathbf{u}_A)_1$ along the line of impact is $(\mathbf{v}_A)_1$. In order for a collision to occur, $(v_A)_1 > (v_B)_1$.

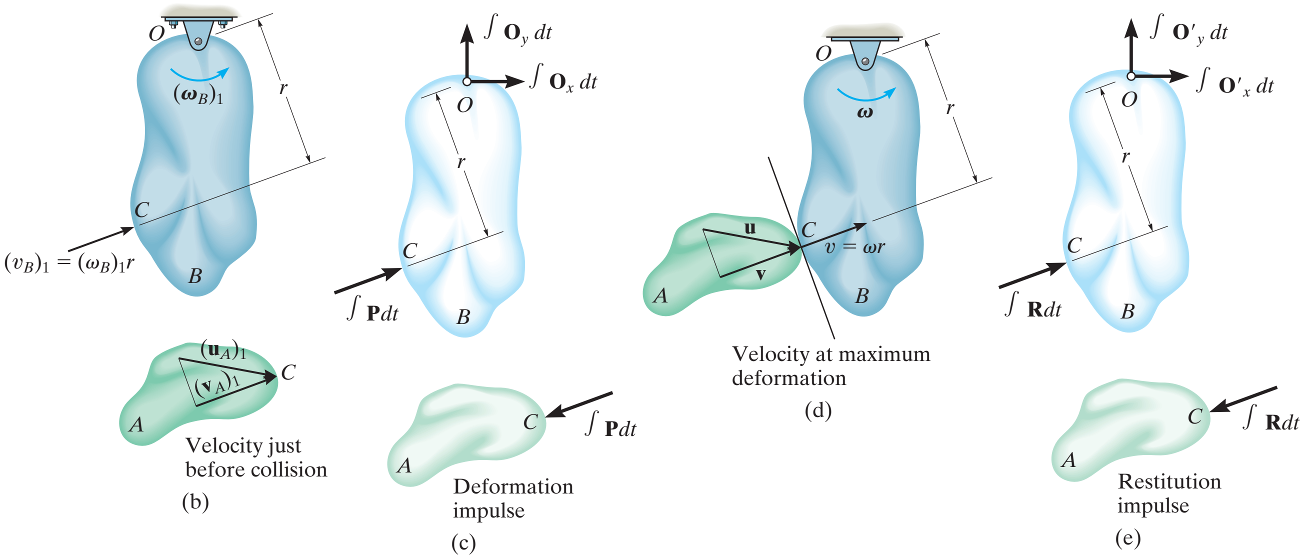
During the impact an equal but opposite impulsive force $\mathbf{P}$ is exerted between the bodies which *deforms* their shapes at the point of contact. The resulting deformation impulse is shown on the impulse diagrams for both bodies, Fig. 19–11*c*. In addition, the impulsive force at point C on the rotating body creates impulsive pin reactions at O. On these diagrams it is assumed that these impulses are much larger than the nonimpulsive weights of the bodies, which are not shown. When the deformation at point C is a maximum, point C on both bodies moves with a common velocity $\mathbf{v}$ along the line of impact, Fig. 19–11*d*. A period of *restitution* then occurs in which the bodies tend to regain their original shapes. The restitution phase creates an equal but opposite restitution impulse between the bodies as shown on the impulse diagram, Fig. 19–11*e*. After restitution the bodies move apart such that point C on body B has a velocity $(\mathbf{v}_B)_2$ and point C on body A has a velocity $(\mathbf{u}_A)_2$, Fig. 19–11*f*, where $(v_B)_2 > (v_A)_2$.

In general, a problem involving the impact of two bodies requires determining the *two unknowns* $(v_A)_2$ and $(v_B)_2$, assuming $(v_A)_1$ and $(v_B)_1$ are known. To solve such problems, two equations must be written. The *first equation* generally involves application of *the conservation of angular momentum.* For example, in the case of bodies A and B, angular momentum is conserved about point O since the impulses at C are internal to the system and the impulses at O create zero moment (or zero angular impulse) about O. The *second equation* can be obtained using the definition of the ***coefficient of restitution***, e, which is a ratio of the restitution impulse to the deformation impulse.



Here is an example of eccentric impact occurring between this bowling ball and pin.

* When these lines coincide, central impact occurs and the problem can be analyzed as discussed in Sec. 15.4.



To establish a useful form of the coefficient of restitution equation we will first apply the principle of angular impulse and momentum about point O to body B. Proceeding from the time just before the collision to the instant of maximum deformation, Figs. 19–11*b*, 19–11*c*, and 19–11*d*, we have

$$(\circlearrowright +) \qquad I_O(\omega_B)_1 + r\int P\,dt = I_O\omega \qquad (19\text{–}18)$$

Here I_O is the moment of inertia of B about point O. Similarly, applying the principle of angular impulse and momentum from the instant of maximum deformation to the time just after the impact, Figs. 19-11*d*, 19-11*e*, and 19-11*f*, yields

$$(\circlearrowright +) \qquad I_O\omega + r\int R\,dt = I_O(\omega_B)_2 \qquad (19\text{–}19)$$

Solving Eqs. 19–18 and 19–19 for $\int P\,dt$ and $\int R\,dt$, respectively, and formulating e, we have

$$e = \frac{\int R\,dt}{\int P\,dt} = \frac{r(\omega_B)_2 - r\omega}{r\omega - r(\omega_B)_1} = \frac{(v_B)_2 - v}{v - (v_B)_1}$$

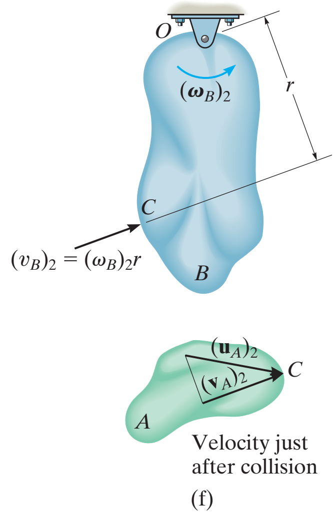


Fig. 19–11

In the same manner, we can write an equation which relates the magnitudes of velocity $(v_A)_1$ and $(v_A)_2$ of body A. The result is

$$e = \frac{v - (v_A)_2}{(v_A)_1 - v}$$

Combining the above two equations by eliminating the common velocity v yields the desired result, i.e.,

$$(+\nearrow) \qquad e = \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1} \qquad (19\text{–}20)$$

This equation is identical to Eq. 15–11, which was derived for the central impact between two particles. It states that the coefficient of restitution is equal to the ratio of the relative velocity of *separation* of the points of contact (C) *just after impact* to the relative velocity at which the points *approach* one another *just* before impact. In deriving this equation, we assumed that the points of contact for both bodies move up and to the right *both* before and after impact. If motion of any one of the contacting points occurs down and to the left, the velocity of this point should be considered a negative quantity in Eq. 19–20.

It is important to realize, however, that this analysis has only a very limited application in engineering, because values of e for eccentric impact have been found to be highly sensitive to the material, geometry, and the velocity of each of the colliding bodies.

During impact the columns of many highway signs are intended to break out of their supports and easily collapse at their joints. This is shown by the slotted connections at their base and the clean breaks at the column's midsection.

EXAMPLE 19.8

The 5-kg slender rod is suspended from the pin at A, Fig. 19–12a. If a 1-kg ball B is thrown at the rod and strikes its center with a velocity of 9 m/s, determine the angular velocity of the rod just after impact. The coefficient of restitution is $e = 0.4$.

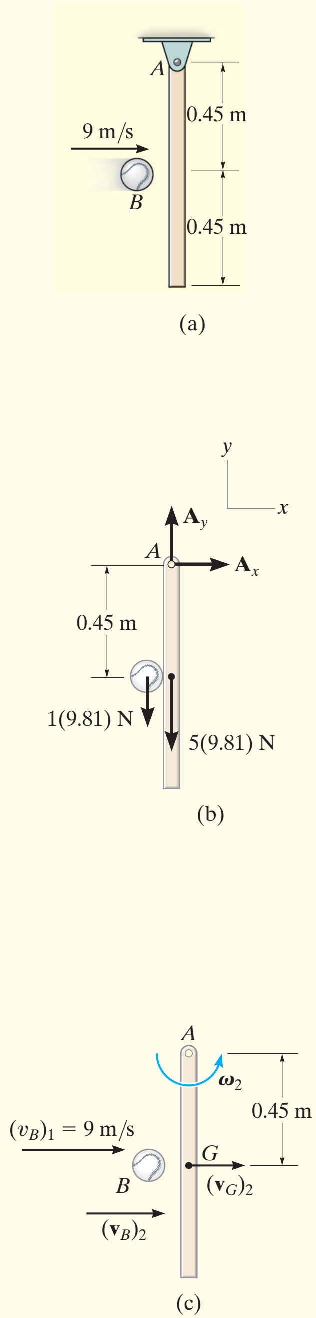


Fig. 19–12

SOLUTION

Conservation of Angular Momentum. If we consider the ball and rod as a single system, Fig. 19–12b, then angular momentum is conserved about point A since the impulsive force between the rod and ball is *internal*. Also, the *weights* of the ball and rod are *nonimpulsive*. Assuming the directions of the velocities of the ball and rod just after impact are as shown on the kinematic diagram, Fig. 19–12c, we require

$$(\circlearrowright +) \qquad (H_A)_1 = (H_A)_2$$

$$m_B(v_B)_1(0.45\text{ m}) = m_B(v_B)_2(0.45\text{ m}) + m_R(v_G)_2(0.45\text{ m}) + I_G\omega_2$$

$$(1\text{ kg})(9\text{ m/s})(0.45\text{ m}) = (1\text{ kg})(v_B)_2(0.45\text{ m}) +$$

$$(5\text{ kg})(v_G)_2(0.45\text{ m}) + \left[\frac{1}{12}(5\text{ kg})(0.9\text{ m})^2\right]\omega_2$$

Since $(v_G)_2 = 0.45\omega_2$ then

$$4.05 = 0.45(v_B)_2 + 1.35\omega_2 \qquad (1)$$

Coefficient of Restitution. With reference to Fig. 19–12c, we have

$$(\xrightarrow{+}) \qquad e = \frac{(v_G)_2 - (v_B)_2}{(v_B)_1 - (v_G)_1} \qquad 0.4 = \frac{(0.45\text{ m})\omega_2 - (v_B)_2}{9\text{ m/s} - 0}$$

$$3.6 = 0.45\omega_2 - (v_B)_2 \qquad (2)$$

Solving Eqs. 1 and 2 yields

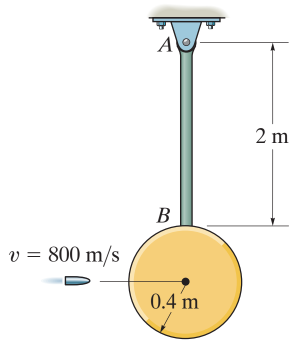
$$(v_B)_2 = -1.957\text{ m/s} = 1.957\text{ m/s} \leftarrow$$

$$\omega_2 = 3.65\text{ rad/s}\circlearrowleft \qquad \textit{Ans.}$$

PROBLEMS

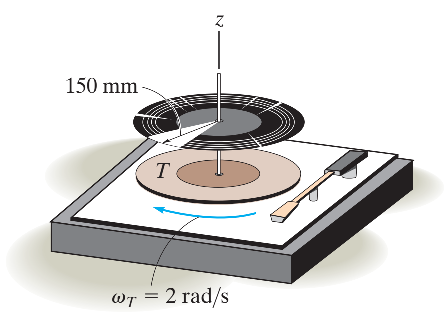
19–29. The 10-g bullet having a velocity of 800 m/s is fired into the 5-kg disk as shown. Determine the angular velocity of the disk just after the bullet becomes embedded. Also, calculate the angle θ the disk will swing when it stops. The disk is originally at rest. Neglect the mass of the rod AB and the final angular momentum of the embedded bullet.

19–30. The 10-g bullet having a velocity of 800 m/s is fired into the 5-kg disk as shown. Determine the angular velocity of the disk just after the bullet becomes embedded. Also, calculate the angle θ the disk will swing when it stops. The disk is originally at rest. The rod AB has a mass of 3 kg. Neglect the final angular momentum of the embedded bullet.



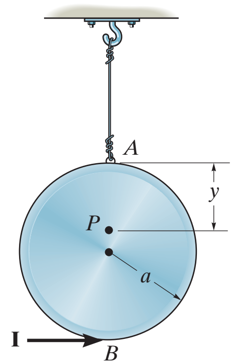
Probs. 19–29/30

19–31. The turntable T of a record player has a mass of 0.75 kg and a radius of gyration $k_z = 125$ mm. It is *turning freely* at $\omega_T = 2$ rad/s when a 50-g record (thin disk) falls on it. Determine the final angular velocity of the turntable just after the record stops slipping on the turntable.



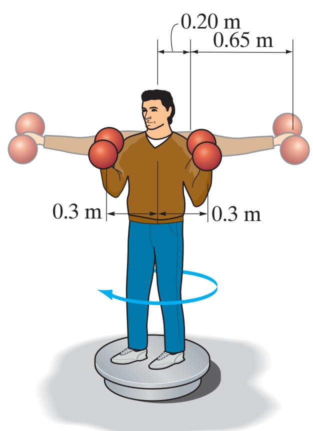
Prob. 19–31

***19–32.** The circular disk has a mass m and is suspended at A by the wire. If it receives a horizontal impulse $\mathbf{I}$ at its edge B, determine the location y of the point P about which the disk appears to rotate during the impact.



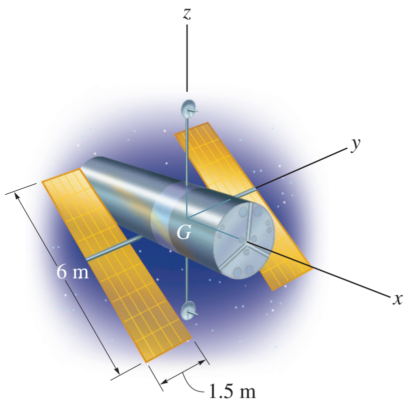
Prob. 19–32

19–33. The 80-kg man is holding two dumbbells while standing on a turntable of negligible mass, which turns freely about a vertical axis. When his arms are fully extended, the turntable is rotating with an angular velocity of 0.5 rev/s. Determine the angular velocity of the man when he retracts his arms to the position shown. When his arms are fully extended, approximate each arm as a uniform 6-kg rod having a length of 650 mm, and his body as a 68-kg solid cylinder of 400-mm diameter. With his arms in the retracted position, assume the man is an 80-kg solid cylinder of 450-mm diameter. Each dumbbell consists of two 5-kg spheres of negligible size.



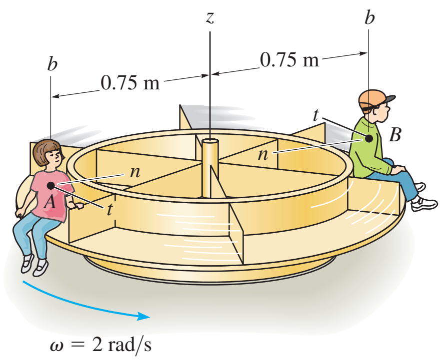
Prob. 19–33

19–34. The Hubble Space Telescope is powered by two solar panels as shown. The body of the telescope has a mass of 11 Mg and radii of gyration $k_x = 1.64$ m and $k_y = 3.85$ m, whereas the solar panels can be considered as thin plates, each having a mass of 54 kg. Due to an internal drive, the panels are given an angular velocity of $\{0.6\mathbf{j}\}$ rad/s, measured relative to the telescope. Determine the angular velocity of the telescope due to the rotation of the panels. Prior to rotating the panels, the telescope was originally traveling at $\mathbf{v}_G = \{-400\mathbf{i} + 250\mathbf{j} + 175\mathbf{k}\}$ m/s. Neglect its orbital rotation.



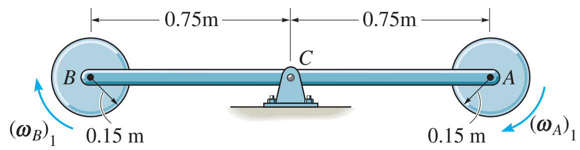
Prob. 19–34

19–35. Two children A and B, each having a mass of 30 kg, sit at the edge of the merry-go-round which is rotating at $\omega = 2$ rad/s. Excluding the children, the merry-go-round has a mass of 180 kg and a radius of gyration $k_z = 0.6$ m. Determine the angular velocity of the merry-go-round if A jumps off horizontally in the $-n$ direction with a speed of 2 m/s, measured with respect to the merry-go-round. What is the merry-go-round's angular velocity if B then jumps off horizontally in the $+t$ direction with a speed of 2 m/s, measured with respect to the merry-go-round? Neglect friction and the size of each child.



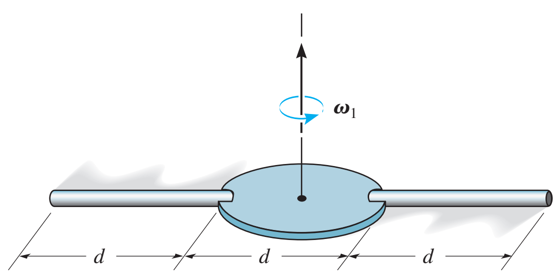
Prob. 19–35

***19–36.** The 2-kg rod ACB supports the two 4-kg disks at its ends. If both disks are given a clockwise angular velocity $(\omega_A)_1 = (\omega_B)_1 = 5$ rad/s while the rod is held stationary and then released, determine the angular velocity of the rod after both disks have stopped spinning relative to the rod due to frictional resistance at the pins A and B. Motion is in the *horizontal plane*. Neglect friction at pin C.



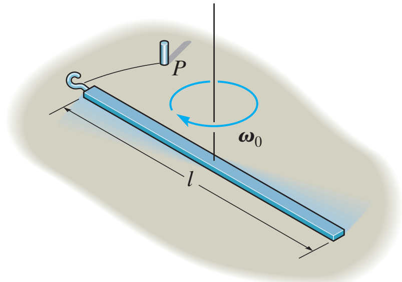
Prob. 19–36

19–37. Each of the two slender rods and the disk have the same mass m. Also, the length of each rod is equal to the diameter d of the disk. If the assembly is rotating with an angular velocity $\boldsymbol{\omega}_1$ when the rods are directed outward, determine the angular velocity of the assembly if by internal means the rods are brought to an upright vertical position.



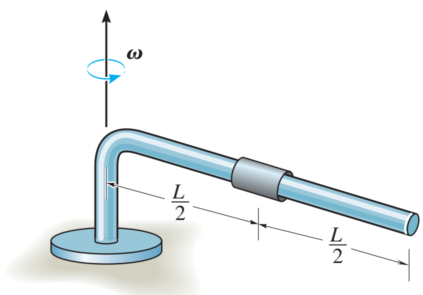
Prob. 19–37

19–38. A thin rod of mass m has an angular velocity $\boldsymbol{\omega}_0$ while rotating on a smooth surface. Determine its new angular velocity just after its end strikes and hooks onto the peg and the rod starts to rotate about P without rebounding. Solve the problem (a) using the parameters given, (b) setting $m = 2$ kg, $\omega_0 = 4$ rad/s, $l = 1.5$ m.



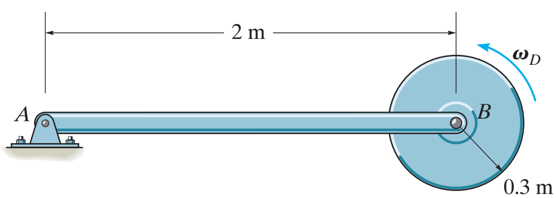
Prob. 19–38

19–39. The rod has a length L and mass m. A smooth collar having a negligible size and one-fourth the mass of the rod is placed on the rod at its midpoint. If the rod is freely rotating at $\boldsymbol{\omega}$ about its end and the collar is released, determine the rod's angular velocity just before the collar flies off the rod. Also, what is the speed of the collar as it leaves the rod?



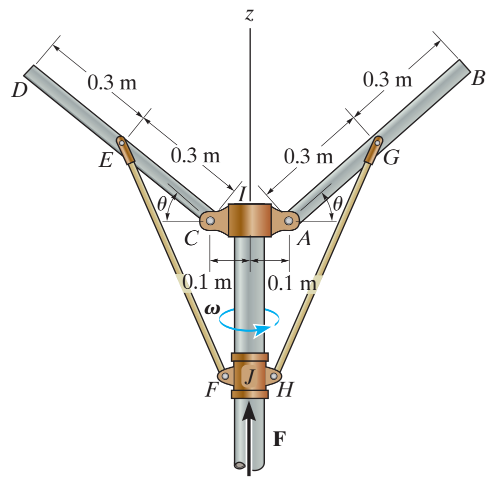
Prob. 19–39

***19–40.** The 12-kg rod AB is pinned to the 40-kg disk. If the disk is given an angular velocity $\omega_D = 100$ rad/s while the rod is held stationary, and the assembly is then released, determine the angular velocity of the rod after the disk has stopped spinning relative to the rod due to frictional resistance at the bearing B. Motion is in the *horizontal plane*. Neglect friction at the pin A.



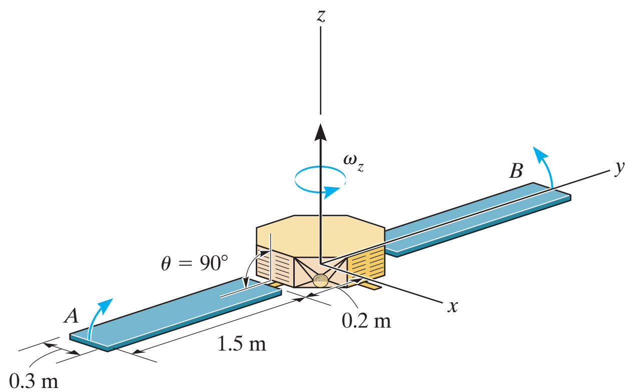
Prob. 19–40

19–41. The vertical shaft is rotating with an angular velocity of 3 rad/s when $\theta = 0°$. If a force **F** is applied to the collar so that $\theta = 90°$, determine the angular velocity of the shaft. Also, find the work done by force **F**. Neglect the mass of rods GH and EF and the collars I and J. The rods AB and CD each have a mass of 10 kg.



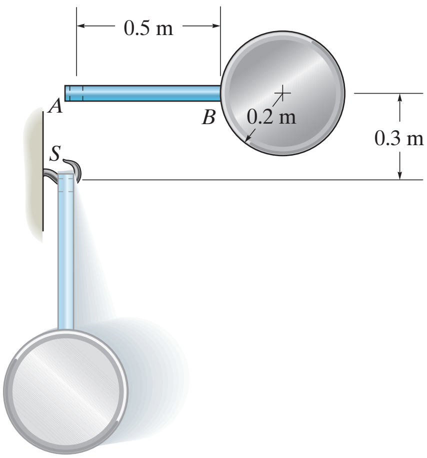
Prob. 19–41

19–42. The satellite has a mass of 200 kg and a radius of gyration about z axis of $k_z = 0.1$ m, excluding the two solar panels A and B. Each solar panel has a mass of 15 kg and can be approximated as a thin plate. If the satellite is originally spinning about the z axis at a constant rate $\omega_z = 0.5$ rad/s when $\theta = 90°$, determine the rate of spin if both panels are raised and reach the upward position, $\theta = 0°$, at the same instant.



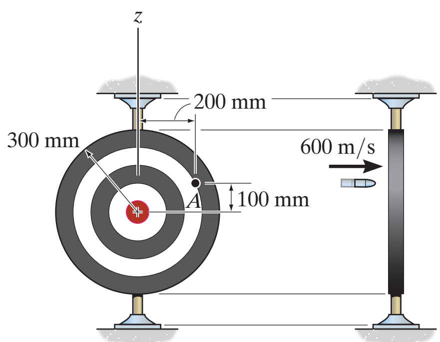
Prob. 19–42

19–43. The pendulum consists of a slender 2-kg rod AB and 5-kg disk. It is released from rest without rotating. When it falls 0.3 m, the end A strikes the hook S, which provides a permanent connection. Determine the angular velocity of the pendulum after it has rotated 90°. Treat the pendulum's weight during impact as a nonimpulsive force.



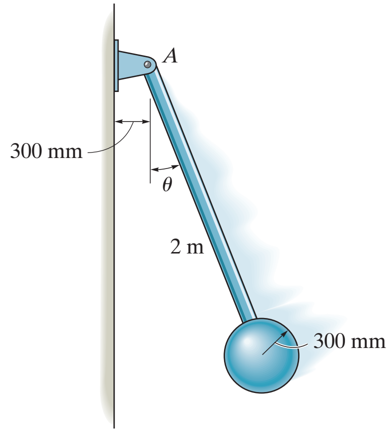
Prob. 19–43

***19–44.** The target is a thin 5-kg circular disk that can rotate freely about the z axis. A 25-g bullet, traveling at 600 m/s, strikes the target at A and becomes embedded in it. Determine the angular velocity of the target after the impact. Initially, it is at rest.



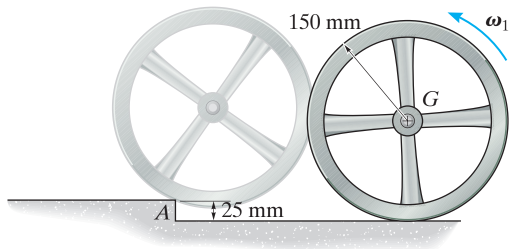
Prob. 19–44

19–45. The pendulum consists of a 15-kg solid ball and 6-kg rod. If it is released from rest when $\theta_1 = 90°$, determine the angle θ_2 after the ball strikes the wall, rebounds, and the pendulum swings up to the point of momentary rest. Take $e = 0.6$.



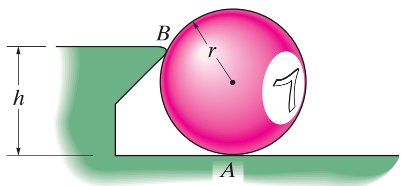
Prob. 19–45

19–46. The wheel has a mass of 50 kg and a radius of gyration of 125 mm about its center of mass G. Determine the minimum value of the angular velocity $\boldsymbol{\omega}_1$ of the wheel, so that it strikes the step at A without rebounding and then rolls over it without slipping.



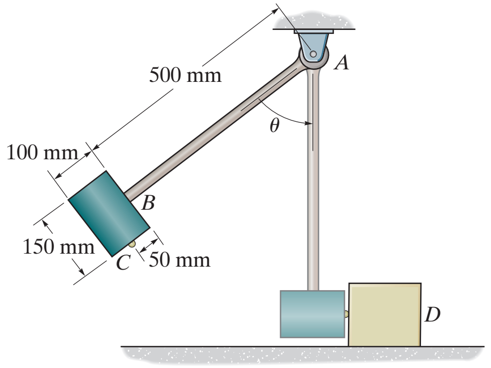
Prob. 19–46

19–47. Determine the height h of the bumper of the pool table, so that when the pool ball of mass m strikes it, no frictional force will be developed between the ball and the table at A. Assume the bumper exerts only a horizontal force on the ball.



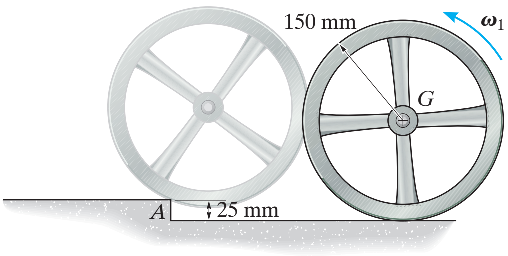
Prob. 19–47

***19–48.** The hammer consists of a 10-kg solid cylinder C and 6-kg uniform slender rod AB. If the hammer is released from rest when $\theta = 90^\circ$ and strikes the 30-kg block D when $\theta = 0^\circ$, determine the velocity of block D and the angular velocity of the hammer immediately after the impact. The coefficient of restitution between the hammer and the block is $e = 0.6$.



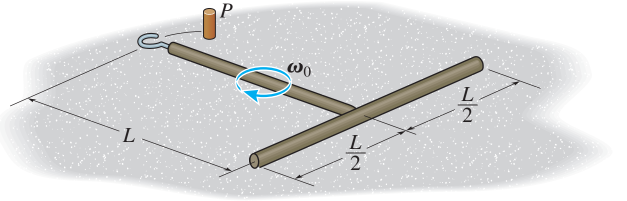
Prob. 19–48

19–49. The wheel has a mass of 50 kg and a radius of gyration of 125 mm about its center of mass G. If it rolls without slipping with an angular velocity of $\boldsymbol{\omega}_1 = 5$ rad/s before it strikes the step at A, determine its angular velocity after it rolls over the step. The wheel does not lose contact with the step when it strikes it.



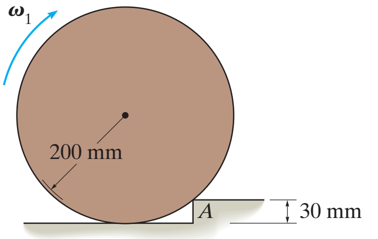
Prob. 19–49

19–50. The uniform rod assembly rotates with an angular velocity of $\boldsymbol{\omega}_0$ on the smooth horizontal plane just before the hook strikes the peg P without rebound. Determine the angular velocity of the assembly immediately after the impact. Each rod has a mass of m.



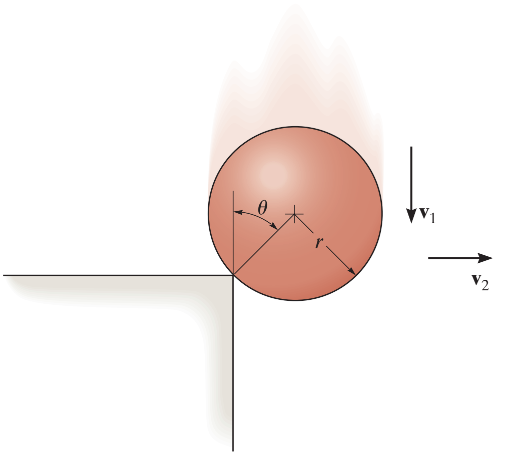
Prob. 19–50

19–51. The 20-kg disk strikes the step without rebounding. Determine the largest angular velocity ω_1 the disk can have without losing contact with the step A.



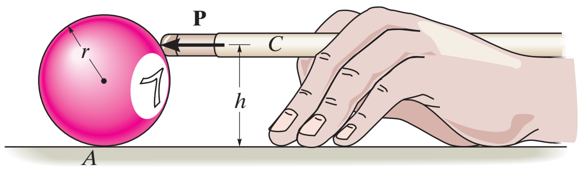
Prob. 19–51

***19–52.** The solid ball of mass m is dropped with a velocity $\mathbf{v}_1$ onto the edge of the rough step. If it rebounds horizontally off the step with a velocity $\mathbf{v}_2$, determine the angle θ at which contact occurs. Assume no slipping when the ball strikes the step. The coefficient of restitution is e.



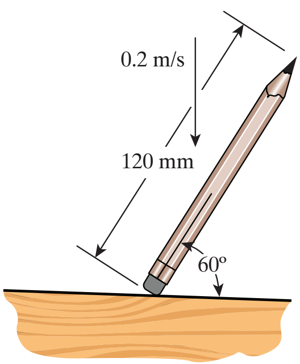
Prob. 19–52

19–53. Determine the height h at which a billiard ball of mass m must be struck so that no frictional force develops between it and the table at A. Assume that the cue C only exerts a horizontal force $\mathbf{P}$ on the ball.



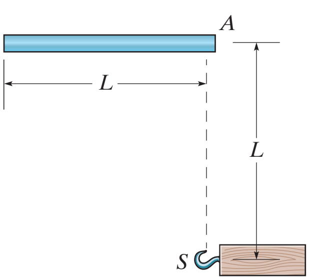
Prob. 19–53

19–54. A 50-g pencil (uniform slender rod) falls down onto the table with a velocity of 0.2 m/s just before impact. If it is at an angle of 60° with the horizontal, and it does not slip during the impact, determine its angular velocity ω and the velocity v_G of its center of mass just after rebounding. The coefficient of restitution is $e = 0.8$.



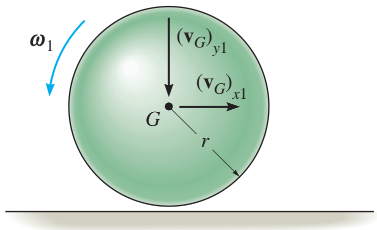
Prob. 19–54

19–55. The rod of mass m and length L is released from rest without rotating. When it falls a distance L, the end A strikes the hook S, which provides a permanent connection. Determine the angular velocity ω of the rod after it has rotated 90°. Treat the rod's weight during impact as a nonimpulsive force.



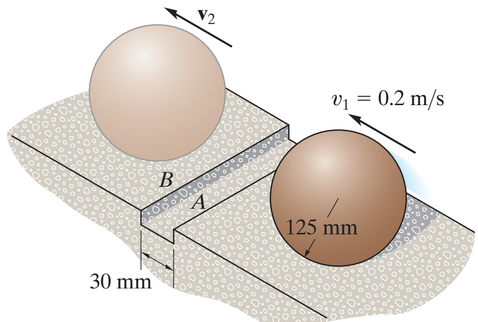
Prob. 19–55

***19–56.** A solid ball with a mass m is thrown on the ground such that at the instant of contact it has an angular velocity $\boldsymbol{\omega}_1$ and velocity components $(\mathbf{v}_G)_{x1}$ and $(\mathbf{v}_G)_{y1}$ as shown. If the ground is rough so no slipping occurs, determine the components of the velocity of its mass center just after impact. The coefficient of restitution is e.



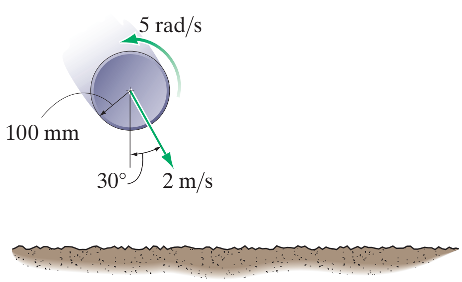
Prob. 19–56

19–57. A ball having a mass of 8 kg and initial speed of $v_1 = 0.2$ m/s rolls over a 30-mm-long depression. Assuming that the ball rolls off the edges of contact, first A then B, without slipping, determine its final velocity v_2 when it reaches the other side.



Prob. 19–57

19–58. The 2-kg disk is thrown down onto the rough surface with the velocity and angular velocity shown. If there is no slipping, and the coefficient of restitution is $e = 0.5$, determine the velocity of the disk and its angular velocity just after rebounding.



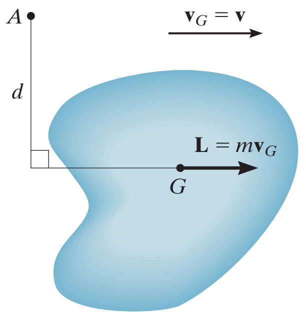
Prob. 19–58

CHAPTER REVIEW

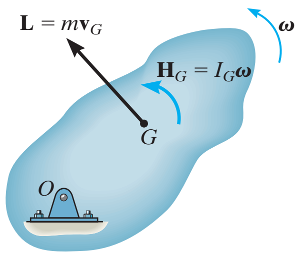
Linear and Angular Momentum

The linear and angular momentum of a rigid body can be referenced to its mass center G.

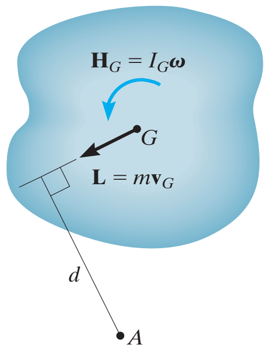
If the angular momentum is to be determined about an axis other than the one passing through the mass center, then the angular momentum is determined by summing $\mathbf{H}_G$ and the moment of $\mathbf{L}$ about this axis.



Translation



Rotation about a fixed axis



General plane motion

Translation	Rotation about a fixed axis	General plane motion
$L = mv_G$	$L = mv_G$	$L = mv_G$
$H_G = 0$	$H_G = I_G\omega$	$H_G = I_G\omega$
$H_A = (mv_G)d$	$H_O = I_O\omega$	$H_A = I_G\omega + (mv_G)d$

Principle of Impulse and Momentum

The principles of linear and angular impulse and momentum are used to solve problems that involve force, velocity, and time. Before applying these equations, it is important to establish the x, y, z inertial coordinate system. The free-body diagram for the body should also be drawn in order to account for all of the forces and couple moments that produce impulses on the body.

$$m(v_{Gx})_1 + \Sigma \int_{t_1}^{t_2} F_x\, dt = m(v_{Gx})_2$$

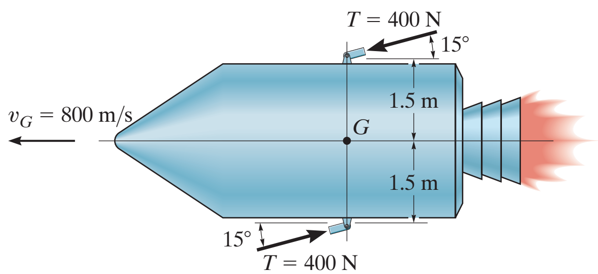
$$m(v_{Gy})_1 + \Sigma \int_{t_1}^{t_2} F_y\, dt = m(v_{Gy})_2$$

$$I_G\omega_1 + \Sigma \int_{t_1}^{t_2} M_G\, dt = I_G\omega_2$$

Conservation of Momentum Provided the sum of the linear impulses acting on a system of connected rigid bodies is zero in a particular direction, then the linear momentum for the system is conserved in this direction. Conservation of angular momentum occurs if the impulses pass through an axis or are parallel to it. Momentum is also conserved if the external forces are small and thereby create nonimpulsive forces on the system. A free-body diagram should accompany any application in order to classify the forces as impulsive or nonimpulsive and to determine an axis about which the angular momentum may be conserved.	$\left(\sum \begin{matrix}\text{syst. linear}\\ \text{momentum}\end{matrix}\right)_1 = \left(\sum \begin{matrix}\text{syst. linear}\\ \text{momentum}\end{matrix}\right)_2$ $\left(\sum \begin{matrix}\text{syst. angular}\\ \text{momentum}\end{matrix}\right)_{O1} = \left(\sum \begin{matrix}\text{syst. angular}\\ \text{momentum}\end{matrix}\right)_{O2}$
Eccentric Impact If the line of impact does not coincide with the line connecting the mass centers of two colliding bodies, then eccentric impact will occur. If the motion of the bodies just after the impact is to be determined, then it is necessary to consider a conservation of momentum equation for the system and use the coefficient of restitution equation.	$e = \dfrac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$

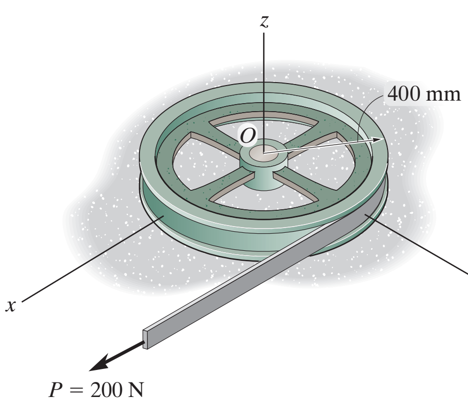
REVIEW PROBLEMS

R19–1. The space capsule has a mass of 1200 kg and a moment of inertia $I_G = 900\ \text{kg}\cdot\text{m}^2$ about an axis passing through G and directed perpendicular to the page. If it is traveling forward with a speed $v_G = 800\ \text{m/s}$ and executes a turn by means of two jets, which provide a constant thrust of 400 N for 0.3 s, determine the capsule's angular velocity just after the jets are turned off.



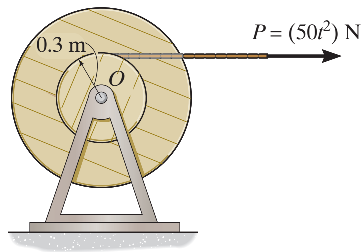
Prob. R19–1

R19–2. The wheel having a mass of 100 kg and a radius of gyration about the z axis of $k_z = 300$ mm rests on the smooth horizontal plane. If the belt is subjected to a force of $P = 200$ N, determine the angular velocity of the wheel and the speed of its center of mass O, three seconds after the force is applied.



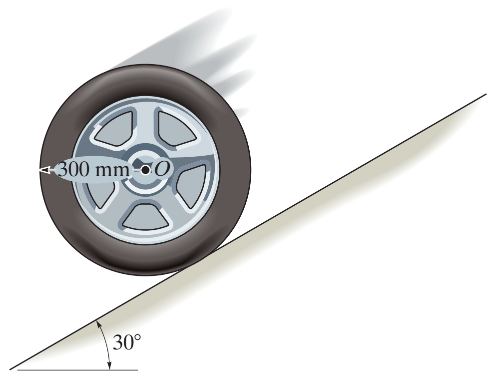
Prob. R19–2

R19–3. The cable is subjected to a force of $P = (50t^2)$ N, where t is in seconds. Determine the angular velocity of the spool 3 s after **P** is applied, starting from rest. The spool has a mass of 75 kg and a radius of gyration of 0.375 m about its center, O.



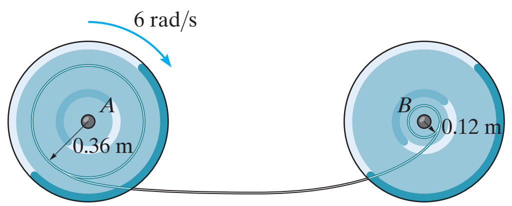
Prob. R19–3

R19–4. The tire has a mass of 9 kg and a radius of gyration $k_O = 225$ mm. If it is released from rest and rolls down the plane without slipping, determine the speed of its center O when $t = 3$ s.



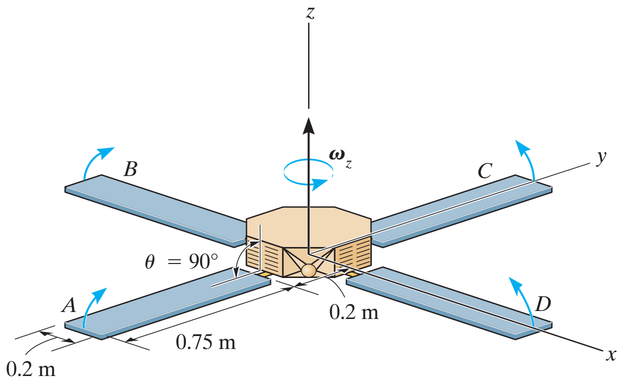
Prob. R19–4

R19–5. Spool B is at rest and spool A is rotating at 6 rad/s when the slack in the cord connecting them is taken up. If the cord does not stretch, determine the angular velocity of each spool immediately after the cord is jerked tight. The spools A and B have masses and radii of gyration $m_A = 15$ kg, $k_A = 0.24$ m, $m_B = 7.5$ kg, $k_B = 0.18$ m, respectively.



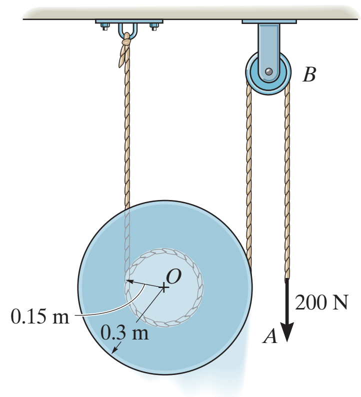
Prob. R19–5

R19–6. The space satellite has a mass of 125 kg and a moment of inertia $I_z = 0.940\ \text{kg}\cdot\text{m}^2$, excluding the four solar panels A, B, C, and D. Each solar panel has a mass of 20 kg and can be approximated as a thin plate. If the satellite is originally spinning about the z axis at a constant rate $\omega_z = 0.5$ rad/s when $\theta = 90°$, determine the rate of spin if all the panels are raised and reach the upward position, $\theta = 0°$, at the same instant.



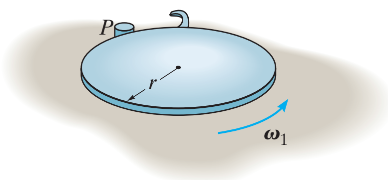
Prob. R19–6

R19–7. The spool has a mass of 15 kg and a radius of gyration $k_O = 0.2$ m. If a force of 200 N is applied to the cord at A, determine the angular velocity of the spool in $t = 3$ s starting from rest. Neglect the mass of the pulley and cord.



Prob. R19–7

R19–8. A thin disk of mass m has an angular velocity $\boldsymbol{\omega}_1$ while rotating on a smooth surface. Determine its new angular velocity just after the hook at its edge strikes the peg P and the disk starts to rotate about P without rebounding.



Prob. R19–8

CHAPTER 20

Operation of this tower crane requires controlling the kinematics of its three-dimensional motion.

THREE-DIMENSIONAL KINEMATICS OF A RIGID BODY

CHAPTER OBJECTIVES

- To analyze the kinematics of a body subjected to rotation about a fixed point and to general plane motion.
- To provide a relative-motion analysis of a rigid body using translating and rotating axes.

20.1 ROTATION ABOUT A FIXED POINT

When a rigid body rotates about a fixed point, the path of motion for a particle on the body lies on the surface of a sphere, having a radius r and centered at the fixed point. Since motion along this path is formed from a series of rotations, we will first develop a familiarity with some of the properties of rotational displacements.

The boom is subjected to rotation about a fixed point because it can rotate up and down, and the frame can turn about a vertical axis.

Euler's Theorem. Euler's theorem states that two "component" rotations about different axes passing through a point are equivalent to a single resultant rotation about an axis passing through the same point. If more than two rotations are applied, they can all be combined into pairs, and each pair can be further reduced and combined into a single rotation.

Finite Rotations. If the component rotations used in Euler's theorem are finite, then it is important that the order in which they are applied be maintained. For example, consider the two finite rotations $\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2$ applied to the block in Fig. 20–1*a*. Each rotation has a magnitude of 90° and a direction defined by the right-hand rule, as indicated by the arrow. The final position of the block is shown at the right. If these two rotations are applied in the reverse order, $\boldsymbol{\theta}_2 + \boldsymbol{\theta}_1$, as shown in Fig. 20–1*b*, then the final position of the block will not be the same as it is in Fig. 20–1*a*. Because finite rotations do not obey the commutative law of addition ($\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 \neq \boldsymbol{\theta}_2 + \boldsymbol{\theta}_1$), they cannot be classified as vectors. Realize that if smaller, yet finite, rotations had been used to illustrate this point, e.g., 10° instead of 90°, the final position of the block would still be different; however, in this case, the difference would only be a small amount.

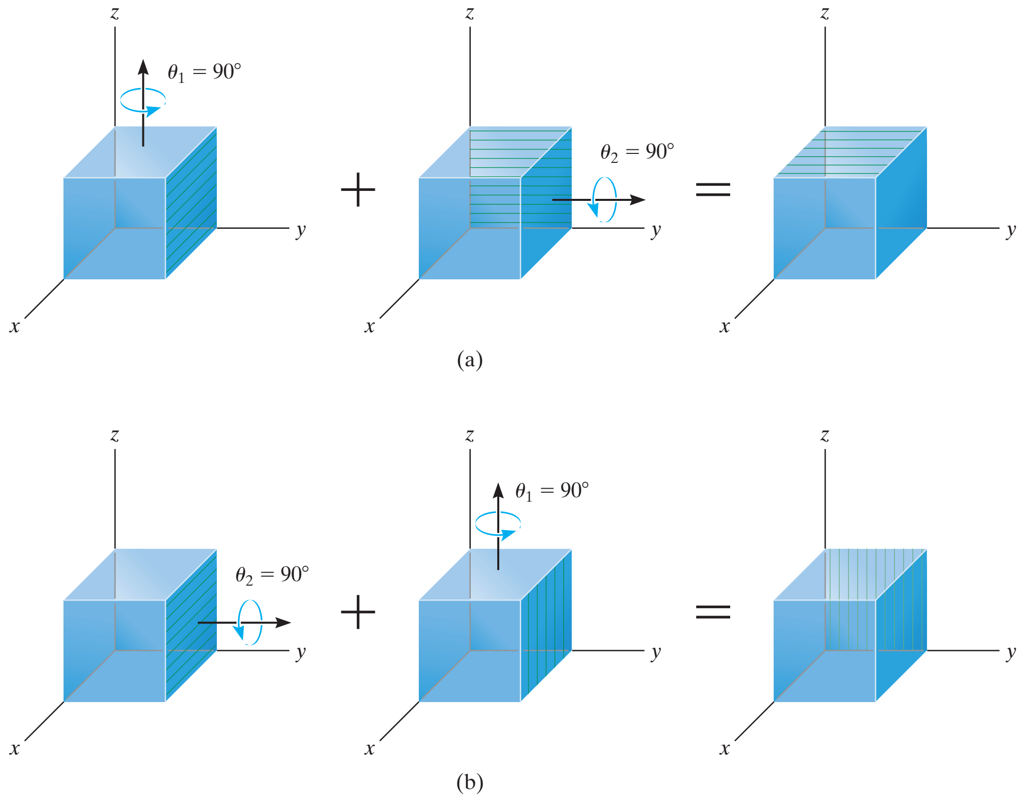


Fig. 20–1

Infinitesimal Rotations. When defining the angular motions of a body subjected to three-dimensional motion, only rotations which are infinitesimally small will be considered. Such rotations can be classified as vectors, since they can be added vectorially in any manner. To show this, for purposes of simplicity let us consider the rigid body itself to be a sphere which is allowed to rotate about its central fixed point O, Fig. 20–2*a*. If we impose two infinitesimal rotations $d\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_2$ on the body, it is seen that point P moves along the path $d\boldsymbol{\theta}_1 \times \mathbf{r} + d\boldsymbol{\theta}_2 \times \mathbf{r}$ and ends up at P'. Had the two successive rotations occurred in the order $d\boldsymbol{\theta}_2 + d\boldsymbol{\theta}_1$, then the resultant displacements of P would have been $d\boldsymbol{\theta}_2 \times \mathbf{r} + d\boldsymbol{\theta}_1 \times \mathbf{r}$ and again P would move to P'. This occurs because the vector cross product obeys the distributive law, i.e., $(d\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_2) \times \mathbf{r} = (d\boldsymbol{\theta}_2 + d\boldsymbol{\theta}_1) \times \mathbf{r}$, and so infinitesimal rotations are vectors, since these quantities have both a magnitude and direction for which the order of (vector) addition is not important, i.e., $d\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_2 = d\boldsymbol{\theta}_2 + d\boldsymbol{\theta}_1$. The two "component" rotations $d\boldsymbol{\theta}_1$ and $d\boldsymbol{\theta}_2$ in Fig. 20–2*a* are therefore equivalent to a single resultant rotation $d\boldsymbol{\theta} = d\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_2$, a consequence of Euler's theorem.

Angular Velocity. If the body is subjected to an angular rotation $d\boldsymbol{\theta}$ about a fixed point, the angular velocity of the body is defined by its time derivative,

$$\boldsymbol{\omega} = \dot{\boldsymbol{\theta}} \tag{20–1}$$

The line specifying the direction of $\boldsymbol{\omega}$, which is collinear with $d\boldsymbol{\theta}$, is referred to as the ***instantaneous axis of rotation***, Fig. 20–2*b*. In general, this axis changes direction during each instant of time. Since $d\boldsymbol{\theta}$ is a vector quantity, so too is $\boldsymbol{\omega}$, and it follows from vector addition that if the body is subjected to two component angular motions, $\boldsymbol{\omega}_1 = \dot{\boldsymbol{\theta}}_1$ and $\boldsymbol{\omega}_2 = \dot{\boldsymbol{\theta}}_2$, the resultant angular velocity is $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$.

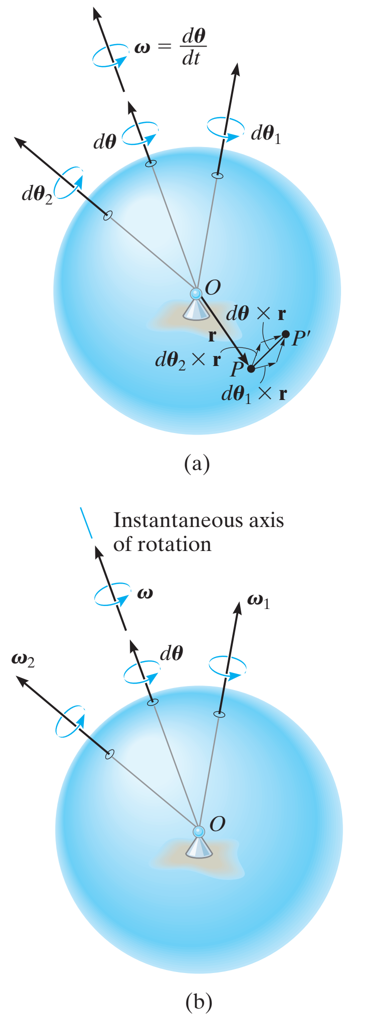


Fig. 20–2

Angular Acceleration. The body's angular acceleration is determined from the time derivative of its angular velocity, i.e.,

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} \tag{20–2}$$

For motion about a fixed point, $\boldsymbol{\alpha}$ must account for a change in *both* the magnitude and direction of $\boldsymbol{\omega}$, so that, in general, $\boldsymbol{\alpha}$ will not be directed along the instantaneous axis of rotation, Fig. 20–3.

As the direction of the instantaneous axis of rotation (or the line of action of $\boldsymbol{\omega}$) changes in space, the locus of the axis generates a fixed ***space cone***, Fig. 20–4. If the change in the direction of this axis is viewed with respect to the rotating body, the locus of the axis generates a ***body cone***.

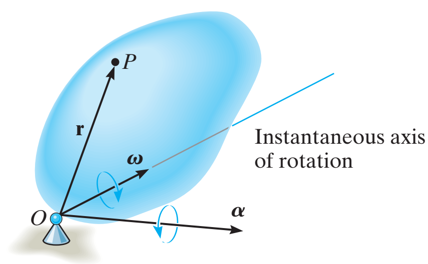


Fig. 20–3

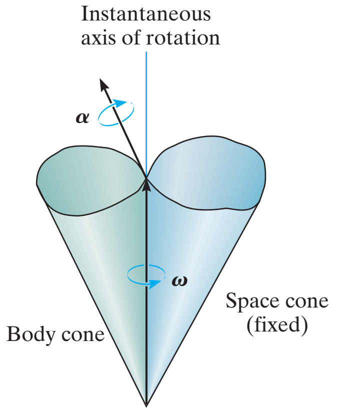


Fig. 20–4

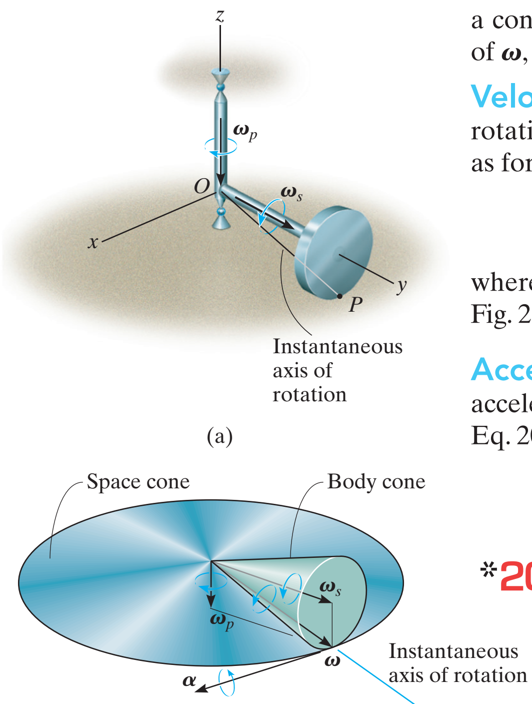


Fig. 20–5

At any given instant, these cones meet along the instantaneous axis of rotation, and when the body is in motion, the body cone appears to roll either on the inside or the outside surface of the space cone. Provided the paths defined by the open ends of the cones are described by the head of the $\boldsymbol{\omega}$ vector, then $\boldsymbol{\alpha}$ must act tangent to these paths at any given instant, since the time rate of change of $\boldsymbol{\omega}$ is equal to $\boldsymbol{\alpha}$, Fig. 20–4.

To illustrate this concept, consider the disk in Fig. 20–5*a* that *spins* about the rod at $\boldsymbol{\omega}_s$, while the rod and disk *precess* about the vertical axis at $\boldsymbol{\omega}_p$. The resultant angular velocity of the disk is therefore $\boldsymbol{\omega} = \boldsymbol{\omega}_s + \boldsymbol{\omega}_p$. Since both point O and the contact point P have zero velocity, then all points on a line between these points must have zero velocity. Thus, both $\boldsymbol{\omega}$ and the instantaneous axis of rotation are along OP. Therefore, as the disk rotates, this axis appears to move along the surface of the fixed space cone shown in Fig. 20–5*b*. If the axis is observed from the rotating disk, the axis then appears to move on the surface of the body cone. If $\boldsymbol{\omega}$ has a constant magnitude, then $\boldsymbol{\alpha}$ indicates only the change in the direction of $\boldsymbol{\omega}$, which is tangent to the cones at the tip of $\boldsymbol{\omega}$ as shown in Fig. 20–5*b*.

Velocity. Once $\boldsymbol{\omega}$ is specified, the velocity of any point on a body rotating about a fixed point can be determined using the same methods as for a body rotating about a fixed axis. Hence, by the cross product,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \tag{20–3}$$

where $\mathbf{r}$ defines the position of the point measured from the fixed point O, Fig. 20–3.

Acceleration. If $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are known at a given instant, the acceleration of a point can then be obtained from the time derivative of Eq. 20–3. This yields

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{20–4}$$

*20.2 THE TIME DERIVATIVE OF A VECTOR MEASURED FROM A FIXED OR TRANSLATING-ROTATING SYSTEM

In many types of problems involving the motion of a body about a fixed point, the angular velocity $\boldsymbol{\omega}$ is specified in terms of its components. Then, if the angular acceleration $\boldsymbol{\alpha}$ of the body is to be determined, it is often easier to find the time derivative of $\boldsymbol{\omega}$ using a coordinate system that has a rotation defined by one or more of the components of $\boldsymbol{\omega}$. For example, in the case of the disk in Fig. 20–5*a*, where $\boldsymbol{\omega} = \boldsymbol{\omega}_s + \boldsymbol{\omega}_p$, the x, y, z axes can be given an angular velocity of $\boldsymbol{\omega}_p$, and then $\boldsymbol{\alpha}$ can be determined relative to these axes. For this reason, and for other uses later, an equation will now be derived, which relates the time derivative of any vector $\mathbf{A}$ defined from a translating-rotating reference to its time derivative defined from a fixed reference.

Consider the x, y, z axes of the moving frame of reference to be rotating with an angular velocity $\mathbf{\Omega}$, which is measured from the fixed X, Y, Z axes, Fig. 20–6*a*. In the following discussion, it will be convenient to express vector $\mathbf{A}$ in terms of its $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ components, which define the directions of the moving axes. Hence,

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

If the time derivative is taken with respect to the moving frame of reference, only the change in the magnitudes of the components of $\mathbf{A}$ must be accounted for, since the directions of the components do not change with respect to the moving reference. Hence,

$$(\dot{\mathbf{A}})_{xyz} = \dot{A}_x\mathbf{i} + \dot{A}_y\mathbf{j} + \dot{A}_z\mathbf{k} \qquad (20\text{–}5)$$

When the time derivative of $\mathbf{A}$ is taken with respect to the fixed frame of reference, then we have

$$\dot{\mathbf{A}} = \dot{A}_x\mathbf{i} + \dot{A}_y\mathbf{j} + \dot{A}_z\mathbf{k} + A_x\dot{\mathbf{i}} + A_y\dot{\mathbf{j}} + A_z\dot{\mathbf{k}}$$

The time derivatives of the unit vectors will now be considered. For example, $\dot{\mathbf{i}} = d\mathbf{i}/dt$ represents only the change in the direction of $\mathbf{i}$ with respect to time, since $\mathbf{i}$ always has a magnitude of 1 unit. As shown in Fig. 20–6*b*, this change, $d\mathbf{i}$, is tangent to the path described by the arrowhead of $\mathbf{i}$, as $\mathbf{i}$ swings due to the rotation $\mathbf{\Omega}$. The rate of change in the magnitude of $d\mathbf{i}$ is $1(d\theta/dt) = 1\Omega$, and so we can account for both the rate of change in the magnitude and direction of $d\mathbf{i}$ using the cross product, $\dot{\mathbf{i}} = \mathbf{\Omega} \times \mathbf{i}$. In general, then*

$$\dot{\mathbf{i}} = \mathbf{\Omega} \times \mathbf{i} \qquad \dot{\mathbf{j}} = \mathbf{\Omega} \times \mathbf{j} \qquad \dot{\mathbf{k}} = \mathbf{\Omega} \times \mathbf{k}$$

Substituting these results into the above equation and using Eq. 20–5, we get

$$\dot{\mathbf{A}} = (\dot{\mathbf{A}})_{xyz} + \mathbf{\Omega} \times \mathbf{A} \qquad (20\text{–}6)$$

This important result will be used throughout Sec. 20.4 and Chapter 21. It states that the time derivative of any vector $\mathbf{A}$ as observed from the fixed X, Y, Z frame of reference is equal to the time rate of change of $\mathbf{A}$ as observed from the x, y, z translating-rotating frame of reference, Eq. 20–5, plus $\mathbf{\Omega} \times \mathbf{A}$, the change of $\mathbf{A}$ caused by the rotation of the x, y, z frame. As a result, Eq. 20–6 should always be used whenever $\mathbf{\Omega}$ produces a change in the direction of $\mathbf{A}$ as seen from the X, Y, Z reference. Notice that if the x, y, z axes are translating, then $\mathbf{\Omega} = \mathbf{0}$, and so $\dot{\mathbf{A}} = (\dot{\mathbf{A}})_{xyz}$. In other words, the time rate of change of $\mathbf{A}$ as observed from both coordinate systems will be the *same*.

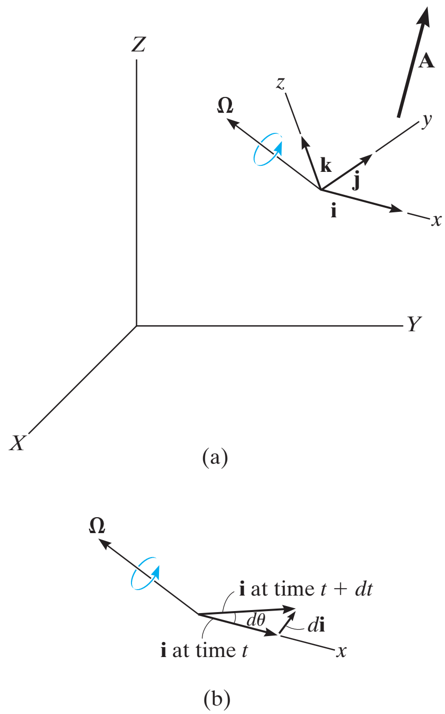


Fig. 20–6

* These formulations were also developed in Sec. 16.8, regarding planar motion of the axes.

20

EXAMPLE 20.1

The disk shown in Fig. 20–7 spins about its axle with a constant angular velocity $\omega_s = 3$ rad/s, while the horizontal platform on which the disk is mounted rotates about the vertical axis at a constant rate $\omega_p = 1$ rad/s. Determine the angular acceleration of the disk and the velocity and acceleration of point A on the disk when it is in the position shown.

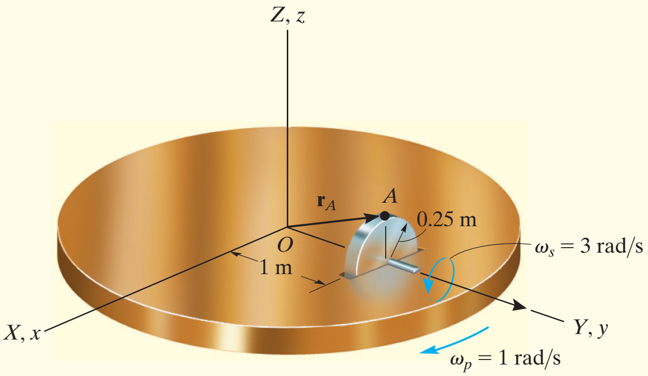


Fig. 20–7

SOLUTION

Point O represents a fixed point of rotation for the disk if one imagines an extension of the disk in the shape of a cone to this point. To determine the velocity and acceleration of point A, it is first necessary to determine the angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$ of the disk, since these vectors are used in Eqs. 20–3 and 20–4.

Angular Velocity. The angular velocity, which is measured from X, Y, Z, is simply the vector addition of its two component motions. Thus,

$$\boldsymbol{\omega} = \boldsymbol{\omega}_s + \boldsymbol{\omega}_p = \{3\mathbf{j} - 1\mathbf{k}\} \text{ rad/s}$$

Angular Acceleration. Since the magnitude of $\boldsymbol{\omega}$ is constant, only a change in its direction, as seen from the fixed reference, creates the angular acceleration $\boldsymbol{\alpha}$ of the disk. One way to obtain $\boldsymbol{\alpha}$ is to calculate the time derivative of *each of the two components* of $\boldsymbol{\omega}$ using Eq. 20–6. At the instant shown in Fig. 20–7, imagine the fixed X, Y, Z and a rotating x, y, z frame to be coincident. If the rotating x, y, z frame is chosen to have an angular velocity of $\boldsymbol{\Omega} = \boldsymbol{\omega}_p = \{-1\mathbf{k}\}$ rad/s, then $\boldsymbol{\omega}_s$ will always be directed along the y (not Y) axis, and the time rate of change of $\boldsymbol{\omega}_s$ as seen from x, y, z is zero; i.e., $(\dot{\boldsymbol{\omega}}_s)_{xyz} = \mathbf{0}$ (the magnitude and direction of $\boldsymbol{\omega}_s$ is constant). Thus,

$$\dot{\boldsymbol{\omega}}_s = (\dot{\boldsymbol{\omega}}_s)_{xyz} + \boldsymbol{\omega}_p \times \boldsymbol{\omega}_s = \mathbf{0} + (-1\mathbf{k}) \times (3\mathbf{j}) = \{3\mathbf{i}\} \text{ rad/s}^2$$

By the same choice of axes rotation, $\boldsymbol{\Omega} = \boldsymbol{\omega}_p$, or even with $\boldsymbol{\Omega} = \mathbf{0}$, the time derivative $(\dot{\boldsymbol{\omega}}_p)_{xyz} = \mathbf{0}$, since $\boldsymbol{\omega}_p$ has a constant magnitude and direction with respect to x, y, z. Hence,

$$\dot{\boldsymbol{\omega}}_p = (\dot{\boldsymbol{\omega}}_p)_{xyz} + \boldsymbol{\omega}_p \times \boldsymbol{\omega}_p = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

The angular acceleration of the disk is therefore

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_s + \dot{\boldsymbol{\omega}}_p = \{3\mathbf{i}\} \text{ rad/s}^2 \qquad \textit{Ans.}$$

Velocity and Acceleration. The velocity and acceleration of point A can be found using Eqs. 20–3 and 20–4. Realizing that $\mathbf{r}_A = \{1\mathbf{j} + 0.25\mathbf{k}\}$ m, Fig. 20–7, we have

$$\mathbf{v}_A = \boldsymbol{\omega} \times \mathbf{r}_A = (3\mathbf{j} - 1\mathbf{k}) \times (1\mathbf{j} + 0.25\mathbf{k}) = \{1.75\mathbf{i}\} \text{ m/s} \qquad \textit{Ans.}$$

$$\begin{aligned}\mathbf{a}_A &= \boldsymbol{\alpha} \times \mathbf{r}_A + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_A) \\ &= (3\mathbf{i}) \times (1\mathbf{j} + 0.25\mathbf{k}) + (3\mathbf{j} - 1\mathbf{k}) \times [(3\mathbf{j} - 1\mathbf{k}) \times (1\mathbf{j} + 0.25\mathbf{k})] \\ &= \{-2.50\mathbf{j} - 2.25\mathbf{k}\} \text{ m/s}^2 \qquad \textit{Ans.}\end{aligned}$$

EXAMPLE 20.2

At the instant $\theta = 60^\circ$, the gyrotop in Fig. 20–8 has three components of angular motion directed as shown and having magnitudes defined as:

Spin: $\omega_s = 10$ rad/s, increasing at the rate of 6 rad/s^2

Nutation: $\omega_n = 3$ rad/s, increasing at the rate of 2 rad/s^2

Precession: $\omega_p = 5$ rad/s, increasing at the rate of 4 rad/s^2

Determine the angular velocity and angular acceleration of the top.

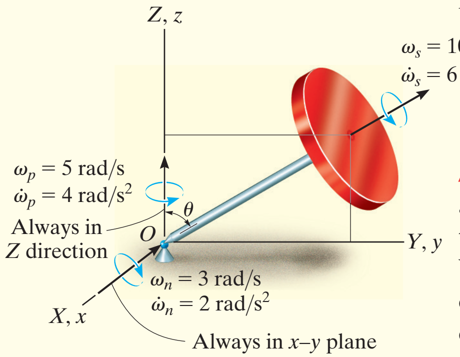


Fig. 20–8

SOLUTION

Angular Velocity. The top rotates about the fixed point O. If the fixed and rotating frames are coincident at the instant shown, then the angular velocity can be expressed in terms of **i**, **j**, **k** components, referenced from the x, y, z frame; i.e.,

$$\boldsymbol{\omega} = -\omega_n\mathbf{i} + \omega_s \sin\theta\mathbf{j} + (\omega_p + \omega_s\cos\theta)\mathbf{k}$$
$$= -3\mathbf{i} + 10\sin 60^\circ\mathbf{j} + (5 + 10\cos 60^\circ)\mathbf{k}$$
$$= \{-3\mathbf{i} + 8.66\mathbf{j} + 10\mathbf{k}\}\text{ rad/s} \qquad \textit{Ans.}$$

Angular Acceleration. As in the solution of Example 20.1, the angular acceleration $\boldsymbol{\alpha}$ will be determined by investigating separately the time rate of change of each of the angular velocity components. We will choose an $\boldsymbol{\Omega}$ for the x, y, z reference so that the component of $\boldsymbol{\omega}$ being considered is viewed as having a constant direction when observed from x, y, z.

Careful examination of the motion of the top reveals that $\boldsymbol{\omega}_s$ has a constant direction relative to x, y, z if these axes rotate at $\boldsymbol{\Omega} = \boldsymbol{\omega}_n + \boldsymbol{\omega}_p$. Thus,

$$\dot{\boldsymbol{\omega}}_s = (\dot{\boldsymbol{\omega}}_s)_{xyz} + (\boldsymbol{\omega}_n + \boldsymbol{\omega}_p) \times \boldsymbol{\omega}_s$$
$$= (6\sin 60^\circ\mathbf{j} + 6\cos 60^\circ\mathbf{k}) + (-3\mathbf{i} + 5\mathbf{k}) \times (10\sin 60^\circ\mathbf{j} + 10\cos 60^\circ\mathbf{k})$$
$$= \{-43.30\mathbf{i} + 20.20\mathbf{j} - 22.98\mathbf{k}\}\text{ rad/s}^2$$

Since $\boldsymbol{\omega}_n$ always lies in the fixed X–Y plane, this vector has a constant direction if the motion is viewed from axes x, y, z having a rotation of $\boldsymbol{\Omega} = \boldsymbol{\omega}_p$ (not $\boldsymbol{\Omega} = \boldsymbol{\omega}_s + \boldsymbol{\omega}_p$). Thus,

$$\dot{\boldsymbol{\omega}}_n = (\dot{\boldsymbol{\omega}}_n)_{xyz} + \boldsymbol{\omega}_p \times \boldsymbol{\omega}_n = -2\mathbf{i} + (5\mathbf{k}) \times (-3\mathbf{i}) = \{-2\mathbf{i} - 15\mathbf{j}\}\text{rad/s}^2$$

Finally, the component $\boldsymbol{\omega}_p$ is always directed along the Z axis so that here it is not necessary to think of x, y, z as rotating, i.e., $\boldsymbol{\Omega} = \mathbf{0}$. We therefore have

$$\dot{\boldsymbol{\omega}}_p = (\dot{\boldsymbol{\omega}}_p)_{xyz} + \mathbf{0} \times \boldsymbol{\omega}_p = \{4\mathbf{k}\}\text{ rad/s}^2$$

Thus, the angular acceleration of the top is

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}_s + \dot{\boldsymbol{\omega}}_n + \dot{\boldsymbol{\omega}}_p = \{-45.3\mathbf{i} + 5.20\mathbf{j} - 19.0\mathbf{k}\}\text{ rad/s}^2 \qquad \textit{Ans.}$$

20.3 GENERAL MOTION

Shown in Fig. 20–9 is a rigid body that has general motion in three dimensions for which the angular velocity is $\boldsymbol{\omega}$ and the angular acceleration is $\boldsymbol{\alpha}$. If point A has a known motion of $\mathbf{v}_A$ and $\mathbf{a}_A$, the motion of any other point B can be determined by using a relative-motion analysis. In this section a translating coordinate system will be used to define the relative motion, and in the next section a reference that is both rotating and translating will be considered.

If the origin of the translating system x, y, z $(\boldsymbol{\Omega} = \mathbf{0})$ is located at the "base point" A, then, at the instant shown, the motion of the body can be regarded as the sum of an instantaneous translation of the body having a motion of $\mathbf{v}_A$, and $\mathbf{a}_A$, and a rotation of the body about an instantaneous axis passing through point A. Since the body is rigid, the motion of point B measured by an observer located at A is therefore the same as the rotation of the body about a fixed point. This relative motion occurs about the instantaneous axis of rotation and is defined by $\mathbf{v}_{B/A} = \boldsymbol{\omega} \times \mathbf{r}_{B/A}$, Eq. 20–3, and $\mathbf{a}_{B/A} = \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A})$, Eq. 20–4. For translating axes, the relative motions are related to absolute motions by $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$ and $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$, Eqs. 16–15 and 16–17, so that the absolute velocity and acceleration of point B can be determined from the equations

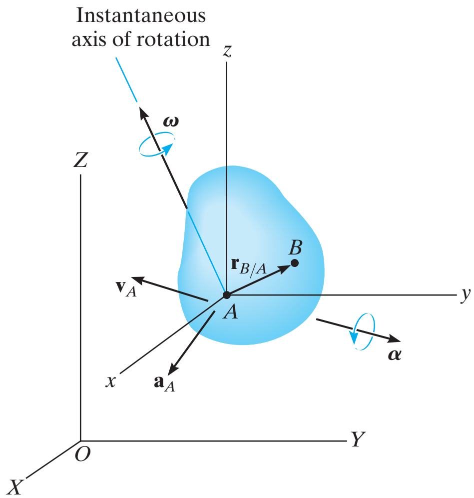


Fig. 20–9

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A} \tag{20–7}$$

and

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A}) \tag{20–8}$$

These two equations are essentially the same as those describing the general plane motion of a rigid body, Eqs. 16–16 and 16–18. However, difficulty in application arises for three-dimensional motion, because $\boldsymbol{\alpha}$ must measure the change in *both* the magnitude and direction of $\boldsymbol{\omega}$.

For many applications, an easy way to obtain $\mathbf{v}_B$ and $\mathbf{a}_B$ is to note that $\mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A$, and so Eq. 20–7 becomes $\mathbf{v}_{B/A} = \boldsymbol{\omega} \times \mathbf{r}_{B/A}$. The cross product indicates that $\mathbf{v}_{B/A}$ is *perpendicular* to both $\mathbf{r}_{B/A}$ and $\boldsymbol{\omega}$, and so, by Eq. B–14 of Appendix B, we require

$$\mathbf{r}_{B/A} \cdot \mathbf{v}_{B/A} = 0 \tag{20–9}$$

Taking the time derivative to obtain $\mathbf{a}_{B/A}$, we have

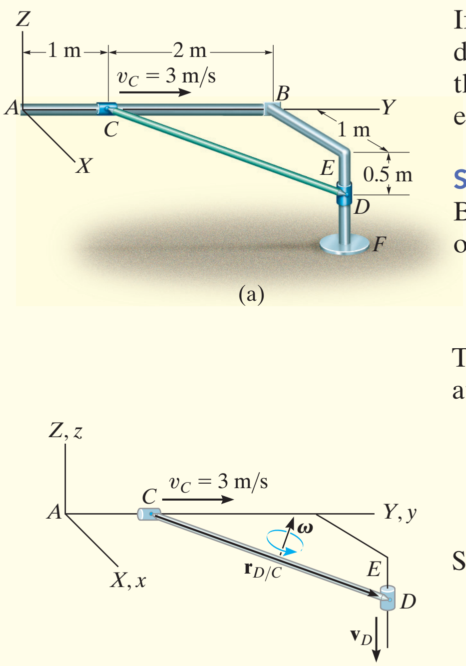
$$\mathbf{v}_{B/A} \cdot \mathbf{v}_{B/A} + \mathbf{r}_{B/A} \cdot \mathbf{a}_{B/A} = 0 \tag{20–10}$$

Then $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$.

Solution II of the following example illustrates application of this idea.

20

EXAMPLE 20.3



(a)

(b)

Fig. 20–10

If the collar at C in Fig. 20–10a moves toward B with a speed of 3 m/s, determine the velocity of the collar at D and the angular velocity of the bar at the instant shown. The bar is connected to the collars at its end points by ball-and-socket joints.

SOLUTION I

Bar CD is subjected to general motion. Why? The velocity of point D on the bar can be related to the velocity of point C by the equation

$$\mathbf{v}_D = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{r}_{D/C}$$

The fixed and translating frames of reference are assumed to coincide at the instant considered, Fig. 20–10b. We have

$$\mathbf{v}_D = -v_D\mathbf{k} \qquad \mathbf{v}_C = \{3\mathbf{j}\}\text{ m/s}$$

$$\mathbf{r}_{D/C} = \{1\mathbf{i} + 2\mathbf{j} - 0.5\mathbf{k}\}\text{ m} \qquad \boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$$

Substituting into the above equation we get

$$-v_D\mathbf{k} = 3\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 1 & 2 & -0.5 \end{vmatrix}$$

Expanding and equating the respective **i**, **j**, **k** components yields

$$-0.5\omega_y - 2\omega_z = 0 \qquad (1)$$

$$0.5\omega_x + 1\omega_z + 3 = 0 \qquad (2)$$

$$2\omega_x - 1\omega_y + v_D = 0 \qquad (3)$$

These equations contain four unknowns.* A fourth equation can be written if the direction of $\boldsymbol{\omega}$ is specified. In this regard, any component of $\boldsymbol{\omega}$ acting along the bar's axis has no effect on moving the collars, rather this component only spins the bar about its axis. Therefore, if $\boldsymbol{\omega}$ is specified as acting perpendicular to the axis of the bar, then $\boldsymbol{\omega}$ must have a unique magnitude to satisfy the above equations. Hence,

$$\boldsymbol{\omega} \cdot \mathbf{r}_{D/C} = (\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}) \cdot (1\mathbf{i} + 2\mathbf{j} - 0.5\mathbf{k}) = 0$$

$$1\omega_x + 2\omega_y - 0.5\omega_z = 0 \qquad (4)$$

*Although this is the case, the magnitude of $\mathbf{v}_D$ can be obtained. For example, solve Eqs. 1 and 2 for ω_y and ω_x in terms of ω_z and substitute this into Eq. 3. You will find ω_z will cancel out, which will allow a solution for v_D.

Solving Eqs. 1 through 4 simultaneously yields

$$\omega_x = -4.86 \text{ rad/s} \quad \omega_y = 2.29 \text{ rad/s} \quad \omega_z = -0.571 \text{ rad/s}$$

$$v_D = 12.0 \text{ m/s, so that} \qquad \omega = 5.40 \text{ rad/s}$$ *Ans.*

SOLUTION II

Applying Eq. 20–9, $\mathbf{v}_{D/C} = \mathbf{v}_D - \mathbf{v}_C = -v_D\mathbf{k} - 3\mathbf{j}$, so that

$$\mathbf{r}_{D/C} \cdot \mathbf{v}_{D/C} = (1\mathbf{i} + 2\mathbf{j} - 0.5\mathbf{k}) \cdot (-v_D\mathbf{k} - 3\mathbf{j}) = 0$$

$$(1)(0) + (2)(-3) + (-0.5)(-v_D) = 0$$

$$v_D = 12 \text{ m/s}$$ *Ans.*

Since $\boldsymbol{\omega}$ is *perpendicular* to $\mathbf{r}_{D/C}$, then from Eq. 20–7, $\mathbf{v}_{D/C} = \boldsymbol{\omega} \times \mathbf{r}_{D/C}$. We have

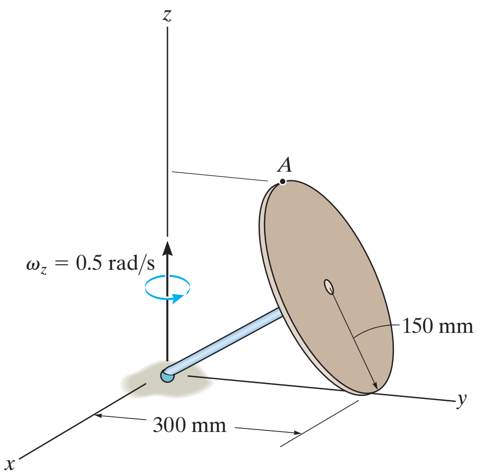
$$v_{D/C} = \omega\, r_{D/C}$$

$$\sqrt{(-12)^2 + (-3)^2} = \omega\sqrt{(1)^2 + (2)^2 + (-0.5)^2}$$

$$\omega = 5.40 \text{ rad/s}$$ *Ans.*

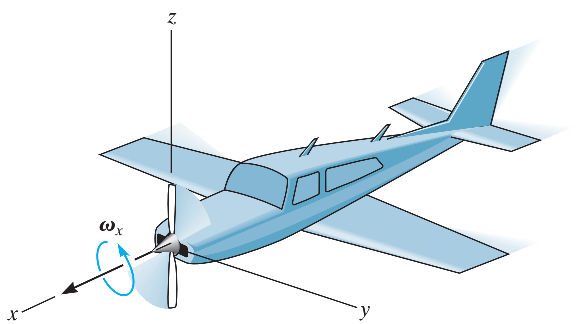
PROBLEMS

20–1. The disk rotates about the z axis at a constant rate $\omega_z = 0.5$ rad/s without slipping on the horizontal plane. Determine the velocity and the acceleration of point A on the disk.



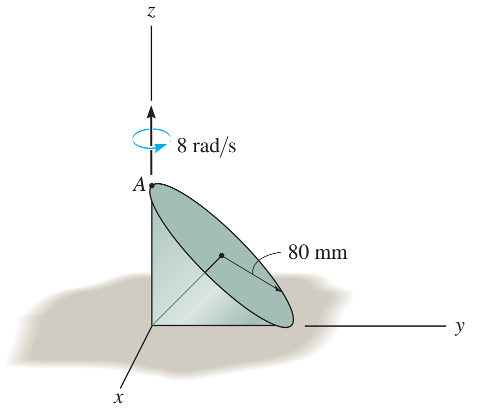
Prob. 20–1

20–2. The propeller of an airplane is rotating at a constant speed $\omega_x \mathbf{i}$, while the plane is undergoing a turn at a constant rate ω_t. Determine the angular acceleration of the propeller if (a) the turn is horizontal, i.e., $\omega_t\mathbf{k}$, and (b) the turn is vertical, downward, i.e., $\omega_t\,\mathbf{j}$.



Prob. 20–2

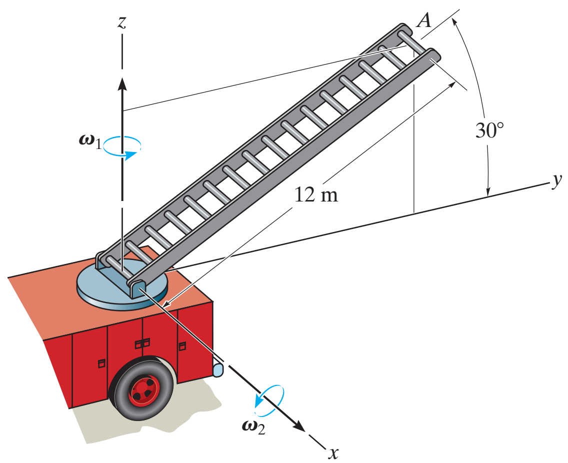
20–3. The cone rolls in a circle and rotates about the z axis at a constant rate $\omega_z = 8$ rad/s. Determine the angular velocity and angular acceleration of the cone if it rolls without slipping. Also, what are the velocity and acceleration of point A?



Prob. 20–3

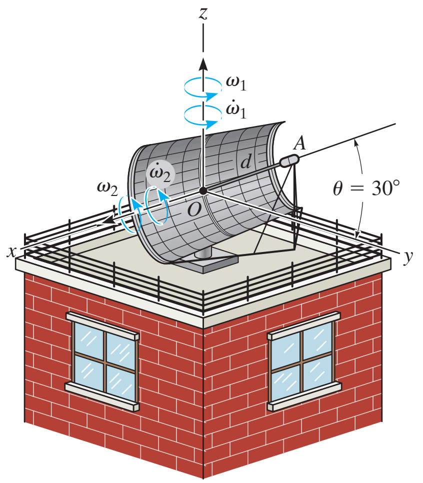
***20–4.** The ladder of the fire truck rotates around the z axis with an angular velocity $\omega_1 = 0.15$ rad/s, which is increasing at 0.8 rad/s^2. At the same instant it is rotating upward at a constant rate $\omega_2 = 0.6$ rad/s. Determine the velocity and acceleration of point A located at the top of the ladder at this instant.

20–5. The ladder of the fire truck rotates around the z axis with an angular velocity of $\omega_1 = 0.15$ rad/s, which is increasing at 0.2 rad/s^2. At the same instant it is rotating upward at $\omega_2 = 0.6$ rad/s while increasing at 0.4 rad/s^2. Determine the velocity and acceleration of point A located at the top of the ladder at this instant.



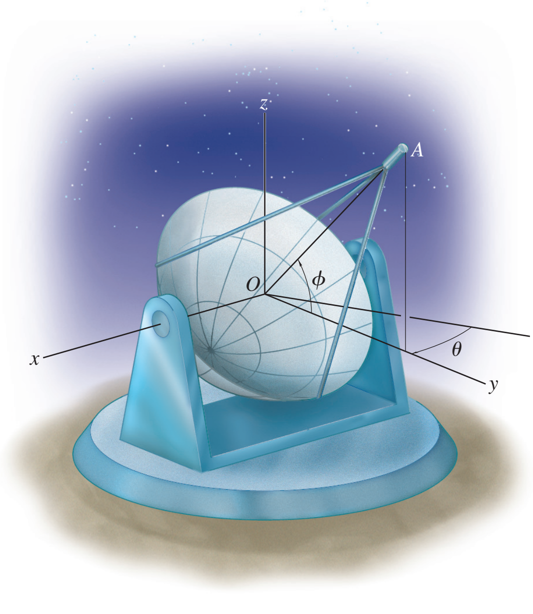
Probs. 20–4/5

20–6. At a given instant, the antenna has an angular motion $\omega_1 = 3$ rad/s and $\dot{\omega}_1 = 2$ rad/s^2 about the z axis. At this same instant $\theta = 30°$, the angular motion about the x axis is $\omega_2 = 1.5$ rad/s, and $\dot{\omega}_2 = 4$ rad/s^2. Determine the velocity and acceleration of the signal horn A at this instant. The distance from O to A is $d = 1$ m.



Prob. 20–6

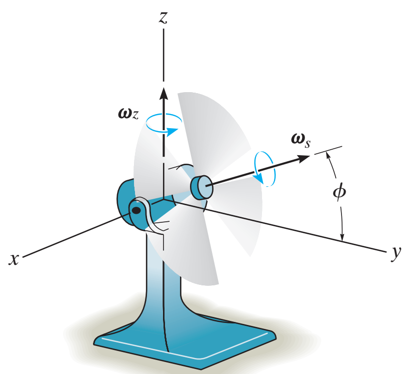
20–7. The antenna is following the motion of a jet plane. At the instant $\theta = 25°$ and $\phi = 75°$, the constant angular rates of change are $\dot{\theta} = 0.4$ rad/s and $\dot{\phi} = 0.6$ rad/s. Determine the velocity and acceleration of the signal horn A at this instant. The distance OA is 0.8 m.



Prob. 20–7

*20–8.** The electric fan is mounted on a swivel support such that the fan rotates about the z axis at a constant rate of $\omega_z = 1$ rad/s and the fan blade is spinning at a constant rate $\omega_s = 60$ rad/s. If $\phi = 45°$ for the motion, determine the angular velocity and the angular acceleration of the blade.

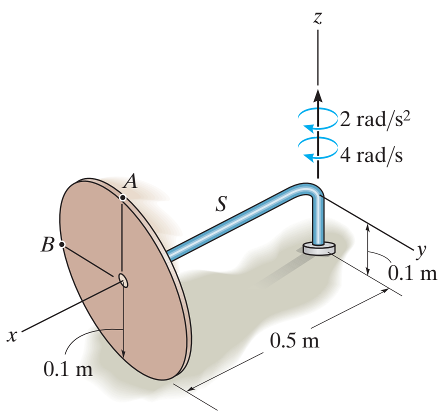
20–9. The electric fan is mounted on a swivel support such that the fan rotates about the z axis at a constant rate of $\omega_z = 1$ rad/s and the fan blade is spinning at a constant rate $\omega_s = 60$ rad/s. If at the instant $\phi = 45°$, $\dot{\phi} = 2$ rad/s for the motion, determine the angular velocity and the angular acceleration of the blade.



Probs. 20–8/9

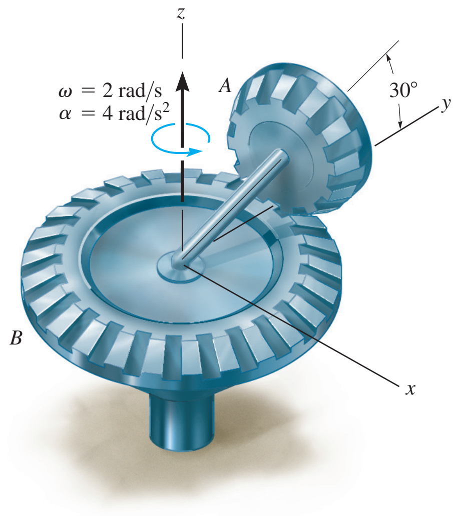
20–10. The disk rotates about the shaft S, while the shaft is turning about the z axis at a rate of $\omega_z = 4$ rad/s, which is increasing at 2 rad/s^2. Determine the velocity and acceleration of point A on the disk at the instant shown. No slipping occurs.

20–11. The disk rotates about the shaft S, while the shaft is turning about the z axis at a rate of $\omega_z = 4$ rad/s, which is increasing at 2 rad/s^2. Determine the velocity and acceleration of point B on the disk at the instant shown. No slipping occurs.



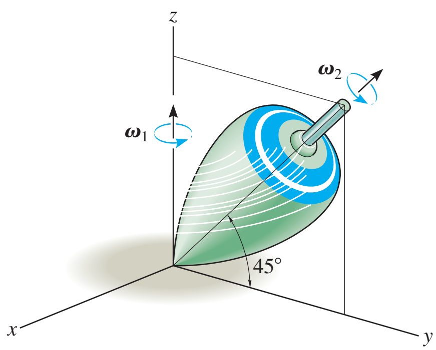
Probs. 20–10/11

*20–12.** The bevel gear A rolls on the fixed gear B. If at the instant shown the shaft to which A is attached is rotating at 2 rad/s and has an angular acceleration of 4 rad/s^2, determine the angular velocity and angular acceleration of gear A.



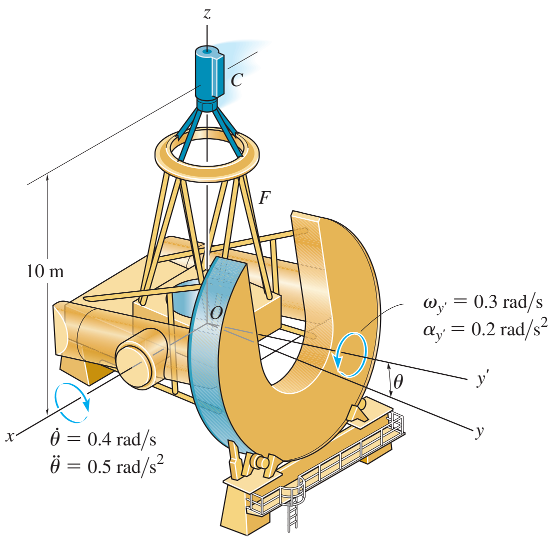
Prob. 20–12

20–13. The motion of the top is such that at the instant shown it rotates about the z axis at $\omega_1 = 0.6$ rad/s, while it spins at $\omega_2 = 8$ rad/s. Determine the angular velocity and angular acceleration of the top at this instant. Express the result as a Cartesian vector.



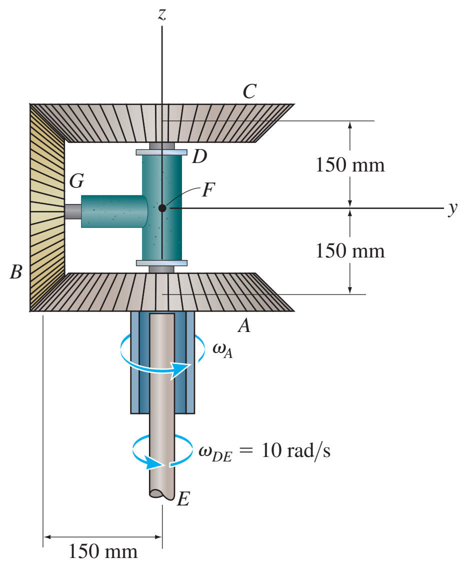
Prob. 20–13

20–14. The telescope is mounted on the frame F that allows it to be directed to any point in the sky. At the instant $\theta = 30^\circ$, the frame has an angular acceleration of $\alpha_{y'} = 0.2\ \text{rad/s}^2$ and an angular velocity of $\omega_{y'} = 0.3$ rad/s about the y' axis, and $\ddot{\theta} = 0.5\ \text{rad/s}^2$ while $\dot{\theta} = 0.4$ rad/s. Determine the velocity and acceleration of the instrument capsule at C at this instant.



Prob. 20–14

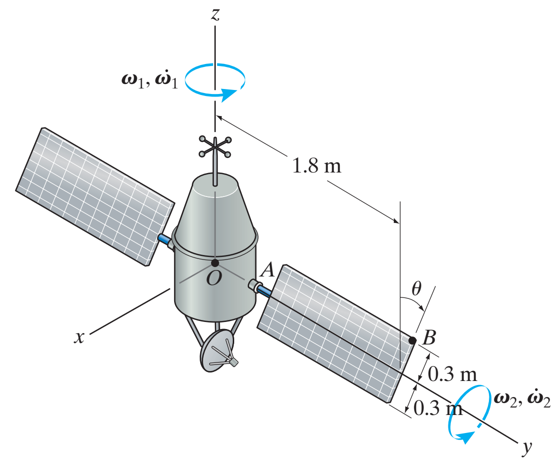
20–15. Gear C is driven by shaft DE, while gear B spins freely about its axle GF, which precesses freely about shaft DE. If gear A is held fixed ($\omega_A = 0$), and shaft DE rotates with a constant angular velocity of $\omega_{DE} = 10$ rad/s, determine the angular velocity of gear B.



Prob. 20–15

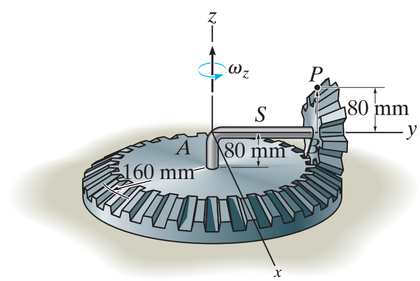
***20–16.** At the instant when $\theta = 90^\circ$, the satellite's body is rotating with an angular velocity of $\omega_1 = 15$ rad/s and angular acceleration of $\dot{\omega}_1 = 3\ \text{rad/s}^2$. Simultaneously, the solar panels rotate with an angular velocity of $\omega_2 = 6$ rad/s and angular acceleration of $\dot{\omega}_2 = 1.5\ \text{rad/s}^2$. Determine the velocity and acceleration of point B on the solar panel at this instant.

20–17. At the instant when $\theta = 90^\circ$, the satellite's body travels in the x direction with a velocity of $\mathbf{v}_O = \{500\mathbf{i}\}$ m/s and acceleration of $\mathbf{a}_O = \{50\mathbf{i}\}\ \text{m/s}^2$. Simultaneously, the body also rotates with an angular velocity of $\omega_1 = 15$ rad/s and angular acceleration of $\dot{\omega}_1 = 3\ \text{rad/s}^2$. At the same time, the solar panels rotate with an angular velocity of $\omega_2 = 6$ rad/s and angular acceleration of $\dot{\omega}_2 = 1.5\ \text{rad/s}^2$. Determine the velocity and acceleration of point B on the solar panel.



Probs. 20–16/17

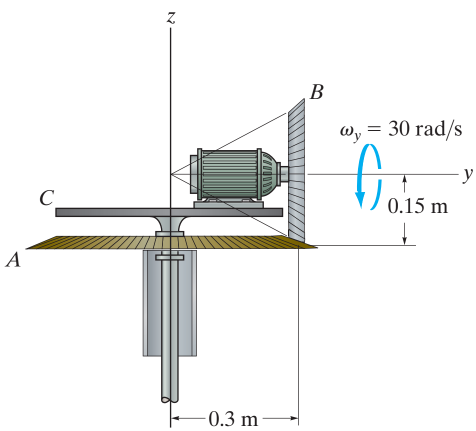
20–18. Gear A is fixed while gear B is free to rotate on the shaft S. If the shaft is turning about the z axis at $\omega_z = 5$ rad/s, while increasing at $2\ \text{rad/s}^2$, determine the velocity and acceleration of point P at the instant shown. The face of gear B lies in a vertical plane.



Prob. 20–18

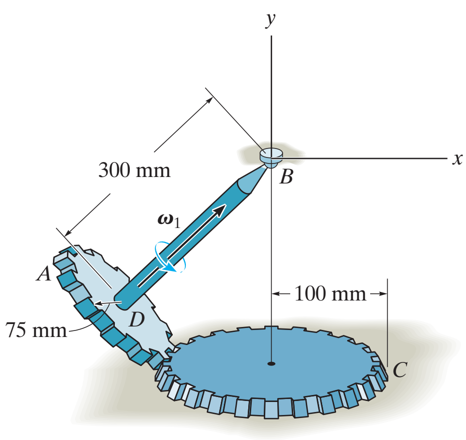
20–19. Gear B is driven by a motor mounted on turntable C. If gear A is held fixed, and the motor shaft rotates with a constant angular velocity of $\omega_y = 30$ rad/s, determine the angular velocity and angular acceleration of gear B.

***20–20.** Gear B is driven by a motor mounted on turntable C. If gear A and the motor shaft rotate with constant angular speeds of $\boldsymbol{\omega}_A = \{10\mathbf{k}\}$ rad/s and $\boldsymbol{\omega}_y = \{30\mathbf{j}\}$ rad/s, respectively, determine the angular velocity and angular acceleration of gear B.



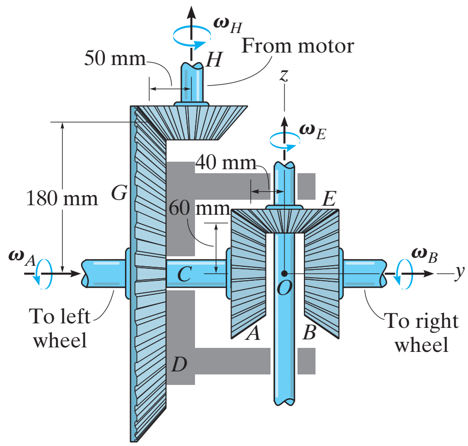
Probs. 20–19/20

20–21. Shaft BD is connected to a ball-and-socket joint at B, and a beveled gear A is attached to its other end. The gear is in mesh with a fixed gear C. If the shaft and gear A are *spinning* with a constant angular velocity $\omega_1 = 8$ rad/s, determine the angular velocity and angular acceleration of gear A.



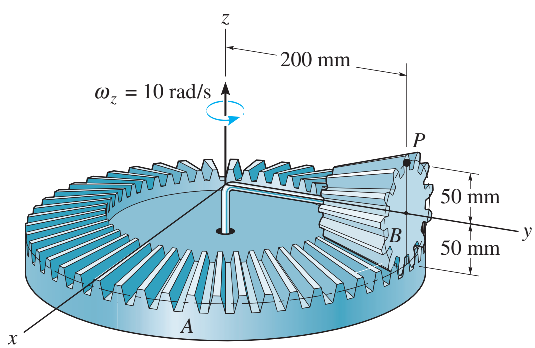
Prob. 20–21

20–22. The differential of an automobile allows the two rear wheels to rotate at different speeds when the automobile travels along a curve. For operation, the rear axles are attached to the wheels at one end and have beveled gears A and B on their other ends. The differential case D is placed over the left axle but can rotate about C independent of the axle. The case supports a pinion gear E on a shaft, which meshes with gears A and B. Finally, a ring gear G is *fixed* to the differential case so that the case rotates with the ring gear when the latter is driven by the drive pinion H. This gear, like the differential case, is free to rotate about the left wheel axle. If the drive pinion is turning at $\omega_H = 100$ rad/s and the pinion gear E is spinning about its shaft at $\omega_E = 30$ rad/s, determine the angular velocity, ω_A and ω_B, of each axle.



Prob. 20–22

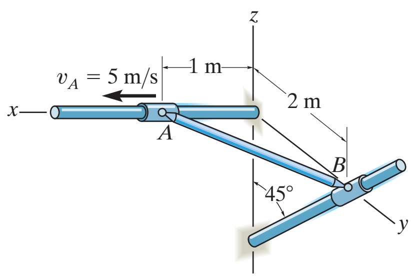
20–23. Gear B is connected to the rotating shaft, while the plate gear A is fixed. If the shaft is turning at a constant rate of $\omega_z = 10$ rad/s about the z axis, determine the magnitudes of the angular velocity and the angular acceleration of gear B. Also, determine the magnitudes of the velocity and acceleration of point P.



Prob. 20–23

***20–24.** Rod AB is attached to collars at its ends by using ball-and-socket joints. If collar A moves along the fixed rod at $v_A = 5$ m/s, determine the angular velocity of the rod and the velocity of collar B at the instant shown. Assume that the rod's angular velocity is directed perpendicular to the axis of the rod.

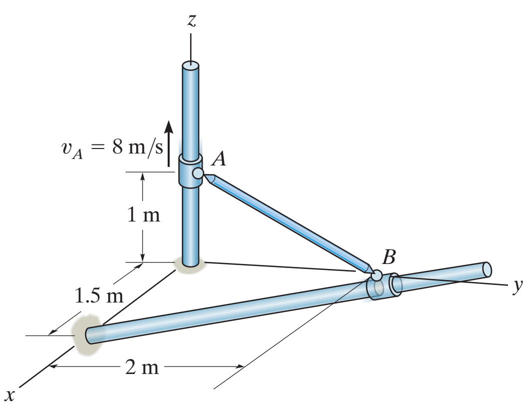
20–25. Rod AB is attached to collars at its ends by using ball-and-socket joints. If collar A moves along the fixed rod with a velocity of $v_A = 5$ m/s and has an acceleration $a_A = 2$ m/s^2 at the instant shown, determine the angular acceleration of the rod and the acceleration of collar B at this instant. Assume that the rod's angular velocity and angular acceleration are directed perpendicular to the axis of the rod.



Probs. 20–24/25

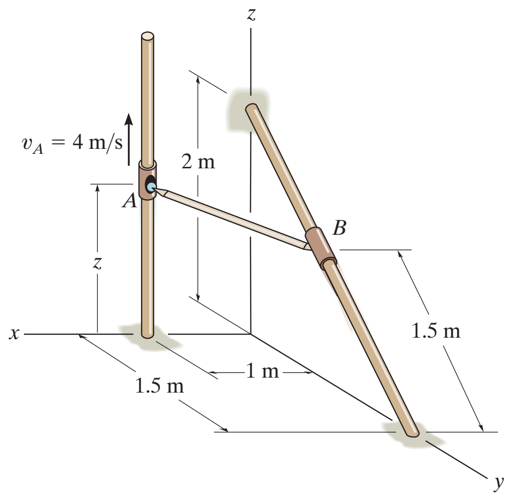
20–26. The rod is attached to smooth collars A and B at its ends using ball-and-socket joints. Determine the speed of B at the instant shown if A is moving at $v_A = 8$ m/s. Also, determine the angular velocity of the rod if it is directed perpendicular to the axis of the rod.

20–27. If the collar A in Prob. 20–26 has a deceleration of $\mathbf{a}_A = \{-5\mathbf{k}\}$ m/s^2, at the instant shown, determine the acceleration of collar B at this instant.



Probs. 20–26/27

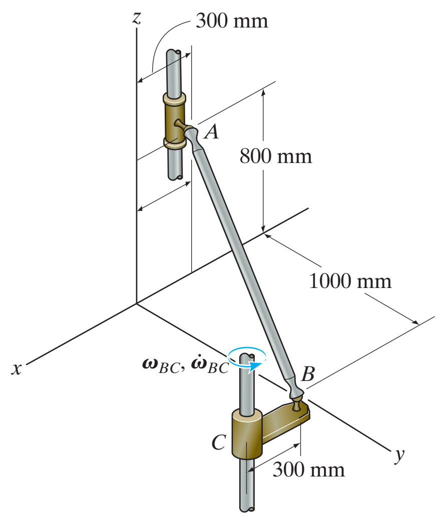
***20–28.** Rod AB is attached to collars at its ends by ball-and-socket joints. If collar A has a speed $v_A = 4$ m/s, determine the speed of collar B at the instant $z = 2$ m. Assume the angular velocity of the rod is directed perpendicular to the rod.



Prob. 20–28

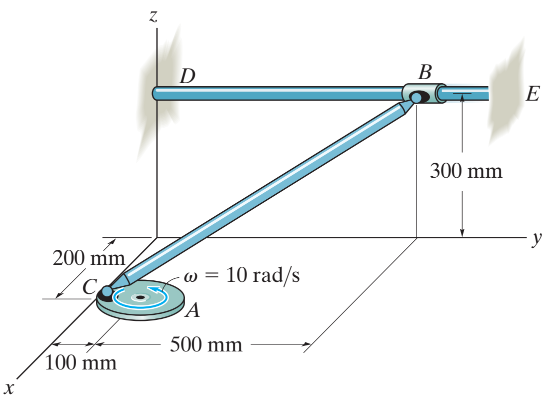
20–29. If crank BC rotates with a constant angular velocity of $\omega_{BC} = 6$ rad/s, determine the velocity of the collar at A. Assume the angular velocity of AB is perpendicular to the rod.

20–30. If crank BC is rotating with an angular velocity of $\omega_{BC} = 6$ rad/s and an angular acceleration of $\dot{\omega}_{BC} = 1.5$ rad/s^2, determine the acceleration of collar A at this instant. Assume the angular velocity and angular acceleration of AB are perpendicular to the rod.



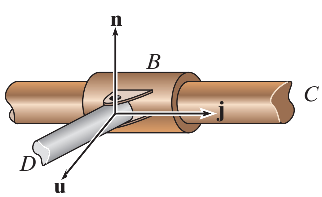
Probs. 20–29/30

20–31. Disk A rotates at a constant angular velocity of 10 rad/s. If rod BC is joined to the disk and a collar by ball-and-socket joints, determine the velocity of collar B at the instant shown. Also, what is the rod's angular velocity $\boldsymbol{\omega}_{BC}$ if it is directed perpendicular to the axis of the rod?



Prob. 20–31

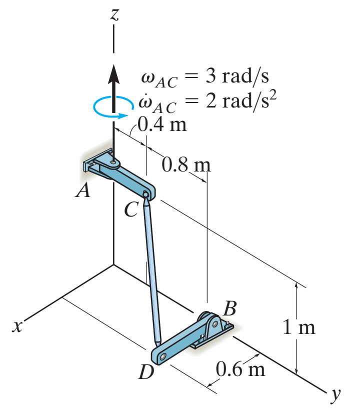
***20–32.** Solve Prob. 20–31 if the connection at B consists of a pin as shown in the figure below, rather than a ball-and-socket joint. *Hint:* The constraint allows rotation of the rod both along the bar (**j** direction) and along the axis of the pin (**n** direction). Since there is no rotational component in the **u** direction, i.e., perpendicular to **n** and **j** where $\mathbf{u} = \mathbf{j} \times \mathbf{n}$, an additional equation for solution can be obtained from $\boldsymbol{\omega} \cdot \mathbf{u} = 0$. The vector **n** is in the same direction as $\mathbf{r}_{B/C} \times \mathbf{r}_{D/C}$.



Prob. 20–32

20–33. Rod CD is attached to the rotating arms using ball-and-socket joints. If AC has the motion shown, determine the angular velocity of link BD at the instant shown.

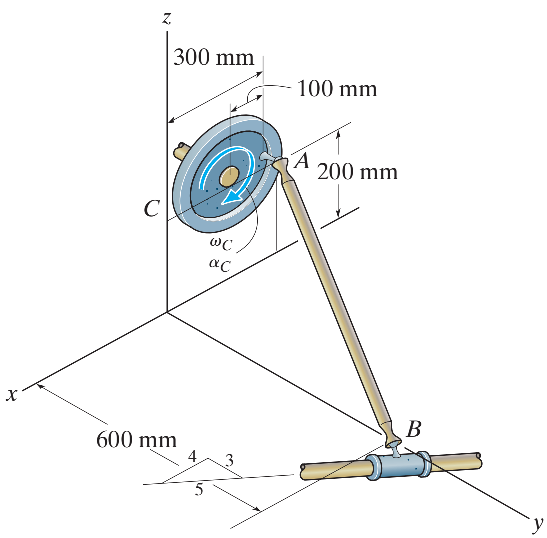
20–34. Rod CD is attached to the rotating arms using ball-and-socket joints. If AC has the motion shown, determine the angular acceleration of link BD at this instant.



Probs. 20–33/34

20–35. If wheel C rotates with a constant angular velocity of $\omega_C = 10$ rad/s, determine the velocity of the collar at B when rod AB is in the position shown. Assume the angular velocity of AB is perpendicular to the rod.

***20–36.** At the instant rod AB is in the position shown wheel C rotates with an angular velocity of $\omega_C = 10$ rad/s and has an angular acceleration of $\alpha_C = 1.5$ rad/s^2. Determine the acceleration of collar B at this instant. Assume the angular velocity and angular acceleration of AB are perpendicular to the rod.



Probs. 20–35/36

*20.4 RELATIVE-MOTION ANALYSIS USING TRANSLATING AND ROTATING AXES

The most general way to analyze the three-dimensional motion of a rigid body requires the use of x, y, z axes that both translate and rotate relative to a second frame X, Y, Z. This analysis also provides a means to determine the motions of two points A and B located on separate members of a mechanism, and the relative motion of one particle with respect to another when one or both particles are moving along *curved paths*.

As shown in Fig. 20–11, the locations of points A and B are specified relative to the X, Y, Z frame of reference by position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$. The base point A represents the origin of the x, y, z coordinate system, which is translating and rotating with respect to X, Y, Z. At the instant considered, the velocity and acceleration of point A are $\mathbf{v}_A$ and $\mathbf{a}_A$, and the angular velocity and angular acceleration of the x, y, z axes are $\boldsymbol{\Omega}$ and $\dot{\boldsymbol{\Omega}} = d\boldsymbol{\Omega}/dt$. All these vectors are *measured* with respect to the X, Y, Z frame of reference, although they can be expressed in Cartesian component form along either set of axes.

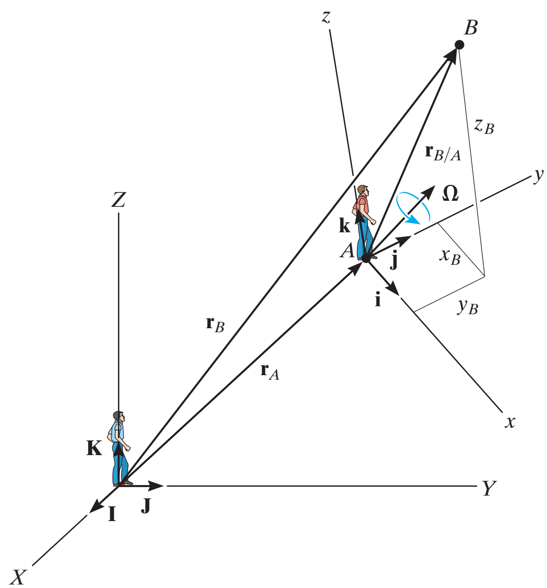


Fig. 20–11

Position. If the position of "B with respect to A" is specified by the ***relative-position vector*** $\mathbf{r}_{B/A}$, Fig. 20–11, then, by vector addition,

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \tag{20–11}$$

where

$\mathbf{r}_B$ = position of B
$\mathbf{r}_A$ = position of the origin A
$\mathbf{r}_{B/A}$ = position of "B with respect to A"

Velocity. The velocity of point B measured from X, Y, Z can be determined by taking the time derivative of Eq. 20–11,

$$\dot{\mathbf{r}}_B = \dot{\mathbf{r}}_A + \dot{\mathbf{r}}_{B/A}$$

The first two terms represent $\mathbf{v}_B$ and $\mathbf{v}_A$. The last term must be evaluated by applying Eq. 20–6, since $\mathbf{r}_{B/A}$ is measured from the rotating reference. Hence,

$$\dot{\mathbf{r}}_{B/A} = (\dot{\mathbf{r}}_{B/A})_{xyz} + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} = (\mathbf{v}_{B/A})_{xyz} + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} \tag{20–12}$$

Therefore,

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz} \tag{20–13}$$

where

$\mathbf{v}_B$ = velocity of B
$\mathbf{v}_A$ = velocity of the origin A of the x, y, z frame of reference
$(\mathbf{v}_{B/A})_{xyz}$ = velocity of "B with respect to A" as measured by an observer attached to the rotating x, y, z frame of reference
$\boldsymbol{\Omega}$ = angular velocity of the x, y, z frame of reference
$\mathbf{r}_{B/A}$ = position of "B with respect to A"

Acceleration. The acceleration of point B measured from X, Y, Z is determined by taking the time derivative of Eq. 20–13.

$$\dot{\mathbf{v}}_B = \dot{\mathbf{v}}_A + \dot{\mathbf{\Omega}} \times \mathbf{r}_{B/A} + \mathbf{\Omega} \times \dot{\mathbf{r}}_{B/A} + \frac{d}{dt}(\mathbf{v}_{B/A})_{xyz}$$

The term on the left and the first term on the right represent $\mathbf{a}_B$ and $\mathbf{a}_A$, respectively. The last term can be evaluated using Eq. 20–12, and the last term is evaluated by applying Eq. 20–6, which yields

$$\frac{d}{dt}(\mathbf{v}_{B/A})_{xyz} = (\dot{\mathbf{v}}_{B/A})_{xyz} + \mathbf{\Omega} \times (\mathbf{v}_{B/A})_{xyz} = (\mathbf{a}_{B/A})_{xyz} + \mathbf{\Omega} \times (\mathbf{v}_{B/A})_{xyz}$$

Here $(\mathbf{a}_{B/A})_{xyz}$ is the acceleration of B with respect to A measured from x, y, z. Substituting this result and Eq. 20–12 into the above equation and simplifying, we have

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\mathbf{\Omega}} \times \mathbf{r}_{B/A} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{B/A}) + 2\mathbf{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz} \tag{20–14}$$

where

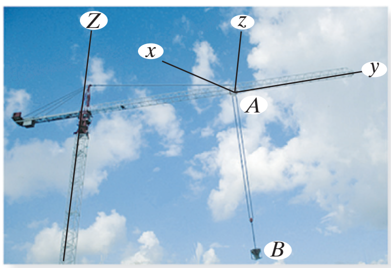
$\mathbf{a}_B$ = acceleration of B

$\mathbf{a}_A$ = acceleration of the origin A of the x, y, z frame of reference

$(\mathbf{a}_{B/A})_{xyz}, (\mathbf{v}_{B/A})_{xyz}$ = relative acceleration and relative velocity of "B with respect to A" as measured by an observer attached to the rotating x, y, z frame of reference

$\dot{\mathbf{\Omega}}, \mathbf{\Omega}$ = angular acceleration and angular velocity of the x, y, z frame of reference

$\mathbf{r}_{B/A}$ = position of "B with respect to A"



Complicated spatial motion of the concrete bucket B occurs due to the rotation of the boom about the Z axis, motion of the carriage A along the boom, and extension and swinging of the cable AB. A translating-rotating x, y, z coordinate system can be established on the carriage, and a relative-motion analysis can then be applied to study this motion.

Equations 20–13 and 20–14 are identical to those used in Sec. 16.8 for analyzing relative plane motion.* In that case, however, application is simplified since $\mathbf{\Omega}$ and $\dot{\mathbf{\Omega}}$ have a *constant direction* which is always perpendicular to the plane of motion. For three-dimensional motion, $\dot{\mathbf{\Omega}}$ must be determined by using Eq. 20–6, since $\dot{\mathbf{\Omega}}$ depends on the change in *both* the magnitude and direction of $\mathbf{\Omega}$.

*Refer to Sec. 16.8 for an interpretation of the terms.

PROCEDURE FOR ANALYSIS

Three-dimensional motion of particles or rigid bodies can be analyzed with Eqs. 20–13 and 20–14 by using the following procedure.

Coordinate Axes.

- Select the location and orientation of the X, Y, Z and x, y, z coordinate axes. Most often solutions can be easily obtained if at the instant considered:

  (1) the origins are coincident

  (2) the axes are collinear

  (3) the axes are parallel

- If several components of angular velocity are involved in a problem, the calculations will be reduced if the x, y, z axes are selected such that only one component of angular velocity is observed with respect to this frame $(\boldsymbol{\Omega}_{xyz})$ and the frame rotates with $\boldsymbol{\Omega}$ defined by the other components of angular velocity.

Kinematic Equations.

- After the origin of the moving reference, A, is defined and the moving point B is specified, Eqs. 20–13 and 20–14 should then be written in symbolic form as

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz}$$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}$$

- If $\mathbf{r}_A$ and $\boldsymbol{\Omega}$ appear to *change direction* when observed from the fixed X, Y, Z reference, then use a set of primed reference axes, x', y', z', having a rotation $\boldsymbol{\Omega}' = \boldsymbol{\Omega}$. Equation 20–6 is then used to determine $\dot{\boldsymbol{\Omega}}$ and the motion $\mathbf{v}_A$ and $\mathbf{a}_A$ of the origin of the moving x, y, z axes.

- If $\mathbf{r}_{B/A}$ and $\boldsymbol{\Omega}_{xyz}$ appear to change direction as observed from x, y, z, then use a set of double-primed reference axes x'', y'', z'', having a rotation $\boldsymbol{\Omega}'' = \boldsymbol{\Omega}_{xyz}$, and apply Eq. 20–6 to determine $\dot{\boldsymbol{\Omega}}_{xyz}$ and the relative motion $(\mathbf{v}_{B/A})_{xyz}$ and $(\mathbf{a}_{B/A})_{xyz}$.

- After the final forms of $\dot{\boldsymbol{\Omega}}$, $\mathbf{v}_A$, $\mathbf{a}_A$, $\dot{\boldsymbol{\Omega}}_{xyz}$, $(\mathbf{v}_{B/A})_{xyz}$, and $(\mathbf{a}_{B/A})_{xyz}$ are obtained, numerical problem data can be substituted and the terms evaluated. The components of all these vectors can be selected either along the X, Y, Z or along the x, y, z axes. The choice is arbitrary, provided a consistent set of unit vectors is used.

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EXAMPLE 20.4

A motor and attached rod AB have the angular motions shown in Fig. 20–12. A collar C on the rod is located 0.25 m from A and is moving downward along the rod with a velocity of 3 m/s and an acceleration of 2 m/s^2. Determine the velocity and acceleration of C at this instant.

SOLUTION

Coordinate Axes. The origin of the fixed X, Y, Z reference is chosen at the center of the platform, and the origin of the moving x, y, z frame is at point A, Fig. 20–12. Since the collar is subjected to two components of angular motion, $\boldsymbol{\omega}_p$ and $\boldsymbol{\omega}_M$, it will be viewed as having an angular velocity of $\boldsymbol{\Omega}_{xyz} = \boldsymbol{\omega}_M$ in x, y, z. Therefore, the x, y, z axes will be attached to the platform so that $\boldsymbol{\Omega} = \boldsymbol{\omega}_p$.

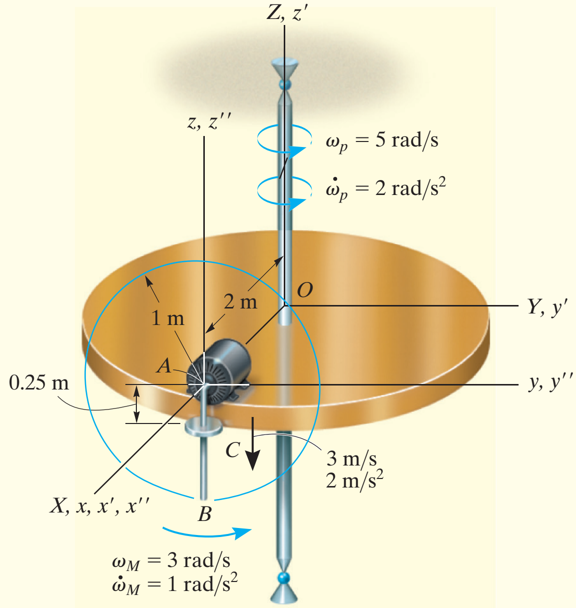

Fig. 20–12

Kinematic Equations. Equations 20–13 and 20–14, applied to points C and A, become

$$\mathbf{v}_C = \mathbf{v}_A + \mathbf{\Omega} \times \mathbf{r}_{C/A} + (\mathbf{v}_{C/A})_{xyz}$$

$$\mathbf{a}_C = \mathbf{a}_A + \dot{\mathbf{\Omega}} \times \mathbf{r}_{C/A} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{C/A}) + 2\mathbf{\Omega} \times (\mathbf{v}_{C/A})_{xyz} + (\mathbf{a}_{C/A})_{xyz}$$

Motion of A. Here $\mathbf{r}_A$ changes direction relative to X, Y, Z. To find the time derivatives of $\mathbf{r}_A$ we will use a set of x', y', z' axes coincident with the X, Y, Z axes that rotate at $\mathbf{\Omega}' = \boldsymbol{\omega}_p$. Thus,

$$\mathbf{\Omega} = \boldsymbol{\omega}_p = \{5\mathbf{k}\} \text{ rad/s } (\mathbf{\Omega} \text{ does not change direction relative to } X, Y, Z.)$$

$$\dot{\mathbf{\Omega}} = \dot{\boldsymbol{\omega}}_p = \{2\mathbf{k}\} \text{ rad/s}^2$$

$$\mathbf{r}_A = \{2\mathbf{i}\} \text{ m}$$

$$\mathbf{v}_A = \dot{\mathbf{r}}_A = (\dot{\mathbf{r}}_A)_{x'y'z'} + \boldsymbol{\omega}_p \times \mathbf{r}_A = \mathbf{0} + 5\mathbf{k} \times 2\mathbf{i} = \{10\mathbf{j}\} \text{ m/s}$$

$$\mathbf{a}_A = \ddot{\mathbf{r}}_A = [(\ddot{\mathbf{r}}_A)_{x'y'z'} + \boldsymbol{\omega}_p \times (\dot{\mathbf{r}}_A)_{x'y'z'}] + \dot{\boldsymbol{\omega}}_p \times \mathbf{r}_A + \boldsymbol{\omega}_p \times \dot{\mathbf{r}}_A$$
$$= [\mathbf{0} + \mathbf{0}] + 2\mathbf{k} \times 2\mathbf{i} + 5\mathbf{k} \times 10\mathbf{j} = \{-50\mathbf{i} + 4\mathbf{j}\} \text{ m/s}^2$$

Motion of C with Respect to A. Here $\mathbf{r}_{C/A}$ changes direction relative to x, y, z, and so to find its time derivatives use a set of x'', y'', z'' axes that rotate at $\mathbf{\Omega}'' = \mathbf{\Omega}_{xyz} = \boldsymbol{\omega}_M$. Thus,

$$\mathbf{\Omega}_{xyz} = \boldsymbol{\omega}_M = \{3\mathbf{i}\} \text{ rad/s } (\mathbf{\Omega}_{xyz} \text{ does not change direction relative to } x, y, z.)$$

$$\dot{\mathbf{\Omega}}_{xyz} = \dot{\boldsymbol{\omega}}_M = \{1\mathbf{i}\} \text{ rad/s}^2$$

$$\mathbf{r}_{C/A} = \{-0.25\mathbf{k}\} \text{ m}$$

$$(\mathbf{v}_{C/A})_{xyz} = (\dot{\mathbf{r}}_{C/A})_{xyz} = (\dot{\mathbf{r}}_{C/A})_{x''y''z''} + \boldsymbol{\omega}_M \times \mathbf{r}_{C/A}$$
$$= -3\mathbf{k} + [3\mathbf{i} \times (-0.25\mathbf{k})] = \{0.75\mathbf{j} - 3\mathbf{k}\} \text{ m/s}$$

$$(\mathbf{a}_{C/A})_{xyz} = (\ddot{\mathbf{r}}_{C/A})_{xyz} = [(\ddot{\mathbf{r}}_{C/A})_{x''y''z''} + \boldsymbol{\omega}_M \times (\dot{\mathbf{r}}_{C/A})_{x''y''z''}] + \dot{\boldsymbol{\omega}}_M \times \mathbf{r}_{C/A} + \boldsymbol{\omega}_M \times (\dot{\mathbf{r}}_{C/A})_{xyz}$$
$$= [-2\mathbf{k} + 3\mathbf{i} \times (-3\mathbf{k})] + (1\mathbf{i}) \times (-0.25\mathbf{k}) + (3\mathbf{i}) \times (0.75\mathbf{j} - 3\mathbf{k})$$
$$= \{18.25\mathbf{j} + 0.25\mathbf{k}\} \text{ m/s}^2$$

Motion of C.

$$\mathbf{v}_C = \mathbf{v}_A + \mathbf{\Omega} \times \mathbf{r}_{C/A} + (\mathbf{v}_{C/A})_{xyz}$$
$$= 10\mathbf{j} + [5\mathbf{k} \times (-0.25\mathbf{k})] + (0.75\mathbf{j} - 3\mathbf{k})$$
$$= \{10.75\mathbf{j} - 3\mathbf{k}\} \text{ m/s} \qquad \textit{Ans.}$$

$$\mathbf{a}_C = \mathbf{a}_A + \dot{\mathbf{\Omega}} \times \mathbf{r}_{C/A} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{C/A}) + 2\mathbf{\Omega} \times (\mathbf{v}_{C/A})_{xyz} + (\mathbf{a}_{C/A})_{xyz}$$
$$= (-50\mathbf{i} + 4\mathbf{j}) + [2\mathbf{k} \times (-0.25\mathbf{k})] + 5\mathbf{k} \times [5\mathbf{k} \times (-0.25\mathbf{k})]$$
$$+ 2[5\mathbf{k} \times (0.75\mathbf{j} - 3\mathbf{k})] + (18.25\mathbf{j} + 0.25\mathbf{k})$$
$$= \{-57.5\mathbf{i} + 22.25\mathbf{j} + 0.25\mathbf{k}\} \text{ m/s}^2 \qquad \textit{Ans.}$$

EXAMPLE 20.5

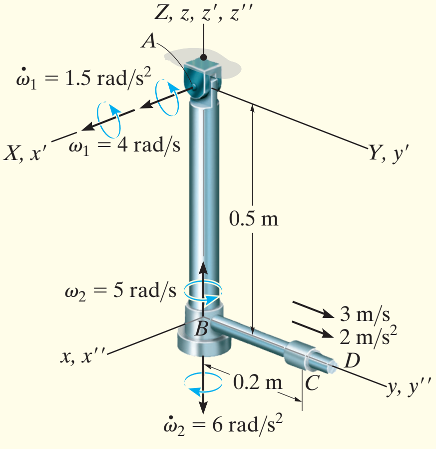

Fig. 20–13

The pendulum shown in Fig. 20–13 consists of two rods; AB is pin supported at A and swings only in the Y–Z plane, whereas a bearing at B allows the attached rod BD to spin about rod AB. At a given instant, the rods have the angular motions shown. Also, a collar C, located 0.2 m from B, has a velocity of 3 m/s and an acceleration of 2 m/s^2 along the rod. Determine the velocity and acceleration of the collar at this instant.

SOLUTION I

Coordinate Axes. The origin of the fixed X, Y, Z frame will be placed at A. Motion of the collar is conveniently observed from B, so the origin of the x, y, z frame is located at this point. We will choose $\mathbf{\Omega} = \boldsymbol{\omega}_1$ and $\mathbf{\Omega}_{xyz} = \boldsymbol{\omega}_2$.

Kinematic Equations.

$$\mathbf{v}_C = \mathbf{v}_B + \mathbf{\Omega} \times \mathbf{r}_{C/B} + (\mathbf{v}_{C/B})_{xyz}$$

$$\mathbf{a}_C = \mathbf{a}_B + \dot{\mathbf{\Omega}} \times \mathbf{r}_{C/B} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{C/B}) + 2\mathbf{\Omega} \times (\mathbf{v}_{C/B})_{xyz} + (\mathbf{a}_{C/B})_{xyz}$$

Motion of *B*. To find the time derivatives of $\mathbf{r}_B$ let the x', y', z' axes rotate with $\mathbf{\Omega}' = \boldsymbol{\omega}_1$. Then

$$\mathbf{\Omega}' = \boldsymbol{\omega}_1 = \{4\mathbf{i}\} \text{ rad/s} \qquad \dot{\mathbf{\Omega}}' = \dot{\boldsymbol{\omega}}_1 = \{1.5\mathbf{i}\} \text{ rad/s}^2$$

$$\mathbf{r}_B = \{-0.5\mathbf{k}\} \text{ m}$$

$$\mathbf{v}_B = \dot{\mathbf{r}}_B = (\dot{\mathbf{r}}_B)_{x'y'z'} + \boldsymbol{\omega}_1 \times \mathbf{r}_B = \mathbf{0} + 4\mathbf{i} \times (-0.5\mathbf{k}) = \{2\mathbf{j}\} \text{ m/s}$$

$$\mathbf{a}_B = \ddot{\mathbf{r}}_B = [(\ddot{\mathbf{r}}_B)_{x'y'z'} + \boldsymbol{\omega}_1 \times (\dot{\mathbf{r}}_B)_{x'y'z'}] + \dot{\boldsymbol{\omega}}_1 \times \mathbf{r}_B + \boldsymbol{\omega}_1 \times \dot{\mathbf{r}}_B$$

$$= [\mathbf{0} + \mathbf{0}] + 1.5\mathbf{i} \times (-0.5\mathbf{k}) + 4\mathbf{i} \times 2\mathbf{j} = \{0.75\mathbf{j} + 8\mathbf{k}\} \text{ m/s}^2$$

Motion of *C* with Respect to *B*. To find the time derivatives of $\mathbf{r}_{C/B}$ relative to x, y, z, let the x'', y'', z'' axes rotate with $\mathbf{\Omega}_{xyz} = \boldsymbol{\omega}_2$. Then

$$\mathbf{\Omega}_{xyz} = \boldsymbol{\omega}_2 = \{5\mathbf{k}\} \text{ rad/s} \qquad \dot{\mathbf{\Omega}}_{xyz} = \dot{\boldsymbol{\omega}}_2 = \{-6\mathbf{k}\} \text{ rad/s}^2$$

$$\mathbf{r}_{C/B} = \{0.2\mathbf{j}\} \text{ m}$$

$$(\mathbf{v}_{C/B})_{xyz} = (\dot{\mathbf{r}}_{C/B})_{xyz} = (\dot{\mathbf{r}}_{C/B})_{x''y''z''} + \boldsymbol{\omega}_2 \times \mathbf{r}_{C/B} = 3\mathbf{j} + 5\mathbf{k} \times 0.2\mathbf{j} = \{-1\mathbf{i} + 3\mathbf{j}\} \text{ m/s}$$

$$(\mathbf{a}_{C/B})_{xyz} = (\ddot{\mathbf{r}}_{C/B})_{xyz} = [(\ddot{\mathbf{r}}_{C/B})_{x''y''z''} + \boldsymbol{\omega}_2 \times (\dot{\mathbf{r}}_{C/B})_{x''y''z''}] + \dot{\boldsymbol{\omega}}_2 \times \mathbf{r}_{C/B} + \boldsymbol{\omega}_2 \times (\dot{\mathbf{r}}_{C/B})_{xyz}$$

$$= (2\mathbf{j} + 5\mathbf{k} \times 3\mathbf{j}) + (-6\mathbf{k} \times 0.2\mathbf{j}) + [5\mathbf{k} \times (-1\mathbf{i} + 3\mathbf{j})]$$

$$= \{-28.8\mathbf{i} - 3\mathbf{j}\} \text{ m/s}^2$$

Motion of *C*.

$$\mathbf{v}_C = \mathbf{v}_B + \mathbf{\Omega} \times \mathbf{r}_{C/B} + (\mathbf{v}_{C/B})_{xyz} = 2\mathbf{j} + 4\mathbf{i} \times 0.2\mathbf{j} + (-1\mathbf{i} + 3\mathbf{j})$$

$$= \{-1\mathbf{i} + 5\mathbf{j} + 0.8\mathbf{k}\} \text{ m/s}$$ *Ans.*

$$\mathbf{a}_C = \mathbf{a}_B + \dot{\mathbf{\Omega}} \times \mathbf{r}_{C/B} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{C/B}) + 2\mathbf{\Omega} \times (\mathbf{v}_{C/B})_{xyz} + (\mathbf{a}_{C/B})_{xyz}$$

$$= (0.75\mathbf{j} + 8\mathbf{k}) + (1.5\mathbf{i} \times 0.2\mathbf{j}) + [4\mathbf{i} \times (4\mathbf{i} \times 0.2\mathbf{j})]$$

$$+ 2[4\mathbf{i} \times (-1\mathbf{i} + 3\mathbf{j})] + (-28.8\mathbf{i} - 3\mathbf{j})$$

$$= \{-28.8\mathbf{i} - 5.45\mathbf{j} + 32.3\mathbf{k}\} \text{ m/s}^2$$ *Ans.*

SOLUTION II

Coordinate Axes. Here we will let the x, y, z axes rotate at

$$\boldsymbol{\Omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = \{4\mathbf{i} + 5\mathbf{k}\} \text{ rad/s}$$

Then $\boldsymbol{\Omega}_{xyz} = \mathbf{0}$.

Motion of *B*. From the constraints of the problem $\boldsymbol{\omega}_1$ does not change direction relative to X, Y, Z; however, the direction of $\boldsymbol{\omega}_2$ is changed by $\boldsymbol{\omega}_1$. Thus, to obtain $\dot{\boldsymbol{\Omega}}$ consider x', y', z' axes coincident with the X, Y, Z axes at A, so that $\boldsymbol{\Omega}' = \boldsymbol{\omega}_1$. Then taking the derivative of the components of $\boldsymbol{\Omega}$,

$$\dot{\boldsymbol{\Omega}} = \dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_2 = [(\dot{\boldsymbol{\omega}}_1)_{x'y'z'} + \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_1] + [(\dot{\boldsymbol{\omega}}_2)_{x'y'z'} + \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2]$$
$$= [1.5\mathbf{i} + \mathbf{0}] + [-6\mathbf{k} + 4\mathbf{i} \times 5\mathbf{k}] = \{1.5\mathbf{i} - 20\mathbf{j} - 6\mathbf{k}\} \text{ rad/s}^2$$

Also, $\boldsymbol{\omega}_1$ changes the direction of $\mathbf{r}_B$ so that the time derivatives of $\mathbf{r}_B$ can be found using the primed axes defined above. Hence,

$$\mathbf{v}_B = \dot{\mathbf{r}}_B = (\dot{\mathbf{r}}_B)_{x'y'z'} + \boldsymbol{\omega}_1 \times \mathbf{r}_B$$
$$= \mathbf{0} + 4\mathbf{i} \times (-0.5\mathbf{k}) = \{2\mathbf{j}\} \text{ m/s}$$
$$\mathbf{a}_B = \ddot{\mathbf{r}}_B = [(\ddot{\mathbf{r}}_B)_{x'y'z'} + \boldsymbol{\omega}_1 \times (\dot{\mathbf{r}}_B)_{x'y'z'}] + \dot{\boldsymbol{\omega}}_1 \times \mathbf{r}_B + \boldsymbol{\omega}_1 \times \dot{\mathbf{r}}_B$$
$$= [\mathbf{0} + \mathbf{0}] + 1.5\mathbf{i} \times (-0.5\mathbf{k}) + 4\mathbf{i} \times 2\mathbf{j} = \{0.75\mathbf{j} + 8\mathbf{k}\} \text{ m/s}^2$$

Motion of *C* with Respect to *B*.

$$\boldsymbol{\Omega}_{xyz} = \mathbf{0}$$
$$\dot{\boldsymbol{\Omega}}_{xyz} = \mathbf{0}$$
$$\mathbf{r}_{C/B} = \{0.2\mathbf{j}\} \text{ m}$$
$$(\mathbf{v}_{C/B})_{xyz} = \{3\mathbf{j}\} \text{ m/s}$$
$$(\mathbf{a}_{C/B})_{xyz} = \{2\mathbf{j}\} \text{ m/s}^2$$

Motion of *C*.

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\Omega} \times \mathbf{r}_{C/B} + (\mathbf{v}_{C/B})_{xyz}$$
$$= 2\mathbf{j} + [(4\mathbf{i} + 5\mathbf{k}) \times (0.2\mathbf{j})] + 3\mathbf{j}$$
$$= \{-1\mathbf{i} + 5\mathbf{j} + 0.8\mathbf{k}\} \text{ m/s} \qquad \textit{Ans.}$$

$$\mathbf{a}_C = \mathbf{a}_B + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/B}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/B})_{xyz} + (\mathbf{a}_{C/B})_{xyz}$$
$$= (0.75\mathbf{j} + 8\mathbf{k}) + [(1.5\mathbf{i} - 20\mathbf{j} - 6\mathbf{k}) \times (0.2\mathbf{j})]$$
$$+ (4\mathbf{i} + 5\mathbf{k}) \times [(4\mathbf{i} + 5\mathbf{k}) \times 0.2\mathbf{j}] + 2[(4\mathbf{i} + 5\mathbf{k}) \times 3\mathbf{j}] + 2\mathbf{j}$$
$$= \{-28.8\mathbf{i} - 5.45\mathbf{j} + 32.3\mathbf{k}\} \text{ m/s}^2 \qquad \textit{Ans.}$$

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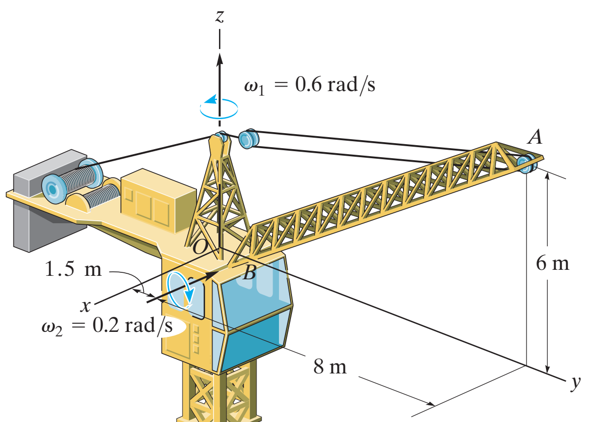
PROBLEMS

20–37. The crane rotates about the z axis with a constant rate $\omega_1 = 0.6$ rad/s, while the boom rotates downward with a constant rate $\omega_2 = 0.2$ rad/s. Determine the velocity and acceleration of point A located at the end of the boom at the instant shown.

20–38. The crane rotates about the z axis with a rate of $\omega_1 = 0.6$ rad/s, which is increasing at $\dot{\omega}_1 = 0.6$ rad/s^2. Also, the boom rotates downward at $\omega_2 = 0.2$ rad/s, which is increasing at $\dot{\omega}_2 = 0.3$ rad/s^2. Determine the velocity and acceleration of point A located at the end of the boom at the instant shown.



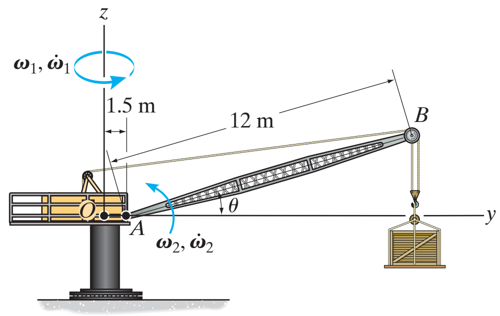
Probs. 20–37/38

20–39. Solve Example 20.5 such that the x, y, z axes move with curvilinear translation, $\mathbf{\Omega} = \mathbf{0}$, in which case the collar appears to have both an angular velocity $\mathbf{\Omega}_{xyz} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ and radial motion.

***20–40.** Solve Example 20.5 by fixing x, y, z axes to rod BD so that $\mathbf{\Omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. In this case the collar appears only to move radially outward along BD; hence $\mathbf{\Omega}_{xyz} = \mathbf{0}$.

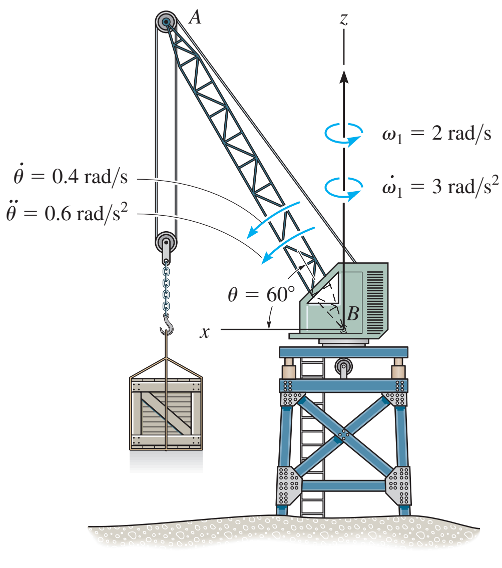
20–41. At the instant $\theta = 30°$, the frame of the crane and the boom AB rotate with a constant angular velocity of $\omega_1 = 1.5$ rad/s and $\omega_2 = 0.5$ rad/s, respectively. Determine the velocity and acceleration of point B at this instant.

20–42. At the instant $\theta = 30°$, the frame of the crane is rotating with an angular velocity of $\omega_1 = 1.5$ rad/s and angular acceleration of $\dot{\omega}_1 = 0.5$ rad/s^2, while the boom AB rotates with an angular velocity of $\omega_2 = 0.5$ rad/s and angular acceleration of $\dot{\omega}_2 = 0.25$ rad/s^2. Determine the velocity and acceleration of point B at this instant.



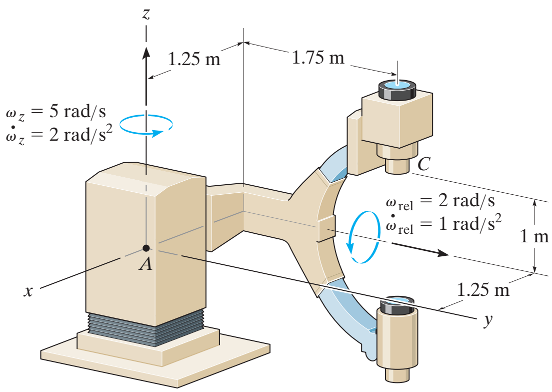
Probs. 20–41/42

20–43. At a given instant the boom AB of the tower crane rotates about the z axis with the motion shown. At this same instant, $\theta = 60°$ and the boom is rotating downward such that $\dot{\theta} = 0.4$ rad/s and $\ddot{\theta} = 0.6$ rad/s^2. Determine the velocity and acceleration of the end of the boom A at this instant. The boom has a length of $l_{AB} = 40$ m.



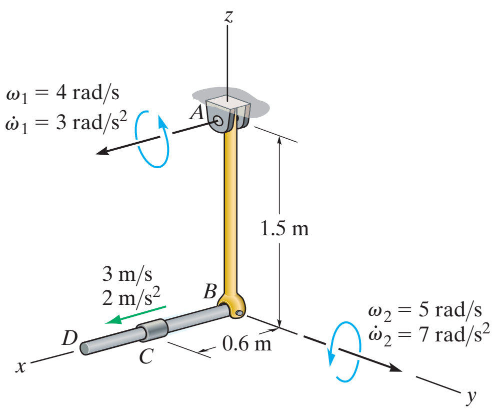
Prob. 20–43

*20–44. During the instant shown the frame of the X-ray camera is rotating about the vertical axis at $\omega_z = 5$ rad/s and $\dot{\omega}_z = 2$ rad/s^2. Relative to the frame the arm is rotating at $\omega_{rel} = 2$ rad/s and $\dot{\omega}_{rel} = 1$ rad/s^2. Determine the velocity and acceleration of the center of the camera C at this instant.



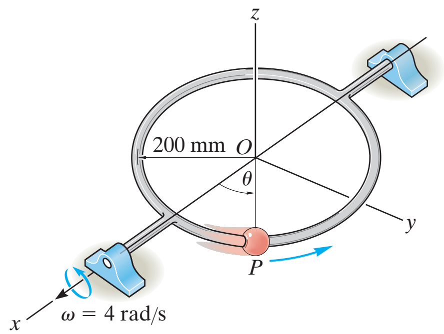
Prob. 20–44

20–45. At the instant shown, the arm AB is rotating about the fixed pin A with an angular velocity $\omega_1 = 4$ rad/s and an angular acceleration $\dot{\omega}_1 = 3$ rad/s^2. At the same instant, rod BD is rotating relative to rod AB with an angular velocity $\omega_2 = 5$ rad/s which is increasing at $\dot{\omega}_2 = 7$ rad/s^2. Also, the collar C is moving along rod BD with a velocity of 3 m/s and an acceleration of 2 m/s^2, both measured relative to the rod. Determine the velocity and acceleration of the collar at this instant.



Prob. 20–45

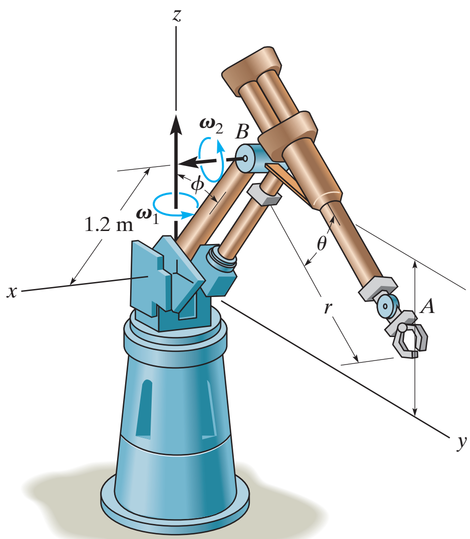
20–46. The particle P slides around the circular hoop with a constant angular velocity of $\dot{\theta} = 6$ rad/s, while the hoop rotates about the x axis at a constant rate of $\omega = 4$ rad/s. If at the instant shown the hoop is in the x–y plane and the angle $\theta = 45°$, determine the velocity and acceleration of the particle at this instant.



Prob. 20–46

20–47. At the instant shown, the industrial manipulator is rotating about the z axis at $\omega_1 = 5$ rad/s, and about joint B at $\omega_2 = 2$ rad/s. Determine the velocity and acceleration of the grip A at this instant, when $\phi = 30°$, $\theta = 45°$, and $r = 1.6$ m.

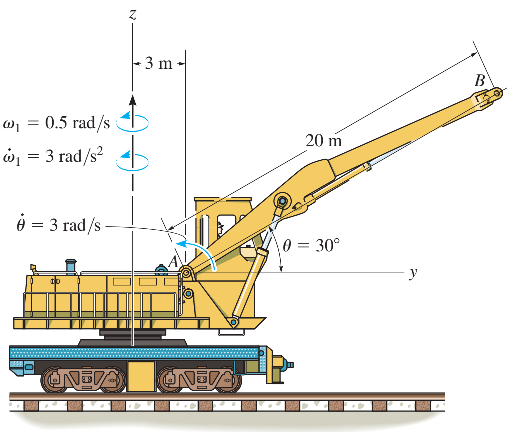
*20–48. At the instant shown, the industrial manipulator is rotating about the z axis at $\omega_1 = 5$ rad/s, and $\dot{\omega}_1 = 2$ rad/s^2; and about joint B at $\omega_2 = 2$ rad/s and $\dot{\omega}_2 = 3$ rad/s^2. Determine the velocity and acceleration of the grip A at this instant, when $\phi = 30°$, $\theta = 45°$, and $r = 1.6$ m.



Probs. 20–47/48

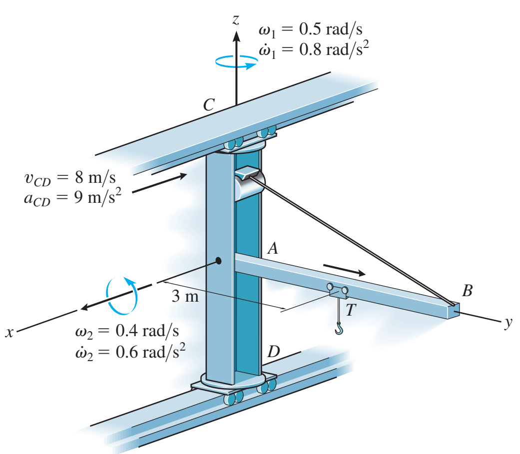
20–49. The boom AB of the locomotive crane is rotating about the z axis with an angular velocity $\omega_1 = 0.5$ rad/s, which is increasing at $\dot{\omega}_1 = 3$ rad/s^2. At this same instant, $\theta = 30°$ and the boom is rotating upward at a constant rate of $\dot{\theta} = 3$ rad/s. Determine the velocity and acceleration of the tip B of the boom at this instant.

20–50. The locomotive crane is traveling to the right at 2 m/s and has an acceleration of 1.5 m/s^2, while the boom is rotating about the z axis with an angular velocity $\omega_1 = 0.5$ rad/s, which is increasing at $\dot{\omega}_1 = 3$ rad/s^2. At this same instant, $\theta = 30°$ and the boom is rotating upward at a constant rate $\dot{\theta} = 3$ rad/s. Determine the velocity and acceleration of the tip B of the boom at this instant.



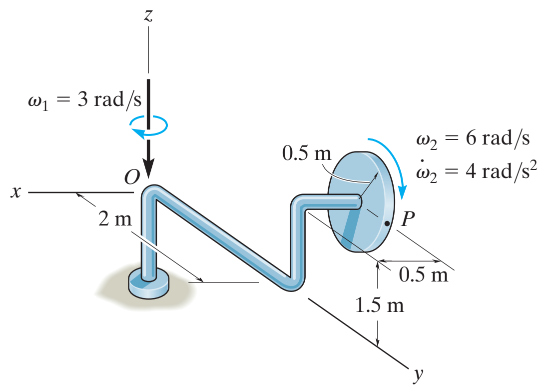
Probs. 20–49/50

20–51. At a given instant, the crane is moving along the track with a velocity $v_{CD} = 8$ m/s and acceleration of 9 m/s^2. Simultaneously, it has the angular motions shown. If the trolley T is moving outwards along the boom AB with a relative speed of 3 m/s and relative acceleration of 5 m/s^2, determine the velocity and acceleration of the trolley.



Prob. 20–51

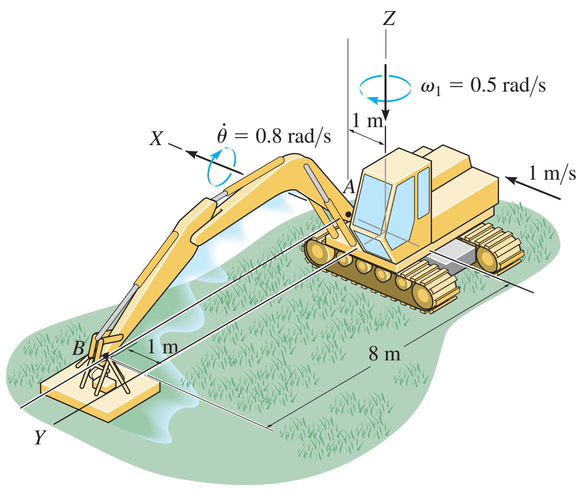
***20–52.** At the given instant, the rod is turning about the z axis with a constant angular velocity $\omega_1 = 3$ rad/s. At this same instant, the disk is spinning at $\omega_2 = 6$ rad/s when $\dot{\omega}_2 = 4$ rad/s^2, both measured *relative* to the rod. Determine the velocity and acceleration of point P on the disk at this instant.



Prob. 20–52

20–53. At the instant shown, the frame of the brush cutter is traveling forward in the x direction with a constant velocity of 1 m/s, and the cab is rotating about the vertical axis with an angular velocity of $\omega_1 = 0.5$ rad/s. At the same instant the boom AB has an angular velocity of $\dot{\theta} = 0.8$ rad/s, in the direction shown. Determine the velocity and acceleration of point B at the connection to the mower at this instant.

20–54. At the instant shown, the frame of the brush cutter is traveling forward in the x direction with a constant velocity of 1 m/s, and the cab is rotating about the vertical axis with an angular velocity of $\omega_1 = 0.5$ rad/s, which is increasing at $\dot{\omega}_1 = 0.4$ rad/s^2. At the same instant the boom AB has an angular velocity of $\dot{\theta} = 0.8$ rad/s, which is increasing at $\ddot{\theta} = 0.9$ rad/s^2. Determine the velocity and acceleration of point B at the connection to the mower at this instant.



Probs. 20–53/54

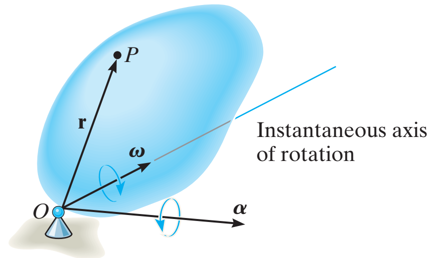
CHAPTER REVIEW

Rotation About a Fixed Point

When a body rotates about a fixed point O, then points on the body follow a path that lies on the surface of a sphere centered at O.

Since the angular acceleration is a time rate of change in the angular velocity, then it is necessary to determine both the magnitude and directional changes of $\boldsymbol{\omega}$ when finding its time derivative. To do this, the angular velocity is often specified in terms of its component motions, such that the direction of some of these components will remain constant relative to rotating x, y, z axes which are rotating at $\boldsymbol{\Omega}$. If this is the case, then the time derivative relative to the fixed axis can be determined using $\dot{\mathbf{A}} = (\dot{\mathbf{A}})_{xyz} + \boldsymbol{\Omega} \times \mathbf{A}$.

Once $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are known, the velocity and acceleration of any point P on the body can then be determined.



$$\mathbf{v}_P = \boldsymbol{\omega} \times \mathbf{r}$$

$$\mathbf{a}_P = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

General Motion

If the body undergoes general motion, then the motion of a point B on the body can be related to the motion of another point A using a relative motion analysis, with translating axes attached to A.

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A})$$

Relative-Motion Analysis Using Translating and Rotating Axes

The motion of two points A and B on a body, on a series of connected bodies, or where each point is located on two different paths, can be related using a relative-motion analysis with rotating and translating axes at A.

When applying the equations to find $\mathbf{v}_B$ and $\mathbf{a}_B$, it is important to account for both the magnitude and directional changes of $\mathbf{r}_A$, $\mathbf{r}_{B/A}$, $\boldsymbol{\Omega}$, and $\boldsymbol{\Omega}_{xyz}$ when taking their time derivatives to find $\mathbf{v}_A$, $\mathbf{a}_A$, $(\mathbf{v}_{B/A})_{xyz}$, $(\mathbf{a}_{B/A})_{xyz}$, $\dot{\boldsymbol{\Omega}}$, and $\dot{\boldsymbol{\Omega}}_{xyz}$. To do this properly, one must use Eq. 20–6.

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz}$$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}$$

CHAPTER 21

The forces acting on this motorcycle can be determined using the equations of motion as discussed in this chapter.

THREE-DIMENSIONAL KINETICS OF A RIGID BODY

CHAPTER OBJECTIVES

- To introduce the methods for finding the moments of inertia and products of inertia of a body about various axes.
- To show how to apply the principles of work and energy and linear and angular impulse and momentum to a rigid body having three-dimensional motion.
- To develop and apply the equations of motion in three dimensions.
- To study gyroscopic and torque-free motion.

*21.1 MOMENTS AND PRODUCTS OF INERTIA

When studying the planar motion of a body, it was necessary to introduce the moment of inertia I_G, which was determined about an axis perpendicular to the plane of motion and passing through the body's mass center G. For the analysis of three-dimensional motion it will sometimes be necessary to calculate six inertial quantities. These terms, called the moments and products of inertia, describe in a particular way the distribution of mass for a body relative to a given coordinate system that has a specified orientation and point of origin.

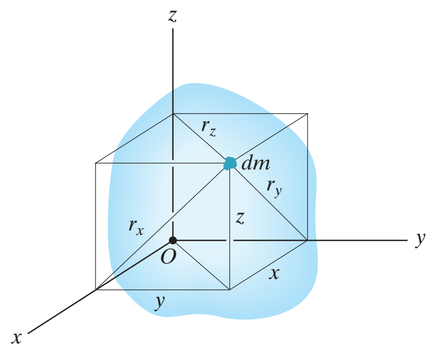


Fig. 21–1

Moment of Inertia. The ***moment of inertia*** for a differential element dm of a body about any one of the three coordinate axes is defined as the product of the mass of the element and the square of the shortest distance from the axis to the element. For example, as noted in Fig. 21–1, $r_x = \sqrt{y^2 + z^2}$, so that the mass moment of inertia of the element about the x axis is

$$dI_{xx} = r_x^2\, dm = (y^2 + z^2)\, dm$$

The moment of inertia I_{xx} for the body can be determined by integrating this expression over the entire mass of the body. Hence, for each of the axes, we can write

$$\begin{aligned} I_{xx} &= \int_m r_x^2 dm = \int_m (y^2 + z^2)\, dm \\ I_{yy} &= \int_m r_y^2 dm = \int_m (x^2 + z^2)\, dm \\ I_{zz} &= \int_m r_z^2 dm = \int_m (x^2 + y^2)\, dm \end{aligned} \qquad (21\text{–}1)$$

In each of these cases the moment of inertia is always a positive quantity, since it is the summation of the product of the mass dm, which is always positive, and the distances squared.

Product of Inertia. The ***product of inertia*** for a differential element dm with respect to a set of two orthogonal planes is defined as the product of the mass of the element and the perpendicular (or shortest) distances from each plane to the element. For example, this distance is x to the y–z plane and it is y to the x–z plane, Fig. 21–1. The product of inertia dI_{xy} for the element is therefore

$$dI_{xy} = xy\, dm$$

Note also that $dI_{yx} = dI_{xy}$. Integrating over the entire mass, the products of inertia of the body with respect to each combination of planes then become

$$\begin{aligned} I_{xy} &= I_{yx} = \int_m xy\, dm \\ I_{yz} &= I_{zy} = \int_m yz\, dm \\ I_{zx} &= I_{xz} = \int_m xz\, dm \end{aligned} \qquad (21\text{–}2)$$

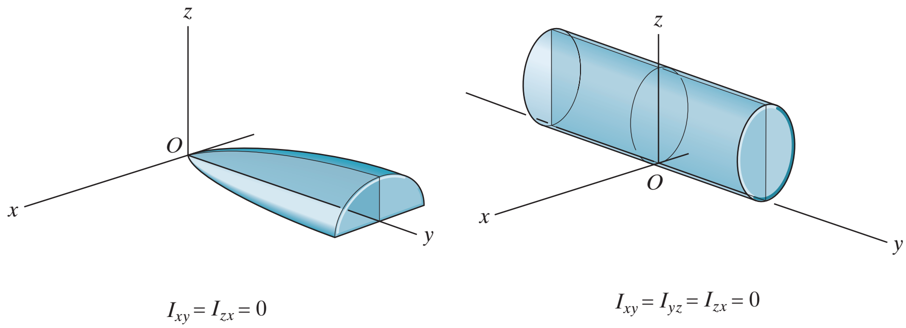


Fig. 21–2

Unlike the moment of inertia, which is always positive, the product of inertia may be positive, negative, or zero. The result depends on the algebraic signs of the two defining coordinates, which vary independently from one another. If either one or both of the orthogonal planes are planes of symmetry for the mass, the product of inertia with respect to these planes will be zero. In such cases, elements of mass will occur in pairs located on each side of the plane of symmetry. On one side of the plane the product of inertia for the element will be positive, while on the other side the product of inertia of the corresponding element will be negative, the sum therefore yielding zero. For example, in Fig. 21–2*a*, the y–z plane is a plane of symmetry, and so $I_{xy} = I_{zx} = 0$. Calculation of I_{yz} will yield a positive result, since all elements of mass are located using only positive y and z coordinates. For the cylinder, with the coordinate axes located as shown in Fig. 21–2*b*, the x–z and y–z planes are both planes of symmetry. Thus, $I_{xy} = I_{yz} = I_{zx} = 0$.

Parallel-Axis and Parallel-Plane Theorems. The techniques of integration used to determine the moment of inertia of a body were described in Sec. 17.1. Also discussed were methods to determine the moment of inertia of a composite body, i.e., a body composed of simpler segments, as tabulated on the inside back cover. In both of these cases the ***parallel-axis theorem*** was often used for the calculations. This theorem transfers the moment of inertia of a body from an axis passing through its mass center G to a parallel axis passing through some other point. If G has coordinates x_G, y_G, z_G, Fig. 21–3, then the parallel-axis equations used to calculate the moments of inertia about the x, y, z axes are

$$\begin{aligned} I_{xx} &= (I_{x'x'})_G + m(y_G^2 + z_G^2) \\ I_{yy} &= (I_{y'y'})_G + m(x_G^2 + z_G^2) \\ I_{zz} &= (I_{z'z'})_G + m(x_G^2 + y_G^2) \end{aligned} \qquad (21\text{–}3)$$

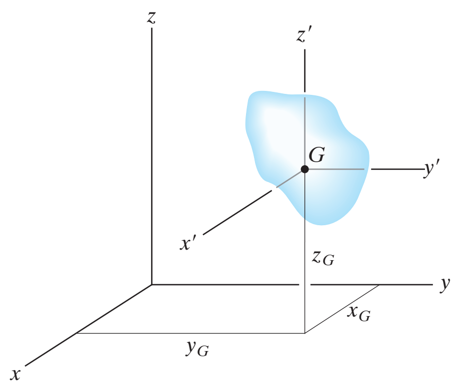


Fig. 21–3

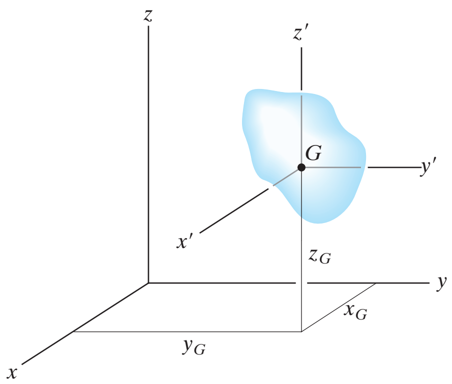


Fig. 21–3 (repeated)

The products of inertia of a composite body are determined in the same manner as the body's moments of inertia. Here, however, the ***parallel-plane theorem*** is important. This theorem is used to transfer the products of inertia of the body with respect to a set of three orthogonal planes passing through the body's mass center to a corresponding set of three parallel planes passing through some other point O. Defining the perpendicular distances between the planes as x_G, y_G, z_G, Fig. 21–3, the parallel-plane theorem for each product of inertia can be written as

$$\begin{aligned} I_{xy} &= (I_{x'y'})_G + m x_G y_G \\ I_{yz} &= (I_{y'z'})_G + m y_G z_G \\ I_{zx} &= (I_{z'x'})_G + m z_G x_G \end{aligned} \tag{21–4}$$

The derivation of these formulas is similar to that given for the parallel-axis equation, Sec. 17.1.

Inertia Tensor. The inertial properties of a body are therefore completely characterized by nine terms, six of which are independent of one another. This set of terms is defined using Eqs. 21–1 and 21–2 and can be written as

$$\begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix}$$

This array is called an ***inertia tensor***.* It has a unique set of values for a body when it is determined for each location of the origin O and orientation of the coordinate axes.

In general, for point O we can always specify a unique axes inclination for which the products of inertia for the body are zero. When this is done, the inertia tensor is said to be "diagonalized" and may be written in the simplified form

$$\begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

Here $I_x = I_{xx}$, $I_y = I_{yy}$, and $I_z = I_{zz}$ are termed the ***principal moments of inertia*** for the body, which are calculated with respect to the ***principal axes of inertia***. Of these three principal moments of inertia, one will be a maximum and another a minimum of the body's moment of inertia.

The dynamics of the space shuttle while it orbits the earth can be predicted only if its moments and products of inertia are known relative to its mass center.

* Negative signs are included here in order to later describe the body's angular momentum. See Eqs. 21–10.

The mathematical determination of the directions of principal axes of inertia will not be discussed here (see Prob. 21–24). However, there are many cases in which the principal axes can be determined by inspection. From the previous discussion it was noted that if the coordinate axes are oriented such that two of the three orthogonal planes containing the axes are planes of symmetry for the body, then all the products of inertia for the body are zero with respect to these coordinate planes, and hence these coordinate axes are principal axes of inertia. For example, the x, y, z axes shown in Fig. 21–2b represent the principal axes of inertia for the cylinder at point O.

Moment of Inertia About an Arbitrary Axis. Consider the body shown in Fig. 21–4, where the nine elements of the inertia tensor have been determined with respect to the x, y, z axes having an origin at O. Here we wish to determine the moment of inertia of the body about the Oa axis, defined by the unit vector $\mathbf{u}_a$. By definition, $I_{Oa} = \int b^2\,dm$, where b is the perpendicular distance from the element dm to Oa. If the position of dm is located using $\mathbf{r}$, then $b = r\sin\theta$, which is the magnitude of the cross product $\mathbf{u}_a \times \mathbf{r}$. The moment of inertia can therefore be expressed as

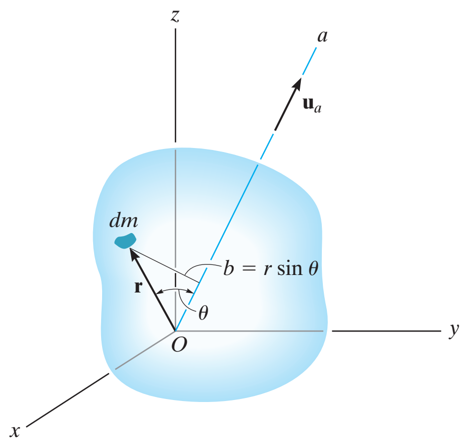


Fig. 21–4

$$I_{Oa} = \int_m |(\mathbf{u}_a \times \mathbf{r})|^2 dm = \int_m (\mathbf{u}_a \times \mathbf{r}) \cdot (\mathbf{u}_a \times \mathbf{r})dm$$

In general, $\mathbf{u}_a = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and so $\mathbf{u}_a \times \mathbf{r} = (u_y z - u_z y)\mathbf{i} + (u_z x - u_x z)\mathbf{j} + (u_x y - u_y x)\mathbf{k}$. And so, after substituting and performing the dot-product operation, we get

$$\begin{aligned} I_{Oa} &= \int_m [(u_y z - u_z y)^2 + (u_z x - u_x z)^2 + (u_x y - u_y x)^2]dm \\ &= u_x^2 \int_m (y^2 + z^2)dm + u_y^2 \int_m (z^2 + x^2)dm + u_z^2 \int_m (x^2 + y^2)\,dm \\ &\quad - 2u_x u_y \int_m xy\,dm - 2u_y u_z \int_m yz\,dm - 2u_z u_x \int_m zx\,dm \end{aligned}$$

Recognizing the integrals to be the moments and products of inertia of the body, Eqs. 21–1 and 21–2, we have

$$I_{Oa} = I_{xx}u_x^2 + I_{yy}u_y^2 + I_{zz}u_z^2 - 2I_{xy}u_x u_y - 2I_{yz}u_y u_z - 2I_{zx}u_z u_x \qquad (21\text{–}5)$$

Thus, if the inertia tensor is specified for the x, y, z axes, the moment of inertia of the body about the inclined Oa axis can be found. For the calculation, the direction cosines u_x, u_y, u_z of the axes must be determined. These terms specify the cosines of the coordinate direction angles α, β, γ made between the positive Oa axis and the positive x, y, z axes, respectively (see Appendix B).

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EXAMPLE 21.1

Fig. 21–5

Determine the moment of inertia of the bent rod shown in Fig. 21–5*a* about the *Oa* axis. The mass of each of the three segments is given in the figure.

SOLUTION

Before applying Eq. 21–5, it is first necessary to determine the moments and products of inertia of the rod with respect to the x, y, z axes. This is done using the formula for the moment of inertia of a slender rod, $I = \frac{1}{12}ml^2$, and the parallel-axis and parallel-plane theorems, Eqs. 21–3 and 21–4. Dividing the rod into three parts and locating the mass center of each segment, Fig. 21–5*b*, we have

$$I_{xx} = \left[\tfrac{1}{12}(2)(0.2)^2 + 2(0.1)^2\right] + [0 + 2(0.2)^2] + \left[\tfrac{1}{12}(4)(0.4)^2 + 4((0.2)^2 + (0.2)^2)\right] = 0.480\ \text{kg}\cdot\text{m}^2$$

$$I_{yy} = \left[\tfrac{1}{12}(2)(0.2)^2 + 2(0.1)^2\right] + \left[\tfrac{1}{12}(2)(0.2)^2 + 2((-0.1)^2 + (0.2)^2)\right] + [0 + 4((-0.2)^2 + (0.2)^2)] = 0.453\ \text{kg}\cdot\text{m}^2$$

$$I_{zz} = [0 + 0] + \left[\tfrac{1}{12}(2)(0.2)^2 + 2(-0.1)^2\right] + \left[\tfrac{1}{12}(4)(0.4)^2 + 4((-0.2)^2 + (0.2)^2)\right] = 0.400\ \text{kg}\cdot\text{m}^2$$

$$I_{xy} = [0 + 0] + [0 + 0] + [0 + 4(-0.2)(0.2)] = -0.160\ \text{kg}\cdot\text{m}^2$$

$$I_{yz} = [0 + 0] + [0 + 0] + [0 + 4(0.2)(0.2)] = 0.160\ \text{kg}\cdot\text{m}^2$$

$$I_{zx} = [0 + 0] + [0 + 2(0.2)(-0.1)] + [0 + 4(0.2)(-0.2)] = -0.200\ \text{kg}\cdot\text{m}^2$$

The *Oa* axis is defined by the unit vector

$$\mathbf{u}_{Oa} = \frac{\mathbf{r}_A}{r_A} = \frac{-0.2\mathbf{i} + 0.4\mathbf{j} + 0.2\mathbf{k}}{\sqrt{(-0.2)^2 + (0.4)^2 + (0.2)^2}} = -0.408\mathbf{i} + 0.816\mathbf{j} + 0.408\mathbf{k}$$

Thus,

$$u_x = -0.408 \quad u_y = 0.816 \quad u_z = 0.408$$

Substituting these results into Eq. 21–5 yields

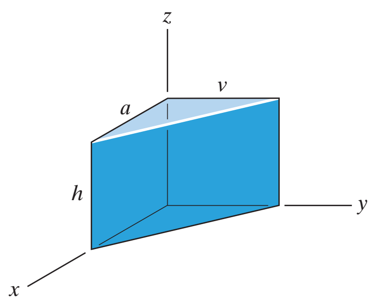
$$\begin{aligned} I_{Oa} &= I_{xx}u_x^2 + I_{yy}u_y^2 + I_{zz}u_z^2 - 2I_{xy}u_xu_y - 2I_{yz}u_yu_z - 2I_{zx}u_zu_x \\ &= 0.480(-0.408)^2 + (0.453)(0.816)^2 + 0.400(0.408)^2 \\ &\quad - 2(-0.160)(-0.408)(0.816) - 2(0.160)(0.816)(0.408) \\ &\quad - 2(-0.200)(0.408)(-0.408) \\ &= 0.169\ \text{kg}\cdot\text{m}^2 \end{aligned}$$

Ans.

PROBLEMS

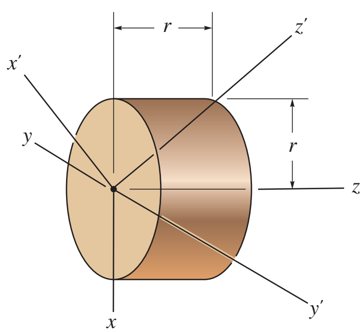
21–1. Determine by direct integration the product of inertia I_{yz} for the homogeneous prism. The density of the material is ρ. Express the result in terms of the total mass m of the prism.

21–2. Determine by direct integration the product of inertia I_{xy} for the homogeneous prism. The density of the material is ρ. Express the result in terms of the total mass m of the prism.



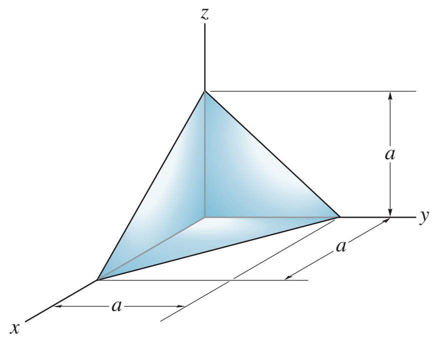
Probs. 21–1/2

21–3. Determine the moments of inertia for the homogeneous cylinder of mass m about the x', y', z' axes.



Prob. 21–3

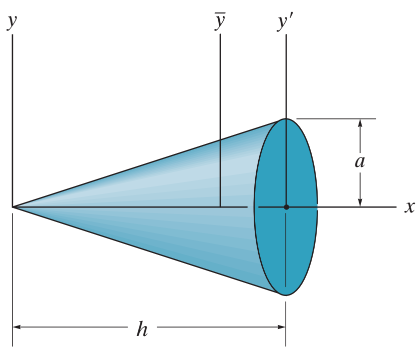
***21–4.** Determine the product of inertia I_{xy} for the homogeneous tetrahedron. The density of the material is ρ. Express the result in terms of the total mass m of the solid. *Suggestion*: Use a triangular element of thickness dz and then express dI_{xy} in terms of the size and mass of the element using the result of Prob. 21–2.



Prob. 21–4

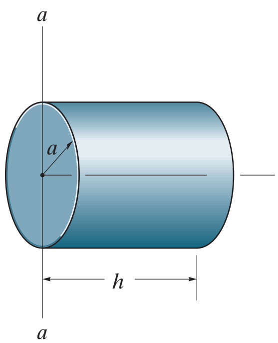
21–5. Show that the sum of the moments of inertia of a body, $I_{xx} + I_{yy} + I_{zz}$, is independent of the orientation of the x, y, z axes and thus depends only on the location of the origin.

21–6. Determine the moment of inertia of the cone with respect to a vertical $\bar{y}$ axis passing through the cone's center of mass. What is the moment of inertia about a parallel axis y' that passes through the diameter of the base of the cone? The cone has a mass m.



Prob. 21–6

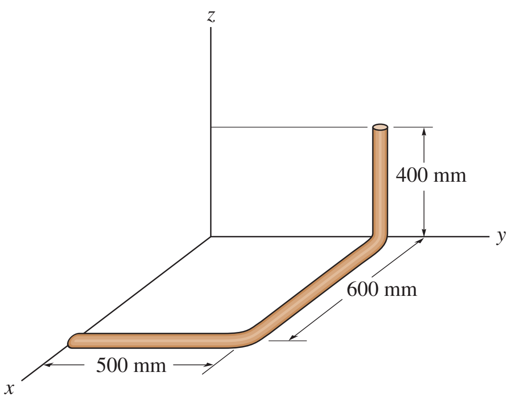
21–7. Determine the moment of inertia of the cylinder with respect to the *a–a* axis of the cylinder. The cylinder has a mass *m*.



Prob. 21–7

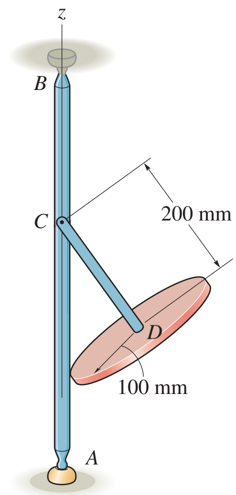
***21–8.** Determine the product of inertia I_{xy} for the bent rod. The rod has a mass per unit length of 2 kg/m.

21–9. Determine the moments of inertia I_{xx}, I_{yy}, I_{zz} for the bent rod. The rod has a mass per unit length of 2 kg/m.



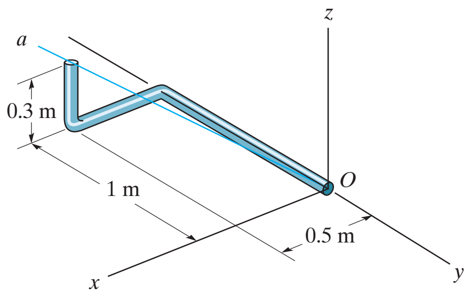
Probs. 21–8/9

21–10. Determine the moment of inertia about the *z* axis of the assembly which consists of the 1.5-kg rod *CD* and the 7-kg disk.



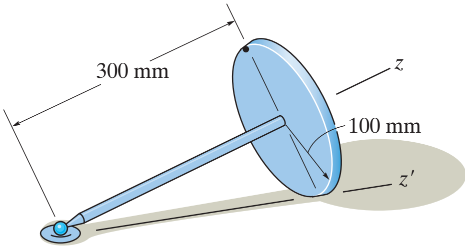
Prob. 21–10

21–11. The bent rod has a mass of 3 kg/m. Determine the moment of inertia of the rod about the *O–a* axis.



Prob. 21–11

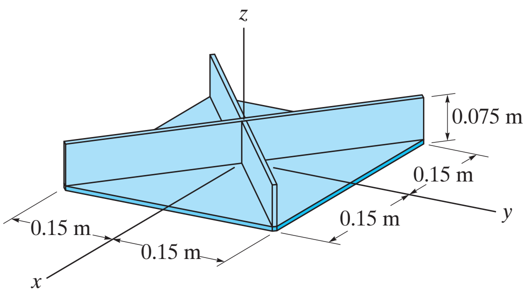
*21–12. Determine the moment of inertia of both the 1.5-kg rod and 4-kg disk about the z' axis.



Prob. 21–12

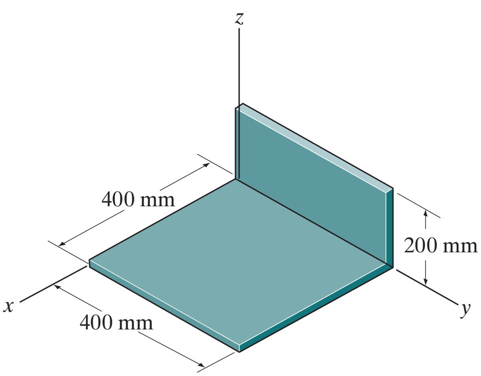
21–13. Determine the moment of inertia I_{xx} of the composite plate assembly. The plates have a density per unit area of 30 kg/m^2.

21–14. Determine the product of inertia I_{yz} of the composite plate assembly. The plates have a density per unit area of 30 kg/m^2.



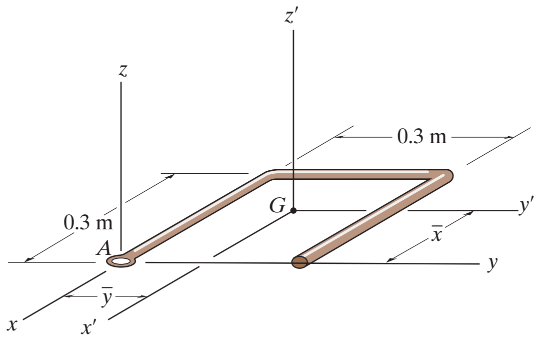
Probs. 21–13/14

21–15. Determine the products of inertia I_{xy}, I_{yz}, and I_{xz} of the thin plate. The material has a density per unit area of 50 kg/m^2.



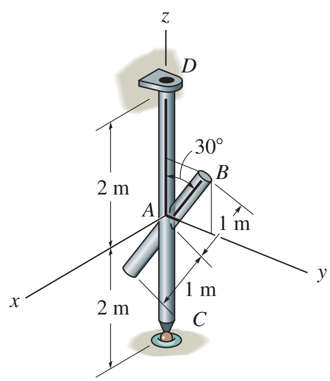
Prob. 21–15

*21–16. The bent rod has a mass per unit length of 2.25 kg/m. Locate the center of gravity $G(\bar{x}, \bar{y})$ and determine the principal moments of inertia $I_{x'}$, $I_{y'}$, and $I_{z'}$ of the rod with respect to the x', y', z' axes.



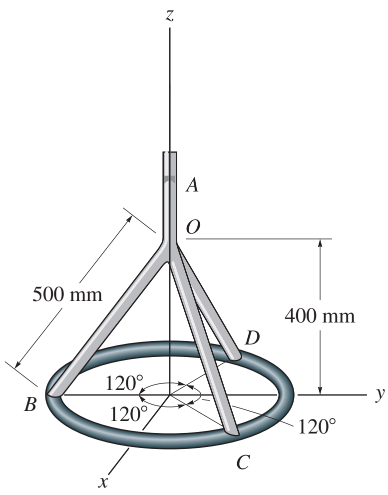
Prob. 21–16

21–17. Determine the moments of inertia about the x, y, z axes of the rod assembly. The rods have a mass of 0.75 kg/m.



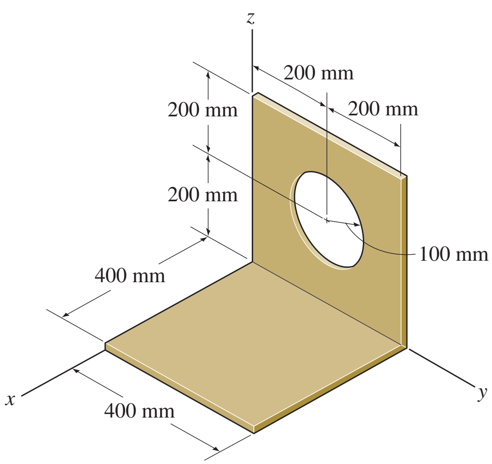
Prob. 21–17

21–18. Determine the moment of inertia of the rod-and-thin-ring assembly about the z axis. The rods and ring have a mass per unit length of 2 kg/m.



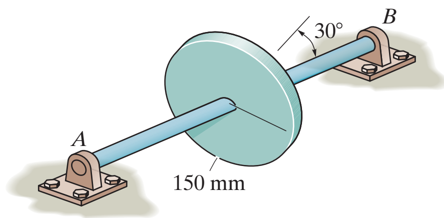
Prob. 21–18

21–19. Determine the products of inertia I_{xy}, I_{yz}, and I_{xz} of the thin plate. The material has a mass per unit area of 50 kg/m^2.



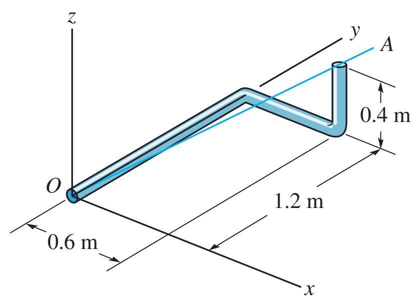
Prob. 21–19

***21–20.** Determine the moment of inertia of the disk about the axis of shaft AB. The disk has a mass of 15 kg.



Prob. 21–20

21–21. The bent rod has a mass of 4 kg/m. Determine the moment of inertia of the rod about the OA axis.



Prob. 21–21

21.2 ANGULAR MOMENTUM

In this section we will develop the necessary equations used to determine the angular momentum of a rigid body about an arbitrary point. These equations will provide a means for developing both the principle of impulse and momentum and the equations of rotational motion for a rigid body.

Consider the rigid body in Fig. 21–6, which has a mass m and center of mass at G. The X, Y, Z coordinate system represents an inertial frame of reference, and hence, its axes are fixed or translate with a constant velocity. The angular momentum as measured from this reference will be determined relative to the arbitrary point A. The position vectors $\mathbf{r}_A$ and $\boldsymbol{\rho}_A$ are drawn from the origin of coordinates to point A and from A to the ith particle of the body. If this particle's mass is m_i, then the angular momentum about point A is

$$(\mathbf{H}_A)_i = \boldsymbol{\rho}_A \times m_i\mathbf{v}_i$$

where $\mathbf{v}_i$ represents the particle's velocity measured from the X, Y, Z coordinate system. If the body has an angular velocity $\boldsymbol{\omega}$ at the instant considered, $\mathbf{v}_i$ may be related to the velocity of A by applying Eq. 20–7, i.e.,

$$\mathbf{v}_i = \mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A$$

Thus,

$$\begin{aligned}(\mathbf{H}_A)_i &= \boldsymbol{\rho}_A \times m_i(\mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A)\\ &= (\boldsymbol{\rho}_A m_i) \times \mathbf{v}_A + \boldsymbol{\rho}_A \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_A)m_i\end{aligned}$$

Summing the moments of all the particles of the body requires an integration. Since $m_i \rightarrow dm$, we have

$$\mathbf{H}_A = \left(\int_m \boldsymbol{\rho}_A dm\right) \times \mathbf{v}_A + \int_m \boldsymbol{\rho}_A \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_A)dm \qquad (21\text{–}6)$$

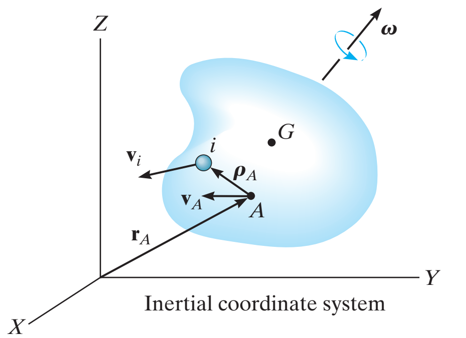


Fig. 21–6

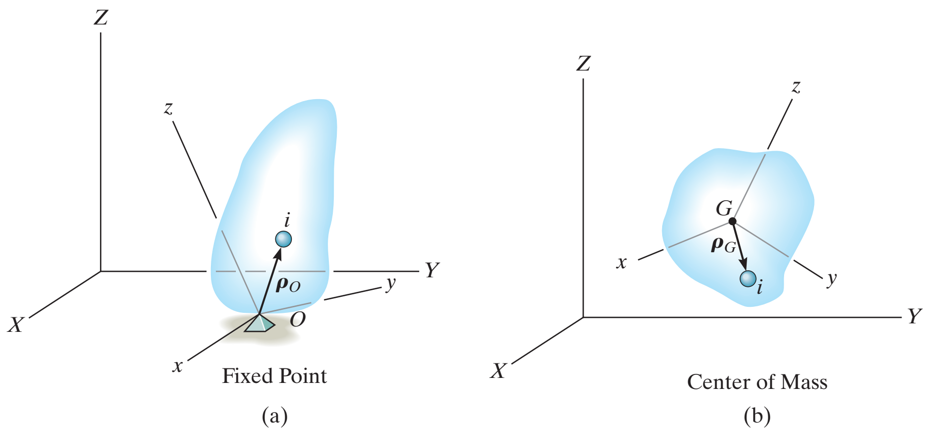


Fig. 21–7

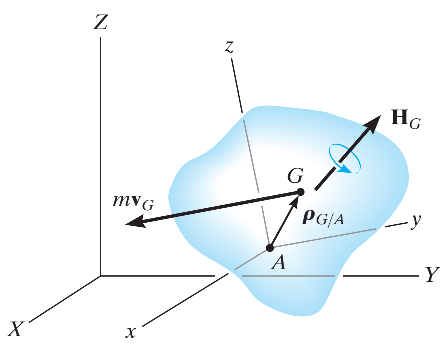
Fixed Point *O*. If A becomes a *fixed point O* in the body, Fig. 21–7*a*, then $\mathbf{v}_A = \mathbf{0}$ and Eq. 21–6 reduces to

$$\mathbf{H}_O = \int_m \boldsymbol{\rho}_O \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_O)\, dm \qquad (21\text{–}7)$$

Center of Mass *G*. If A is located at the *center of mass G* of the body, Fig. 21–7*b*, then $\int_m \boldsymbol{\rho}_A\, dm = \mathbf{0}$ and

$$\mathbf{H}_G = \int_m \boldsymbol{\rho}_G \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_G)\, dm \qquad (21\text{–}8)$$

Arbitrary Point *A*. In general, A can be a point other than O or G, Fig. 21–7*c*, in which case Eq. 21–6 may nevertheless be simplified to the following form (see Prob. 21–22).



Arbitrary Point
(c)

$$\mathbf{H}_A = \boldsymbol{\rho}_{G/A} \times m\mathbf{v}_G + \mathbf{H}_G \qquad (21\text{–}9)$$

Here the angular momentum consists of two parts—the moment of the linear momentum $m\mathbf{v}_G$ of the body about point A added (vectorially) to the angular momentum $\mathbf{H}_G$. Equation 21–9 can also be used to determine the angular momentum of the body about a fixed point O. The results, of course, will be the same as those found using the more convenient Eq. 21–7.

Rectangular Components of *H*. To make practical use of Eqs. 21–7 through 21–9, the angular momentum must be expressed in terms of its scalar components. For a general formulation, note that Eqs. 21–7 and 21–8 are both of the form

$$\mathbf{H} = \int_m \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})dm$$

Expressing $\mathbf{H}$, $\boldsymbol{\rho}$, and $\boldsymbol{\omega}$ in terms of their x, y, z components, we have

$$H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k} = \int_m (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times [(\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})]dm$$

Expanding the cross products and combining terms yields

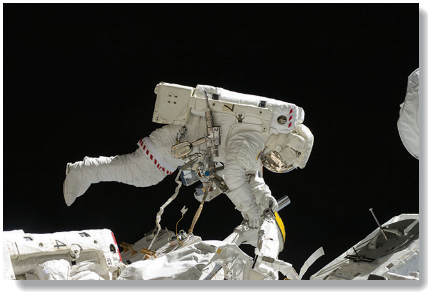
$$\begin{aligned} H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k} = &\left[\omega_x \int_m (y^2 + z^2)dm - \omega_y \int_m xy\,dm - \omega_z \int_m xz\,dm\right]\mathbf{i} \\ &+ \left[-\omega_x \int_m xy\,dm + \omega_y \int_m (x^2 + z^2)dm - \omega_z \int_m yz\,dm\right]\mathbf{j} \\ &+ \left[-\omega_x \int_m zx\,dm - \omega_y \int_m yz\,dm + \omega_z \int_m (x^2 + y^2)dm\right]\mathbf{k} \end{aligned}$$

Equating the respective $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ components and recognizing that the integrals represent the moments and products of inertia, we obtain

$$\begin{aligned} H_x &= I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ H_y &= -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\ H_z &= -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z \end{aligned} \qquad (21\text{–}10)$$

These equations can be simplified further if the x, y, z coordinate axes are oriented such that they become *principal axes of inertia* for the body at the point. When these axes are used, the products of inertia $I_{xy} = I_{yz} = I_{zx} = 0$, and if the principal moments of inertia about the x, y, z axes are represented as $I_x = I_{xx}$, $I_y = I_{yy}$, $I_z = I_{zz}$, then the three components of angular momentum become

$$H_x = I_x\omega_x \quad H_y = I_y\omega_y \quad H_z = I_z\omega_z \qquad (21\text{–}11)$$



The motion of the astronaut is controlled by use of small directional jets attached to his or her space suit. The impulses these jets provide must be carefully specified in order to prevent tumbling and loss of orientation. (© NASA)

Principle of Impulse and Momentum. Now that the formulation of the angular momentum for a body has been developed, the ***principle of impulse and momentum***, as discussed in Sec. 19.2, can be used to solve kinetic problems which involve *force, velocity, and time*. For this case, the following two vector equations are available:

$$m(\mathbf{v}_G)_1 + \Sigma\int_{t_1}^{t_2} \mathbf{F}\,dt = m(\mathbf{v}_G)_2 \tag{21–12}$$

$$(\mathbf{H}_O)_1 + \Sigma\int_{t_1}^{t_2} \mathbf{M}_O\,dt = (\mathbf{H}_O)_2 \tag{21–13}$$

In three dimensions each vector term can be represented by its three scalar components, and therefore a total of six scalar equations can be written. Three equations relate the linear impulse and momentum in the x, y, z directions, and the other three equations relate the body's angular impulse and momentum about the x, y, z axes. Before applying Eqs. 21–12 and 21–13 to the solution of problems, the material in Secs. 19.2 and 19.3 should be reviewed.

21.3 KINETIC ENERGY

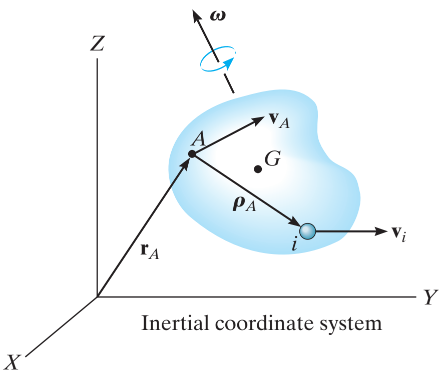


Fig. 21–8

In order to apply the principle of work and energy to solve problems involving general rigid body motion, it is first necessary to formulate expressions for the kinetic energy of the body. If the ith particle of the body in Fig. 21–8 has a mass m_i and velocity $\mathbf{v}_i$, measured relative to the inertial X, Y, Z frame of reference, then its kinetic energy is

$$T_i = \tfrac{1}{2}m_i v_i^2 = \tfrac{1}{2}m_i(\mathbf{v}_i \cdot \mathbf{v}_i)$$

Provided the velocity of an arbitrary point A in the body is known, $\mathbf{v}_i$ can be related to $\mathbf{v}_A$ by the equation $\mathbf{v}_i = \mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A$. Using this expression, the kinetic energy for the particle becomes

$$\begin{aligned} T_i &= \tfrac{1}{2}m_i(\mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A)\cdot(\mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A) \\ &= \tfrac{1}{2}(\mathbf{v}_A \cdot \mathbf{v}_A)m_i + \mathbf{v}_A \cdot (\boldsymbol{\omega} \times \boldsymbol{\rho}_A)m_i + \tfrac{1}{2}(\boldsymbol{\omega} \times \boldsymbol{\rho}_A)\cdot(\boldsymbol{\omega} \times \boldsymbol{\rho}_A)m_i \end{aligned}$$

The kinetic energy for the entire body is obtained by summing the kinetic energies of all the particles of the body. This requires an integration. Since $m_i \rightarrow dm$, we get

$$T = \tfrac{1}{2}m(\mathbf{v}_A \cdot \mathbf{v}_A) + \mathbf{v}_A \cdot \left(\boldsymbol{\omega} \times \int_m \boldsymbol{\rho}_A dm\right) + \tfrac{1}{2}\int_m (\boldsymbol{\omega} \times \boldsymbol{\rho}_A)\cdot(\boldsymbol{\omega} \times \boldsymbol{\rho}_A)dm$$

The last term on the right can be rewritten using the vector identity $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$, where $\mathbf{a} = \boldsymbol{\omega}$, $\mathbf{b} = \boldsymbol{\rho}_A$, and $\mathbf{c} = \boldsymbol{\omega} \times \boldsymbol{\rho}_A$. The final result is

$$T = \tfrac{1}{2}m(\mathbf{v}_A \cdot \mathbf{v}_A) + \mathbf{v}_A \cdot \left(\boldsymbol{\omega} \times \int_m \boldsymbol{\rho}_A dm\right) + \tfrac{1}{2}\boldsymbol{\omega} \cdot \int_m \boldsymbol{\rho}_A \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_A) dm \qquad (21\text{–}14)$$

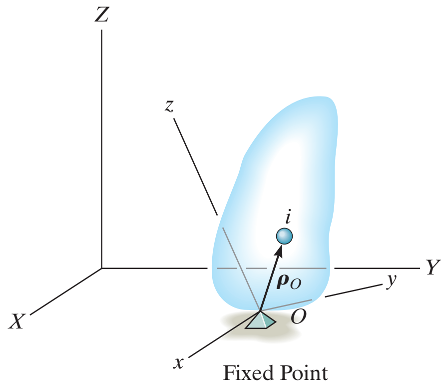
Simplification occurs if the reference point A is either a fixed point or the center of mass.

Fixed Point *O*. If A is a *fixed point O* in the body, Fig. 21–7*a*, then $\mathbf{v}_A = \mathbf{0}$, and using Eq. 21–7, we can express Eq. 21–14 as

$$T = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{H}_O$$

If the x, y, z axes represent the principal axes of inertia for the body, then $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$ and $\mathbf{H}_O = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$. Substituting and performing the dot-product operations yields

$$T = \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 + \tfrac{1}{2}I_z\omega_z^2 \qquad (21\text{–}15)$$



Fixed Point

(a)

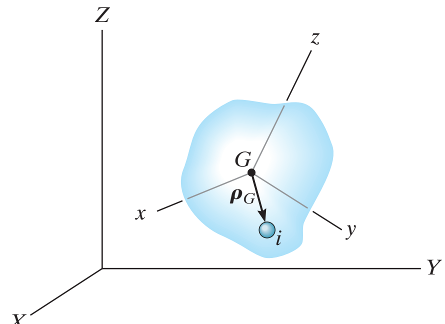
Fig. 21–7 (repeated)

Center of Mass *G*. If A is located at the *center of mass G* of the body, Fig. 21-7*b*, then $\int \boldsymbol{\rho}_A \, dm = \mathbf{0}$ and, using Eq. 21–8, we can write Eq. 21–14 as

$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{H}_G$$

In a manner similar to that for a fixed point, the last term on the right side can be represented in scalar form, in which case

$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 + \tfrac{1}{2}I_z\omega_z^2 \qquad (21\text{–}16)$$



Center of Mass

(b)

Fig. 21–7 (repeated)

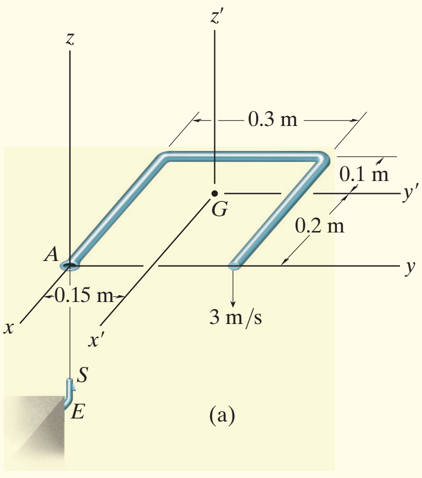
Here it is seen that the kinetic energy consists of two parts; namely, the translational kinetic energy of the mass center, $\frac{1}{2}mv_G^2$, and the body's rotational kinetic energy.

Principle of Work and Energy. Having formulated the kinetic energy for a body, the ***principle of work and energy*** can be applied to solve kinetic problems which involve *force, velocity, and displacement*. For this case only one scalar equation can be written for each body, namely,

$$T_1 + \Sigma U_{1-2} = T_2 \qquad (21\text{–}17)$$

Before applying this equation, the material in Secs. 18.1 through 18.4 should be reviewed.

EXAMPLE 21.2



(a)

The rod in Fig. 21–9*a* has a mass per unit length of 2.25 kg/m. Determine its angular velocity just after the end *A* falls onto the hook at *E*. The hook provides a permanent connection for the rod due to the spring-lock mechanism *S*. Just before striking the hook the rod is falling downward with a speed $(v_G)_1 = 3$ m/s.

SOLUTION

The principle of impulse and momentum will be used since impact occurs.

Impulse and Momentum Diagrams. Fig. 21–9*b*. During the short time Δt, the impulsive force $\mathbf{F}$ acting at *A* changes the momentum of the rod. (The impulse created by the rod's weight $\mathbf{W}$ during this time is small compared to $\int \mathbf{F}\,dt$, so that it can be neglected, i.e., the weight is a nonimpulsive force.) Hence, the angular momentum of the rod is *conserved* about point *A* since the moment of $\int \mathbf{F}\,dt$ about *A* is zero.

Conservation of Angular Momentum. Equation 21–9 must be used to find the angular momentum of the rod, since *A* does not become a *fixed point* until *after* the impulsive interaction with the hook. With reference to Fig. 21–9*b*, $(\mathbf{H}_A)_1 = (\mathbf{H}_A)_2$, or

$$\mathbf{r}_{G/A} \times m(\mathbf{v}_G)_1 = \mathbf{r}_{G/A} \times m(\mathbf{v}_G)_2 + (\mathbf{H}_G)_2 \qquad (1)$$

From Fig. 21–9*a*, $\mathbf{r}_{G/A} = \{-0.2\mathbf{i} + 0.15\mathbf{j}\}$ m. Furthermore, the primed axes are principal axes of inertia for the rod because $I_{x'y'} = I_{x'z'} = I_{z'y'} = 0$. Hence, from Eqs. 21–11, $(\mathbf{H}_G)_2 = I_{x'}\omega_x\mathbf{i} + I_{y'}\omega_y\mathbf{j} + I_{z'}\omega_z\mathbf{k}$. The principal moments of inertia are $I_{x'} = 0.0354\ \text{kg}\cdot\text{m}^2$, $I_{y'} = 0.0202\ \text{kg}\cdot\text{m}^2$, $I_{z'} = 0.0557\ \text{kg}\cdot\text{m}^2$ (see Prob. 21–16). Substituting into Eq. 1, we have

$$(-0.2\mathbf{i} + 0.15\mathbf{j}) \times [(2.25 \times 0.9)(-3\mathbf{k})] = (-0.2\mathbf{i} + 0.15\mathbf{j}) \times [(2.25 \times 0.9)(-v_G)_2\mathbf{k}]$$
$$+ 0.0354\omega_x\mathbf{i} + 0.0202\omega_y\mathbf{j} + 0.0557\omega_z\mathbf{k}$$

Expanding and equating the respective $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components yields

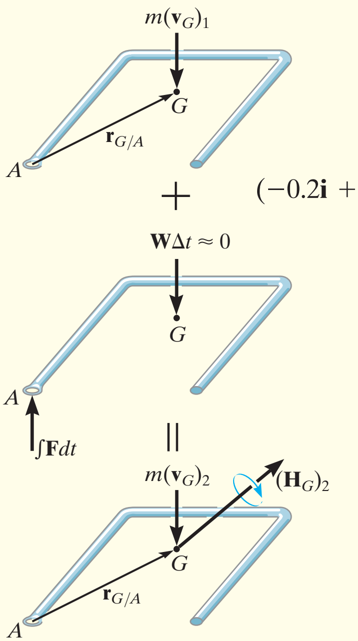
$$-0.9112 = -0.3037(v_G)_2 + 0.0354\omega_x \qquad (2)$$
$$-1.2150 = -0.4050(v_G)_2 + 0.0202\omega_y \qquad (3)$$
$$0 = 0.0557\omega_z \qquad (4)$$

Kinematics. There are four unknowns in the above equations; however, another equation may be obtained by relating $\boldsymbol{\omega}$ to $(\mathbf{v}_G)_2$ using kinematics. Since $\omega_z = 0$ (Eq. 4) and after impact the rod rotates about the fixed point *A*, Eq. 20–3 can be applied, in which case $(\mathbf{v}_G)_2 = \boldsymbol{\omega} \times \mathbf{r}_{G/A}$, or

$$-(v_G)_2\mathbf{k} = (\omega_x\mathbf{i} + \omega_y\mathbf{j}) \times (-0.2\mathbf{i} + 0.15\mathbf{j})$$
$$-(v_G)_2 = 0.15\omega_x + 0.2\omega_y \qquad (5)$$

Solving Eqs. 2, 3, and 5 simultaneously yields

$$(\mathbf{v}_G)_2 = \{-2.52\mathbf{k}\}\ \text{m/s} \qquad \boldsymbol{\omega} = \{-4.09\mathbf{i} - 9.55\mathbf{j}\}\ \text{rad/s}$$ *Ans.*

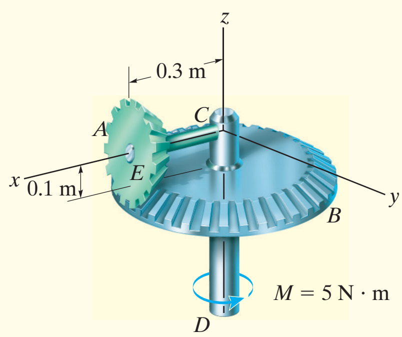


(b)

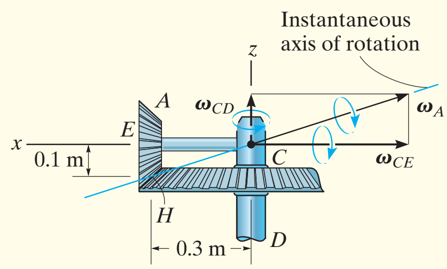
Fig. 21–9

EXAMPLE 21.3

A 5-N · m torque is applied to the vertical shaft *CD* shown in Fig. 21–10*a*, which causes the 10-kg gear *A* to turn freely about the axle *CE*. Assuming that *A* starts from rest, determine the angular velocity of *CD* after it has turned two revolutions. Neglect the mass of *CD* and *CE* and assume that *A* can be approximated by a thin disk. Gear *B* is fixed.



(a)



(b)

Fig. 21–10

SOLUTION

The principle of work and energy can be used for the solution. Why?

Work. If *CD*, *CE*, and *A* are considered as a system of connected bodies, only the applied torque **M** does work. For two revolutions of *CD*, this work is $\Sigma U_{1-2} = (5\text{ N}\cdot\text{m})(4\pi\text{ rad}) = 62.83\text{ J}$.

Kinetic Energy. Since the gear is initially at rest, its initial kinetic energy is zero. A kinematic diagram for the gear when it is moving is shown in Fig. 21–10*b*. If the angular velocity of *CD* is taken as $\boldsymbol{\omega}_{CD}$, then the angular velocity of *A* is $\boldsymbol{\omega}_A = \boldsymbol{\omega}_{CD} + \boldsymbol{\omega}_{CE}$. The gear may be imagined as a portion of a massless extended body which is rotating about the *fixed point C*. The instantaneous axis of rotation for this body is along line *CH*, because both points *C* and *H* on the body (gear) have zero velocity and must therefore lie on this axis. This requires that the components $\boldsymbol{\omega}_{CD}$ and $\boldsymbol{\omega}_{CE}$ be related by the equation $\omega_{CD}/0.1\text{ m} = \omega_{CE}/0.3\text{ m}$ or $\omega_{CE} = 3\omega_{CD}$. Thus,

$$\boldsymbol{\omega}_A = -\omega_{CE}\mathbf{i} + \omega_{CD}\mathbf{k} = -3\omega_{CD}\mathbf{i} + \omega_{CD}\mathbf{k} \qquad (1)$$

The *x*, *y*, *z* axes in Fig. 21–10*a* represent *principal axes of inertia* at *C* for the gear. Since point *C* is a fixed point of rotation, Eq. 21–15 may be applied to determine the kinetic energy, i.e.,

$$T = \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 + \tfrac{1}{2}I_z\omega_z^2 \qquad (2)$$

Using the parallel-axis theorem, the moments of inertia of the gear about point *C* are as follows:

$$I_x = \tfrac{1}{2}(10\text{ kg})(0.1\text{ m})^2 = 0.05\text{ kg}\cdot\text{m}^2$$
$$I_y = I_z = \tfrac{1}{4}(10\text{ kg})(0.1\text{ m})^2 + 10\text{ kg}(0.3\text{ m})^2 = 0.925\text{ kg}\cdot\text{m}^2$$

Since $\omega_x = -3\omega_{CD}$, $\omega_y = 0$, $\omega_z = \omega_{CD}$, Eq. 2 becomes

$$T_A = \tfrac{1}{2}(0.05)(-3\omega_{CD})^2 + 0 + \tfrac{1}{2}(0.925)(\omega_{CD})^2 = 0.6875\omega_{CD}^2$$

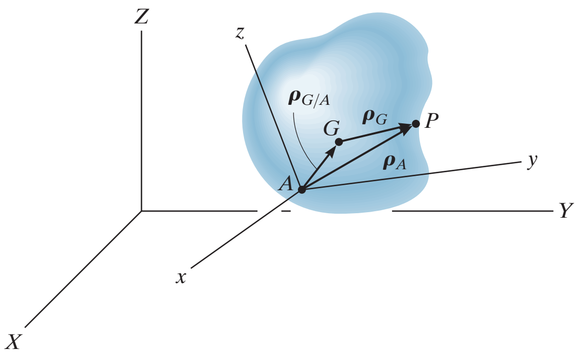
Principle of Work and Energy. Applying the principle of work and energy, we obtain

$$T_1 + \Sigma U_{1-2} = T_2$$
$$0 + 62.83 = 0.6875\omega_{CD}^2$$
$$\omega_{CD} = 9.56\text{ rad/s} \qquad \textit{Ans.}$$

21

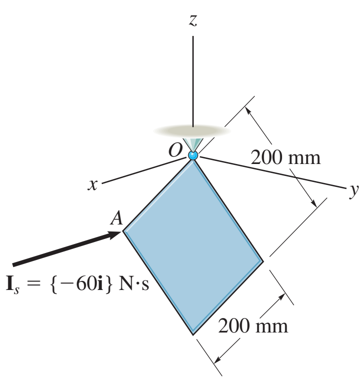
PROBLEMS

21–22. Show that if the angular momentum of a body is determined with respect to an arbitrary point A, then $\mathbf{H}_A$ can be expressed by Eq. 21–9. This requires substituting $\boldsymbol{\rho}_A = \boldsymbol{\rho}_G + \boldsymbol{\rho}_{G/A}$ into Eq. 21–6 and expanding, noting that $\int \boldsymbol{\rho}_G \, dm = \mathbf{0}$ by definition of the mass center and $\mathbf{v}_G = \mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_{G/A}$.



Prob. 21–22

21–23. A thin plate, having a mass of 4 kg is suspended from one of its corners by a ball-and-socket joint O. If a stone strikes the plate perpendicular to its surface at an adjacent corner A with an impulse of $\mathbf{I}_s = \{-60\mathbf{i}\}\ \text{N}\cdot\text{s}$, determine the instantaneous axis of rotation for the plate and the impulse created at O.

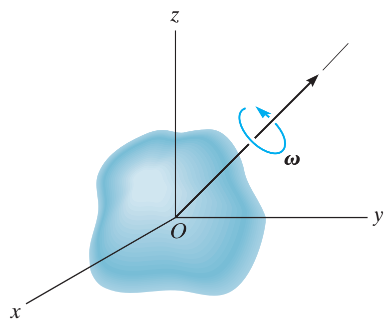


Prob. 21–23

***21–24.** If a body contains *no planes of symmetry*, the principal moments of inertia can be determined mathematically. To show how this is done, consider the rigid body which is spinning with an angular velocity $\boldsymbol{\omega}$, directed along one of its principal axes of inertia. If the principal moment of inertia about this axis is I, the angular momentum can be expressed as $\mathbf{H} = I\boldsymbol{\omega} = I\omega_x\mathbf{i} + I\omega_y\mathbf{j} + I\omega_z\mathbf{k}$. The components of $\mathbf{H}$ may also be expressed by Eqs. 21–10, where the inertia tensor is assumed to be known. Equate the $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ components of both expressions for $\mathbf{H}$ and consider ω_x, ω_y, and ω_z to be unknown. The solution of these three equations is obtained provided the determinant of the coefficients is zero. Show that this determinant, when expanded, yields the cubic equation

$$I^3 - (I_{xx} + I_{yy} + I_{zz})I^2$$
$$+ (I_{xx}I_{yy} + I_{yy}I_{zz} + I_{zz}I_{xx} - I_{xy}^2 - I_{yz}^2 - I_{zx}^2)I$$
$$- (I_{xx}I_{yy}I_{zz} - 2I_{xy}I_{yz}I_{zx} - I_{xx}I_{yz}^2 - I_{yy}I_{zx}^2 - I_{zz}I_{xy}^2) = 0$$

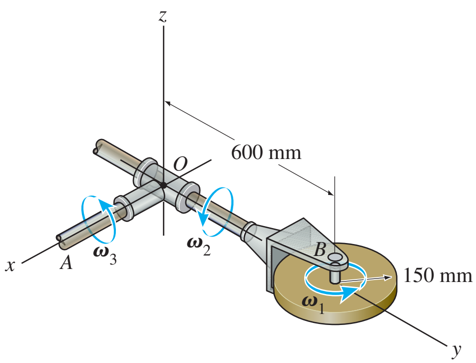
The three positive roots of I, obtained from the solution of this equation, represent the principal moments of inertia I_x, I_y, and I_z.



Prob. 21–24

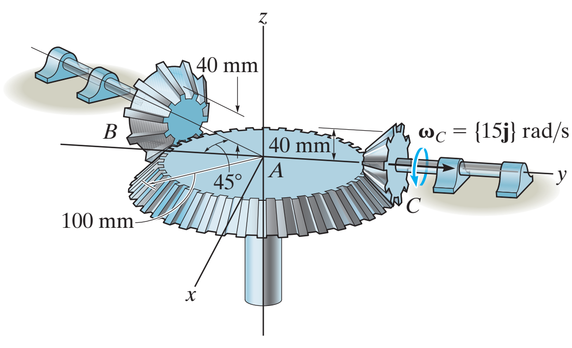
21–25. The 10-kg circular disk spins about its axle with a constant angular velocity of $\omega_1 = 15$ rad/s. Simultaneously, arm OB and shaft OA rotate about their axes with constant angular velocities of $\omega_2 = 0$ and $\omega_3 = 6$ rad/s, respectively. Determine the angular momentum of the disk about point O, and its kinetic energy.

21–26. The 10-kg circular disk spins about its axle with a constant angular velocity of $\omega_1 = 15$ rad/s. Simultaneously, arm OB and shaft OA rotate about their axes with constant angular velocities of $\omega_2 = 10$ rad/s and $\omega_3 = 6$ rad/s, respectively. Determine the angular momentum of the disk about point O, and its kinetic energy.



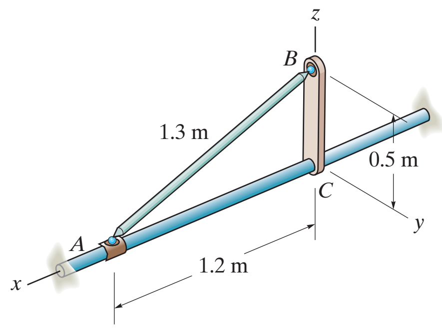
Probs. 21–25/26

21–27. The large gear has a mass of 5 kg and a radius of gyration of $k_z = 75$ mm. Gears B and C each have a mass of 200 g and a radius of gyration about the axis of their connecting shaft of 15 mm. If the gears are in mesh and C has an angular velocity of $\boldsymbol{\omega}_C = \{15\mathbf{j}\}$ rad/s, determine the total angular momentum for the system of three gears about point A.



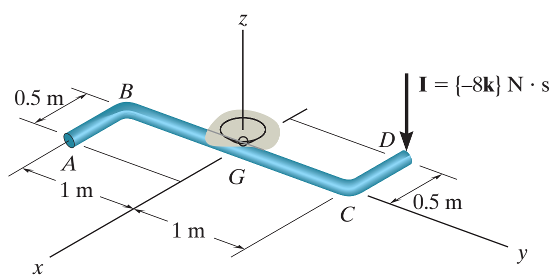
Prob. 21–27

***21–28.** The 4-kg rod AB is attached to the 1-kg collar at A and a 2-kg link BC using ball-and-socket joints. If the rod is released from rest in the position shown, determine the angular velocity of the link after it has rotated 180°.



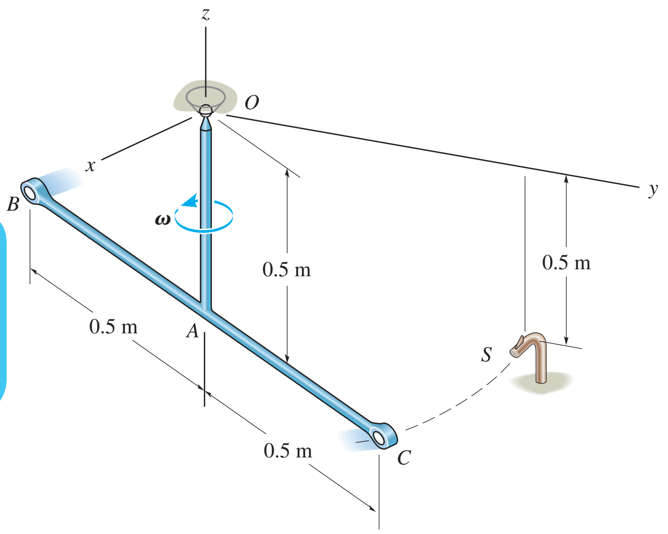
Prob. 21–28

21–29. The rod assembly is supported at G by a ball-and-socket joint. Each segment has a mass of 0.5 kg/m. If the assembly is originally at rest and an impulse of $\mathbf{I} = \{-8\mathbf{k}\}$ N·s is applied at D, determine the angular velocity of the assembly just after the impact.



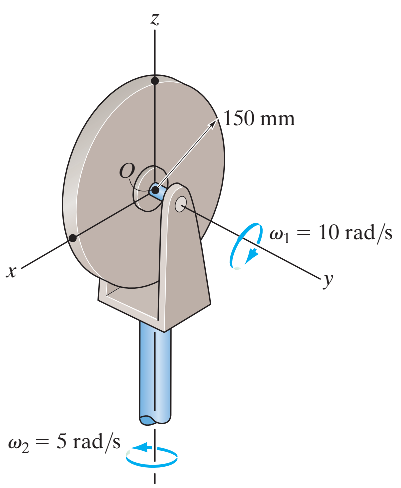
Prob. 21–29

21–30. The rod assembly has a mass of 2.5 kg/m and is rotating with a constant angular velocity of $\boldsymbol{\omega} = \{2\mathbf{k}\}$ rad/s when the looped end at C encounters a hook at S, which provides a permanent connection. Determine the angular velocity of the assembly immediately after impact.



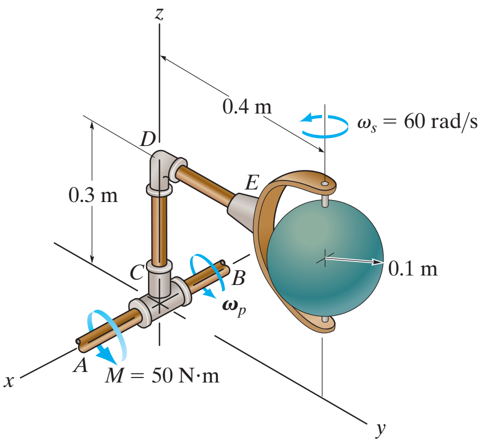
Prob. 21–30

21–31. The 15-kg circular disk spins about its axle with a constant angular velocity of $\omega_1 = 10$ rad/s. Simultaneously, the yoke is rotating with a constant angular velocity of $\omega_2 = 5$ rad/s. Determine the angular momentum of the disk about its center of mass O, and its kinetic energy.



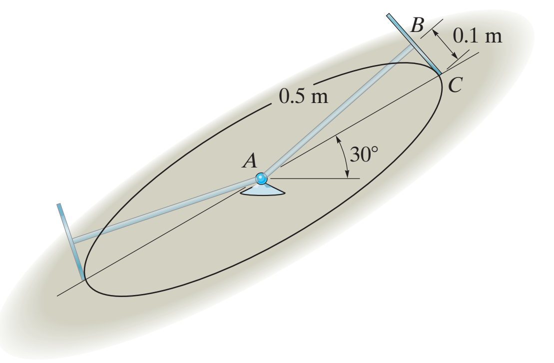
Prob. 21–31

***21–32.** The 20-kg sphere rotates about the axle with a constant angular velocity of $\omega_s = 60$ rad/s. If shaft AB is subjected to a torque of $M = 50$ N·m, causing it to rotate, determine the value of $\boldsymbol{\omega}_p$ after the shaft has turned 90° from the position shown. Initially, $\omega_p = 0$. Neglect the mass of arm CDE.



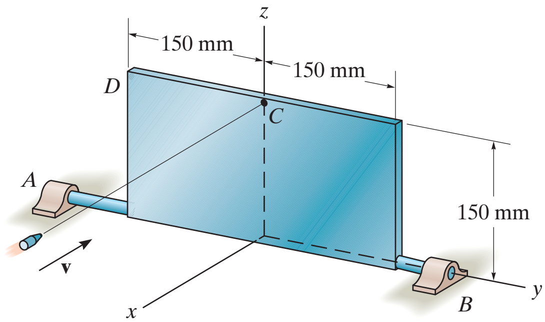
Prob. 21–32

21–33. The 2-kg thin disk is connected to the slender rod which is fixed to the ball-and-socket joint at A. If it is released from rest in the position shown, determine the spin of the disk about the rod when the disk reaches its lowest position. Neglect the mass of the rod. The disk rolls without slipping.



Prob. 21–33

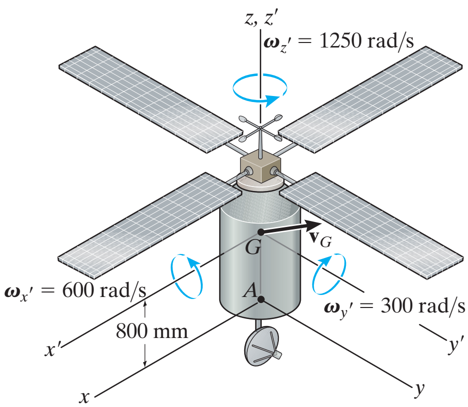
21–34. The 15-kg rectangular plate is free to rotate about the y axis because of the bearing supports at A and B. When the plate is balanced in the vertical plane, a 3-g bullet is fired into it, perpendicular to its surface, with a velocity $\mathbf{v} = \{-2000\mathbf{i}\}$ m/s. Determine the angular velocity of the plate at the instant it has rotated 180°. If the bullet strikes corner D with the same velocity $\mathbf{v}$, instead of at C, does the angular velocity remain the same? Why or why not?



Prob. 21–34

21–35. The 200-kg satellite has its center of mass at point G. Its radii of gyration about the z', x', y' axes are $k_{z'} = 300$ mm, $k_{x'} = k_{y'} = 500$ mm, respectively. At the instant shown, the satellite rotates about the x', y', and z' axes with the angular velocity shown, and its center of mass G has a velocity of $\mathbf{v}_G = \{-250\mathbf{i} + 200\mathbf{j} + 120\mathbf{k}\}$ m/s. Determine the angular momentum of the satellite about point A at this instant.

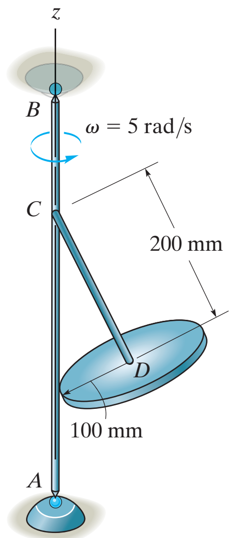
***21–36.** The 200-kg satellite has its center of mass at point G. Its radii of gyration about the z', x', y' axes are $k_{z'} = 300$ mm, $k_{x'} = k_{y'} = 500$ mm, respectively. At the instant shown, the satellite rotates about the x', y', and z' axes with the angular velocity shown, and its center of mass G has a velocity of $\mathbf{v}_G = \{-250\mathbf{i} + 200\mathbf{j} + 120\mathbf{k}\}$ m/s. Determine the kinetic energy of the satellite at this instant.



Probs. 21–35/36

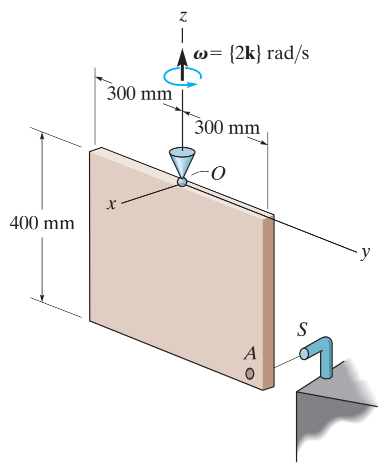
21–37. Determine the kinetic energy of the 7-kg disk and 1.5-kg rod when the assembly is rotating about the z axis at $\omega = 5$ rad/s.

21–38. Determine the angular momentum $\mathbf{H}_z$ of the 7-kg disk and 1.5-kg rod when the assembly is rotating about the z axis at $\omega = 5$ rad/s.



Probs. 21–37/38

21–39. The 5-kg thin plate is suspended at O using a ball-and-socket joint. It is rotating with a constant angular velocity $\boldsymbol{\omega} = \{2\mathbf{k}\}$ rad/s when the corner A strikes the hook at S, which provides a permanent connection. Determine the angular velocity of the plate immediately after impact.



Prob. 21–39

21

*21.4 EQUATIONS OF MOTION

Having become familiar with the techniques used to describe both the inertial properties and the angular momentum of a body, we can now write the equations which describe the motion of the body in their most useful forms.

Equations of Translational Motion. The *translational motion* of a body is defined in terms of the acceleration of the body's mass center, which is measured from an inertial X, Y, Z reference. This equation can be written in vector form as

$$\Sigma \mathbf{F} = m\mathbf{a}_G \tag{21–18}$$

or by the three scalar equations

$$\begin{aligned} \Sigma F_x &= m(a_G)_x \\ \Sigma F_y &= m(a_G)_y \\ \Sigma F_z &= m(a_G)_z \end{aligned} \tag{21–19}$$

Here, $\Sigma \mathbf{F} = \Sigma F_x \mathbf{i} + \Sigma F_y \mathbf{j} + \Sigma F_z \mathbf{k}$ represents the sum of all the external forces acting on the body.

Equations of Rotational Motion. In Sec. 15.6, we developed Eq. 15–17, namely,

$$\Sigma \mathbf{M}_O = \dot{\mathbf{H}}_O \tag{21–20}$$

which states that the sum of the moments of all the external forces acting on a system of particles contained in a rigid body about a *fixed point O* is equal to the time rate of change of the total angular momentum of the particles about point O. When moments of the external forces are summed about the system's *mass center G*, one again obtains the same simple form of Eq. 21–20, relating the moment summation $\Sigma \mathbf{M}_G$ to the angular momentum $\mathbf{H}_G$.

To show this, consider the system of particles in Fig. 21–11, where the x, y, z axes have their origin at G, and translate relative to the X, Y, Z inertial coordinate system. In general, G is *accelerating*, so by definition the translating frame is *not* an inertial reference. The angular momentum of the ith particle with respect to this frame is, however,

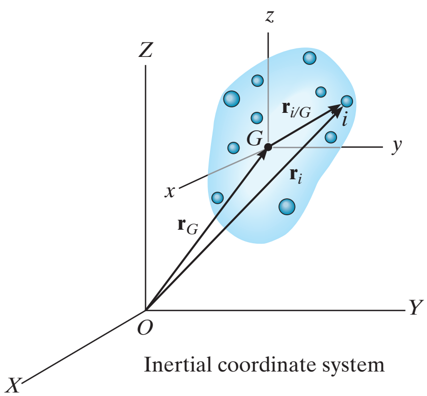


Fig. 21–11

$$(\mathbf{H}_i)_G = \mathbf{r}_{i/G} \times m_i \mathbf{v}_{i/G}$$

where $\mathbf{r}_{i/G}$ and $\mathbf{v}_{i/G}$ represent the position and velocity of the ith particle with respect to G. Taking the time derivative we have

$$(\dot{\mathbf{H}}_i)_G = \dot{\mathbf{r}}_{i/G} \times m_i \mathbf{v}_{i/G} + \mathbf{r}_{i/G} \times m_i \dot{\mathbf{v}}_{i/G}$$

By definition, $\mathbf{v}_{i/G} = \dot{\mathbf{r}}_{i/G}$. Thus, the first term on the right side is zero since the cross product of the same vectors is zero. Also, $\mathbf{a}_{i/G} = \dot{\mathbf{v}}_{i/G}$, so that

$$(\dot{\mathbf{H}}_i)_G = (\mathbf{r}_{i/G} \times m_i \mathbf{a}_{i/G})$$

Similar expressions can be written for the other particles of the body. When the results are summed, we get

$$\dot{\mathbf{H}}_G = \Sigma(\mathbf{r}_{i/G} \times m_i \mathbf{a}_{i/G})$$

Here $\dot{\mathbf{H}}_G$ is the time rate of change of the total angular momentum of the body about G.

The relative acceleration for the ith particle is defined by the equation $\mathbf{a}_{i/G} = \mathbf{a}_i - \mathbf{a}_G$, where $\mathbf{a}_i$ and $\mathbf{a}_G$ represent, respectively, the accelerations of the ith particle and point G measured with respect to the inertial frame of reference. Substituting and expanding, using the distributive property of the vector cross product, yields

$$\dot{\mathbf{H}}_G = \Sigma(\mathbf{r}_{i/G} \times m_i \mathbf{a}_i) - (\Sigma m_i \mathbf{r}_{i/G}) \times \mathbf{a}_G$$

By definition of the mass center, the sum $(\Sigma m_i \mathbf{r}_{i/G}) = (\Sigma m_i)\bar{\mathbf{r}}$ is equal to zero, since the position vector $\bar{\mathbf{r}}$ relative to G is zero. Hence, the last term in the above equation is zero. Using the equation of motion, the product $m_i \mathbf{a}_i$ can be replaced by the resultant external force $\mathbf{F}_i$ acting on the ith particle. Denoting $\Sigma \mathbf{M}_G = \Sigma(\mathbf{r}_{i/G} \times \mathbf{F}_i)$, the final result can be written as

$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \qquad (21\text{–}21)$$

For some problems it is convenient to allow the x, y, z axes to *rotate* rather then just translate. If these axes have an angular velocity $\mathbf{\Omega}$, then the time derivative $\dot{\mathbf{H}} = d\mathbf{H}/dt$, as used in Eqs. 21–20 and 21–21, must account for this rotation as measured from the inertial X, Y, Z axes. This requires application of Eq. 20–6, in which case Eqs. 21–20 and 21–21 become

$$\begin{aligned} \Sigma \mathbf{M}_O &= (\dot{\mathbf{H}}_O)_{xyz} + \mathbf{\Omega} \times \mathbf{H}_O \\ \Sigma \mathbf{M}_G &= (\dot{\mathbf{H}}_G)_{xyz} + \mathbf{\Omega} \times \mathbf{H}_G \end{aligned} \qquad (21\text{–}22)$$

Here $(\dot{\mathbf{H}})_{xyz}$ is the time rate of change of $\mathbf{H}$ measured from the x, y, z reference.

There are three ways in which one can define the motion of the x, y, z axes. Obviously, motion of this reference should be chosen so that it will yield the simplest set of moment equations for the solution of a particular problem.

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x, y, z Axes Having Motion $\boldsymbol{\Omega} = \mathbf{0}$. The x, y, z axes can be chosen such that the axes only *translate* relative to the inertial X, Y, Z frame of reference. Doing this simplifies Eq. 21–22, since $\boldsymbol{\Omega} = \mathbf{0}$. However, the body may have a rotation $\boldsymbol{\omega}$ about these axes, and if this is the case then the moments and products of inertia of the body will have to be expressed as *functions of time*. In most cases this would be a difficult task, so that such a choice of axes has restricted application.

x, y, z Axes Having Motion $\boldsymbol{\Omega} = \boldsymbol{\omega}$. The x, y, z axes can be chosen such that they are *fixed in and move with the body*. The moments and products of inertia of the body relative to these axes will then be *constant* during the motion. Since $\boldsymbol{\Omega} = \boldsymbol{\omega}$, Eqs. 21–22 become

21

$$\begin{aligned} \Sigma \mathbf{M}_O &= (\dot{\mathbf{H}}_O)_{xyz} + \boldsymbol{\omega} \times \mathbf{H}_O \\ \Sigma \mathbf{M}_G &= (\dot{\mathbf{H}}_G)_{xyz} + \boldsymbol{\omega} \times \mathbf{H}_G \end{aligned} \tag{21–23}$$

We can express each of these vector equations as three scalar equations using Eqs. 21–10. Neglecting the subscripts O and G, we have

$$\begin{aligned} \Sigma M_x &= I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z - I_{xy}(\dot{\omega}_y - \omega_z\omega_x) \\ &\quad - I_{yz}(\omega_y^2 - \omega_z^2) - I_{zx}(\dot{\omega}_z + \omega_x\omega_y) \\ \Sigma M_y &= I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x - I_{yz}(\dot{\omega}_z - \omega_x\omega_y) \\ &\quad - I_{zx}(\omega_z^2 - \omega_x^2) - I_{xy}(\dot{\omega}_x + \omega_y\omega_z) \\ \Sigma M_z &= I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y - I_{zx}(\dot{\omega}_x - \omega_y\omega_z) \\ &\quad - I_{xy}(\omega_x^2 - \omega_y^2) - I_{yz}(\dot{\omega}_y + \omega_z\omega_x) \end{aligned} \tag{21–24}$$

If the x, y, z axes are chosen as *principal axes of inertia*, the products of inertia are zero, $I_{xx} = I_x$, etc., and the above equations become

$$\begin{aligned} \Sigma M_x &= I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z \\ \Sigma M_y &= I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x \\ \Sigma M_z &= I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y \end{aligned} \tag{21–25}$$

This set of equations is known historically as the ***Euler equations of motion***, named after the Swiss mathematician Leonhard Euler, who first developed them. They apply *only* for moments summed about either point O or G.

When applying these equations it should be realized that $\dot{\omega}_x$, $\dot{\omega}_y$, $\dot{\omega}_z$ represent the time derivatives of the magnitudes of the x, y, z components of $\boldsymbol{\omega}$ as observed from x, y, z. To determine them, it is first necessary to find $\omega_x, \omega_y, \omega_z$ when the x, y, z axes are oriented in a *general position* and *then* take the time derivative, i.e., $(\dot{\boldsymbol{\omega}})_{xyz}$. However, for this case the x, y, z axes are rotating at $\boldsymbol{\Omega} = \boldsymbol{\omega}$, and so Eq. 20–6 becomes $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz} + \boldsymbol{\omega} \times \boldsymbol{\omega}$. Since $\boldsymbol{\omega} \times \boldsymbol{\omega} = \mathbf{0}$, then $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. This important result indicates that the time derivative of $\boldsymbol{\omega}$ with respect to the X, Y, Z axes, that is $\dot{\boldsymbol{\omega}}$, can also be used to obtain $(\dot{\boldsymbol{\omega}})_{xyz}$. Generally this is the easiest way to determine the result, as shown in Example 21.5.

x, y, z Axes Having Motion $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$. To simplify the calculations for finding $\dot{\boldsymbol{\omega}}$, it is often convenient to choose the x, y, z axes having an angular velocity $\boldsymbol{\Omega}$ which is different from the angular velocity $\boldsymbol{\omega}$ of the body. This is particularly suitable for the analysis of spinning tops and gyroscopes which are *symmetrical* about their spinning axes.* When this is the case, the moments and products of inertia will remain constant about the axis of spin.

Equations 21–22 are applicable for such a set of axes. Each of these two vector equations can be reduced to a set of three scalar equations which are derived in a manner similar to Eqs. 21–25,† i.e.,

$$
\begin{aligned}
\Sigma M_x &= I_x\dot{\omega}_x - I_y\Omega_z\omega_y + I_z\Omega_y\omega_z \\
\Sigma M_y &= I_y\dot{\omega}_y - I_z\Omega_x\omega_z + I_x\Omega_z\omega_x \\
\Sigma M_z &= I_z\dot{\omega}_z - I_x\Omega_y\omega_x + I_y\Omega_x\omega_y
\end{aligned}
\tag{21–26}
$$

Here Ω_x, Ω_y, Ω_z represent the x, y, z components of $\boldsymbol{\Omega}$, measured from the inertial frame of reference, and $\dot{\omega}_x$, $\dot{\omega}_y$, $\dot{\omega}_z$ must be determined relative to the x, y, z axes that have the rotation $\boldsymbol{\Omega}$. See Example 21.6.

Any one of these sets of moment equations, Eqs. 21–24, 21–25, or 21–26, represents a series of three first-order nonlinear differential equations that are "coupled," since the angular-velocity components are present in all the terms. Success in determining the solution for a particular problem therefore depends upon what is unknown in these equations. Difficulty certainly arises when one attempts to solve for the unknown components of $\boldsymbol{\omega}$ when the external moments are functions of time. Further complications can arise if the moment equations are coupled to the three scalar equations of translational motion, Eqs. 21–19. For example, this can happen when the rotation of the body is related to the translation of its mass center, as in the case of a hoop which rolls without slipping. Problems such as this, that require the simultaneous

* A detailed discussion of such devices is given in Sec. 21.5.

† See Prob. 21–43.

21

solution of differential equations, are generally solved using numerical methods with the aid of a computer. In many engineering problems, however, we are given information about the motion of the body and are required to determine the applied moments acting on the body. Most of these problems have direct solutions, so that there is no need to resort to computer techniques.

PROCEDURE FOR ANALYSIS

Problems involving the three-dimensional motion of a rigid body can be solved using the following procedure.

Free-Body Diagram.

- Draw a *free-body diagram* of the body at the instant considered and specify the x, y, z coordinate system. The origin of this reference must be located either at the body's mass center G, or at point O, considered fixed in an inertial reference frame and located either in the body or on a massless extension of the body.
- Unknown reactive force components can be shown having a positive sense of direction.
- Depending on the nature of the problem, decide what type of rotational motion $\mathbf{\Omega}$ the x, y, z coordinate system should have, i.e., $\mathbf{\Omega} = \mathbf{0}$, $\mathbf{\Omega} = \boldsymbol{\omega}$, or $\mathbf{\Omega} \neq \boldsymbol{\omega}$. When choosing, keep in mind that the moment equations are simplified when the axes move in such a manner that they represent principal axes of inertia for the body at all times.
- Calculate the necessary moments and products of inertia for the body relative to the x, y, z axes.

Kinematics.

- Determine the x, y, z components of the body's angular velocity and find the time derivatives of $\boldsymbol{\omega}$.
- Note that if $\mathbf{\Omega} = \boldsymbol{\omega}$, then $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. Therefore we can either find the time derivative of $\boldsymbol{\omega}$ with respect to the X, Y, Z axes, $\dot{\boldsymbol{\omega}}$, and then determine its components $\dot{\omega}_x$, $\dot{\omega}_y$, $\dot{\omega}_z$, or we can find the components of $\boldsymbol{\omega}$ along the x, y, z axes, when the axes are oriented in a general position, and then take the time derivative of the magnitudes of these components, $(\dot{\boldsymbol{\omega}})_{xyz}$.

Equations of Motion.

- Apply either the two vector equations 21–18 and 21–22 or the six scalar component equations appropriate for the x, y, z coordinate axes chosen for the problem.

EXAMPLE 21.4

The gear shown in Fig. 21–12*a* has a mass of 10 kg and is mounted at an angle of 10° with the rotating shaft having negligible mass. If $I_z = 0.1\ \text{kg}\cdot\text{m}^2$, $I_x = I_y = 0.05\ \text{kg}\cdot\text{m}^2$, and the shaft is rotating with a constant angular velocity of $\omega = 30$ rad/s, determine the components of reaction that the thrust bearing *A* and journal bearing *B* exert on the shaft at the instant shown.

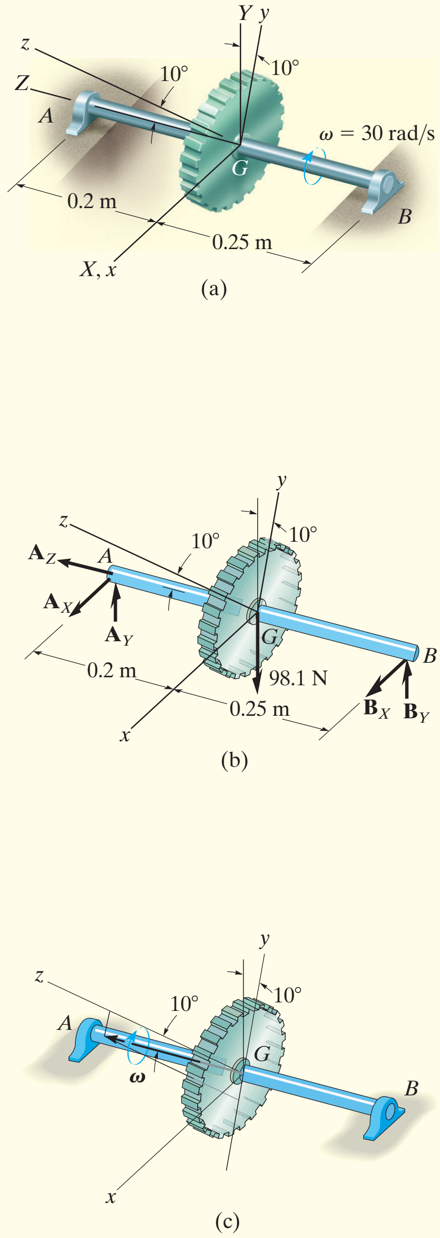


Fig. 21–12

SOLUTION

Free-Body Diagram. Fig. 21–12*b*. The origin of the *x*, *y*, *z* coordinate system is located at the gear's center of mass *G*, which is also a fixed point. The axes are fixed in and rotate with the gear so that these axes always represent the principal axes of inertia for the gear. Hence $\boldsymbol{\Omega} = \boldsymbol{\omega}$.

Kinematics. As shown in Fig. 21–12*c*, the angular velocity $\boldsymbol{\omega}$ of the gear is constant in magnitude and is always directed along the axis of the shaft *AB*. Since this vector is measured from the *X*, *Y*, *Z* inertial frame of reference, for any position of the *x*, *y*, *z* axes,

$$\omega_x = 0 \quad \omega_y = -30 \sin 10^\circ \quad \omega_z = 30 \cos 10^\circ$$

These components remain constant for any general orientation of the *x*, *y*, *z* axes, and so relative to the *x*, *y*, *z* axes $\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$. Also note that since $\boldsymbol{\Omega} = \boldsymbol{\omega}$, then $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. Therefore, we can also find these time derivatives of $\boldsymbol{\omega}$ relative to the *X*, *Y*, *Z* axes. Since $\boldsymbol{\omega}$ has a constant magnitude and direction (+*Z*) then $\dot{\boldsymbol{\omega}} = \mathbf{0}$, and so again $\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$. Furthermore, since *G* is a fixed point, $(a_G)_x = (a_G)_y = (a_G)_z = 0$.

Equations of Motion. Applying Eqs. 21–25 ($\boldsymbol{\Omega} = \boldsymbol{\omega}$) yields

$$\Sigma M_x = I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z$$
$$-(A_Y)(0.2) + (B_Y)(0.25) = 0 - (0.05 - 0.1)(-30 \sin 10^\circ)(30 \cos 10^\circ)$$
$$-0.2A_Y + 0.25B_Y = -7.70 \qquad (1)$$
$$\Sigma M_y = I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x$$
$$A_X(0.2) \cos 10^\circ - B_X(0.25) \cos 10^\circ = 0 - 0$$
$$A_X = 1.25B_X \qquad (2)$$
$$\Sigma M_z = I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y$$
$$A_X(0.2) \sin 10^\circ - B_X(0.25) \sin 10^\circ = 0 - 0$$
$$A_X = 1.25B_X (\text{check})$$

Applying Eqs. 21–19, we have

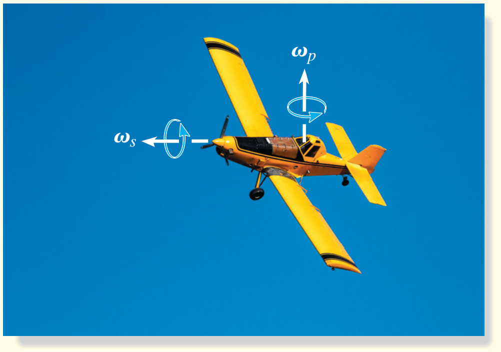
$$\Sigma F_X = m(a_G)_X; \qquad A_X + B_X = 0 \qquad (3)$$
$$\Sigma F_Y = m(a_G)_Y; \qquad A_Y + B_Y - 98.1 = 0 \qquad (4)$$
$$\Sigma F_Z = m(a_G)_Z; \qquad A_Z = 0 \qquad \textit{Ans.}$$

Solving Eqs. 1 through 4 simultaneously gives

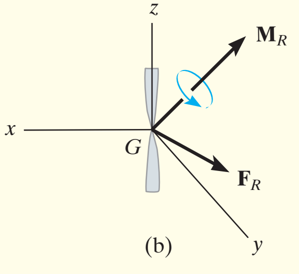
$$A_X = B_X = 0 \quad A_Y = 71.6\ \text{N} \quad B_Y = 26.5\ \text{N} \qquad \textit{Ans.}$$

EXAMPLE 21.5

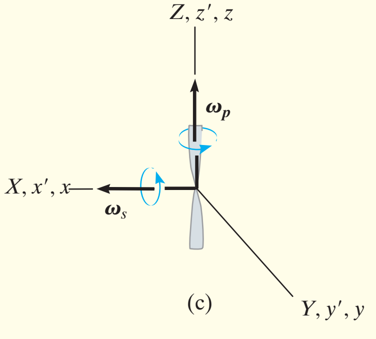
The airplane shown in Fig. 21–13*a* is in the process of making a steady *horizontal* turn at the rate of ω_p. During this motion, the propeller is spinning at the rate of ω_s. If the propeller has two blades, determine the moments which the propeller shaft exerts on the propeller at the instant the blades are in the vertical position. For simplicity, assume the blades to be a uniform slender bar having a moment of inertia I about an axis perpendicular to the blades passing through the center of the bar, and having zero moment of inertia about a longitudinal axis.



(a)



(b)



(c)

Fig. 21–13

SOLUTION

Free-Body Diagram. Fig. 21–13*b*. The reactions of the connecting shaft on the propeller are indicated by the resultants $\mathbf{F}_R$ and $\mathbf{M}_R$. (The propeller's weight is assumed to be negligible.) The x, y, z axes will be taken fixed to the propeller, such that these axes always represent the principal axes of inertia for the propeller. The moments of inertia I_x and I_y are equal ($I_x = I_y = I$) and $I_z = 0$.

Kinematics. The angular velocity of the propeller observed from the X, Y, Z axes, coincident with the x, y, z axes, Fig. 21–13*c*, is $\boldsymbol{\omega} = \boldsymbol{\omega}_s + \boldsymbol{\omega}_p = \omega_s\mathbf{i} + \omega_p\mathbf{k}$, so that the x, y, z components of $\boldsymbol{\omega}$ are

$$\omega_x = \omega_s \qquad \omega_y = 0 \qquad \omega_z = \omega_p$$

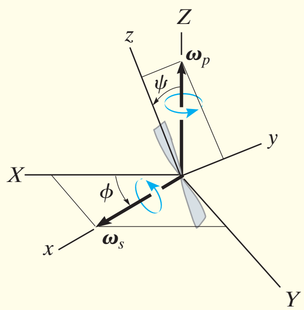
Since $\boldsymbol{\Omega} = \boldsymbol{\omega}$, then $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. To find $\dot{\boldsymbol{\omega}}$, which is the time derivative with respect to the fixed X, Y, Z axes, we must use Eq. 20–6 since $\boldsymbol{\omega}$ changes direction relative to X, Y, Z. The time rate of change of each component $\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_s + \dot{\boldsymbol{\omega}}_p$ relative to the X, Y, Z axes can be obtained by introducing a third coordinate system x', y', z', which has an angular velocity $\boldsymbol{\Omega}' = \boldsymbol{\omega}_p$ and is coincident with the X, Y, Z axes at the instant shown. Thus

$$\begin{aligned}\dot{\boldsymbol{\omega}} &= (\dot{\boldsymbol{\omega}})_{x'y'z'} + \boldsymbol{\omega}_p \times \boldsymbol{\omega}\\ &= (\dot{\boldsymbol{\omega}}_s)_{x'y'z'} + (\dot{\boldsymbol{\omega}}_p)_{x'y'z'} + \boldsymbol{\omega}_p \times (\boldsymbol{\omega}_s + \boldsymbol{\omega}_p)\\ &= \mathbf{0} + \mathbf{0} + \boldsymbol{\omega}_p \times \boldsymbol{\omega}_s + \boldsymbol{\omega}_p \times \boldsymbol{\omega}_p\\ &= \mathbf{0} + \mathbf{0} + \omega_p\mathbf{k} \times \omega_s\mathbf{i} + \mathbf{0} = \omega_p\omega_s\mathbf{j}\end{aligned}$$

Since the X, Y, Z axes are coincident with the x, y, z axes at the instant shown, the components of $\dot{\boldsymbol{\omega}}$ along x, y, z are therefore

$$\dot{\omega}_x = 0 \quad \dot{\omega}_y = \omega_p\omega_s \quad \dot{\omega}_z = 0$$

These same results can also be determined by direct calculation of $(\dot{\boldsymbol{\omega}})_{xyz}$; however, this will involve a bit more work. To do this, it will be necessary to view the propeller (or the x, y, z axes) in some *general position* such as shown in Fig. 21–13*d*. Here the plane has turned through an angle ϕ (phi) and the propeller has turned through an angle ψ (psi) relative to the plane. Notice that $\boldsymbol{\omega}_p$ is always directed along the fixed Z axis and $\boldsymbol{\omega}_s$ follows the x axis. Thus the general components of $\boldsymbol{\omega}$ are



(d)

Fig. 21–13

$$\omega_x = \omega_s \quad \omega_y = \omega_p \sin\psi \quad \omega_z = \omega_p \cos\psi$$

Since ω_s and ω_p are constant, the time derivatives of these components become

$$\dot{\omega}_x = 0 \quad \dot{\omega}_y = \omega_p \cos\psi\,\dot{\psi} \quad \omega_z = -\omega_p \sin\psi\,\dot{\psi}$$

But $\phi = \psi = 0°$ and $\dot{\psi} = \omega_s$ at the instant considered. Thus,

$$\begin{aligned}\omega_x &= \omega_s \quad & \omega_y &= 0 \quad & \omega_z &= \omega_p\\ \dot{\omega}_x &= 0 \quad & \dot{\omega}_y &= \omega_p\omega_s \quad & \dot{\omega}_z &= 0\end{aligned}$$

which are the same results as those obtained previously.

Equations of Motion. Using Eqs. 21–25, we have

$$\Sigma M_x = I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z = I(0) - (I - 0)(0)\omega_p$$

$$M_x = 0 \qquad \textit{Ans.}$$

$$\Sigma M_y = I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x = I(\omega_p\omega_s) - (0 - I)\omega_p\omega_s$$

$$M_y = 2I\omega_p\omega_s \qquad \textit{Ans.}$$

$$\Sigma M_z = I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y = 0(0) - (I - I)\omega_s(0)$$

$$M_z = 0 \qquad \textit{Ans.}$$

EXAMPLE 21.6

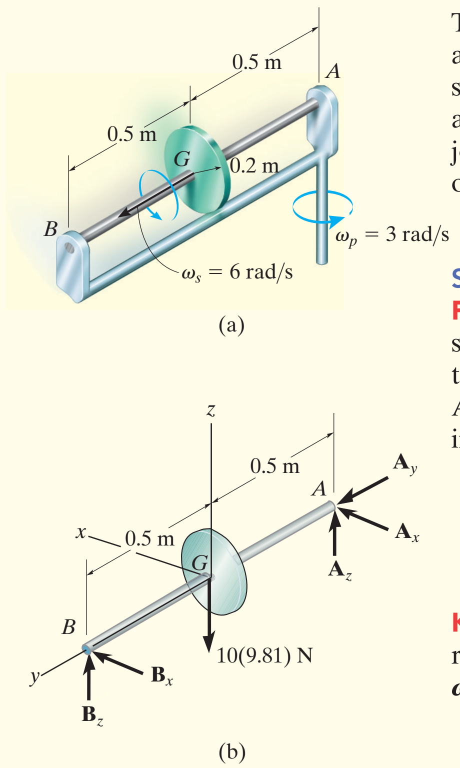


Fig. 21–14

The 10-kg flywheel (or thin disk) shown in Fig. 21–14*a* rotates (spins) about the shaft at a constant angular velocity of $\omega_s = 6$ rad/s. At the same time, the shaft rotates (precesses) about the bearing at A with an angular velocity of $\omega_p = 3$ rad/s. If A is a thrust bearing and B is a journal bearing, determine the components of force reaction at each of these supports due to the motion.

SOLUTION I

Free-Body Diagram. Fig. 21–14*b*. The origin of the x, y, z coordinate system is located at the center of mass G of the flywheel. Here we will let these coordinates have an angular velocity of $\mathbf{\Omega} = \boldsymbol{\omega}_p = \{3\mathbf{k}\}$ rad/s. Although the flywheel spins relative to these axes, the moments of inertia *remain constant*,* i.e.,

$$I_x = I_z = \tfrac{1}{4}(10\text{ kg})(0.2\text{ m})^2 = 0.1\text{ kg}\cdot\text{m}^2$$

$$I_y = \tfrac{1}{2}(10\text{ kg})(0.2\text{ m})^2 = 0.2\text{ kg}\cdot\text{m}^2$$

Kinematics. From the coincident inertial X, Y, Z frame of reference, Fig. 21–14*c*, the flywheel has an angular velocity of $\boldsymbol{\omega} = \{6\mathbf{j} + 3\mathbf{k}\}$ rad/s, so that

$$\omega_x = 0 \quad \omega_y = 6\text{ rad/s} \quad \omega_z = 3\text{ rad/s}$$

The time derivative of $\boldsymbol{\omega}$ must be determined relative to the x, y, z axes. In this case both $\boldsymbol{\omega}_p$ and $\boldsymbol{\omega}_s$ do not change their magnitude or direction, and so

$$\dot{\omega}_x = 0 \quad \dot{\omega}_y = 0 \quad \dot{\omega}_z = 0$$

Equations of Motion. Applying Eqs. 21–26 ($\mathbf{\Omega} \neq \boldsymbol{\omega}$) yields

$$\Sigma M_x = I_x\dot{\omega}_x - I_y\Omega_z\omega_y + I_z\Omega_y\omega_z$$

$$-A_z(0.5) + B_z(0.5) = 0 - (0.2)(3)(6) + 0 = -3.6$$

$$\Sigma M_y = I_y\dot{\omega}_y - I_z\Omega_x\omega_z + I_x\Omega_z\omega_x$$

$$0 = 0 - 0 + 0$$

$$\Sigma M_z = I_z\dot{\omega}_z - I_x\Omega_y\omega_x + I_y\Omega_x\omega_y$$

$$A_x(0.5) - B_x(0.5) = 0 - 0 + 0$$

* This would not be true for the propeller in Example 21.5.

Applying Eqs. 21–19, we have

$$\Sigma F_X = m(a_G)_X; \qquad A_x + B_x = 0$$

$$\Sigma F_Y = m(a_G)_Y; \qquad A_y = -10(0.5)(3)^2$$

$$\Sigma F_Z = m(a_G)_Z; \qquad A_z + B_z - 10(9.81) = 0$$

Solving these equations, we obtain

$$A_x = 0 \quad A_y = -45.0 \text{ N} \quad A_z = 52.6 \text{ N} \qquad \textit{Ans.}$$
$$B_x = 0 \qquad B_z = 45.4 \text{ N} \qquad \textit{Ans.}$$

NOTE: If the precession $\boldsymbol{\omega}_p$ had not occurred, the z component of force at A and B would be equal to 49.05 N. In this case, however, the difference in these components is caused by the "gyroscopic moment" created whenever a spinning body precesses about another axis. We will study this effect in detail in the next section.

SOLUTION II

This example can also be solved using Euler's equations of motion, Eqs. 21–25. In this case $\boldsymbol{\Omega} = \boldsymbol{\omega} = \{6\mathbf{j} + 3\mathbf{k}\}$ rad/s, and the time derivative $(\dot{\boldsymbol{\omega}})_{xyz}$ can also be obtained with reference to the fixed X, Y, Z axes since $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. This calculation can be performed by choosing x', y', z' axes to have an angular velocity of $\boldsymbol{\Omega}' = \boldsymbol{\omega}_p$, Fig. 21–14*c*. Applying Eq. 20–6,

$$\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{x'y'z'} + \boldsymbol{\omega}_p \times \boldsymbol{\omega} = \mathbf{0} + 3\mathbf{k} \times (6\mathbf{j} + 3\mathbf{k}) = \{-18\mathbf{i}\} \text{ rad/s}^2$$

$$\dot{\omega}_x = -18 \text{ rad/s} \quad \dot{\omega}_y = 0 \quad \dot{\omega}_z = 0$$

The moment equations then become

$$\Sigma M_x = I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z$$

$$-A_z(0.5) + B_z(0.5) = 0.1(-18) - (0.2 - 0.1)(6)(3) = -3.6$$

$$\Sigma M_y = I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x$$

$$0 = 0 - 0$$

$$\Sigma M_z = I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y$$

$$A_x(0.5) - B_x(0.5) = 0 - 0$$

The solution then proceeds as before.

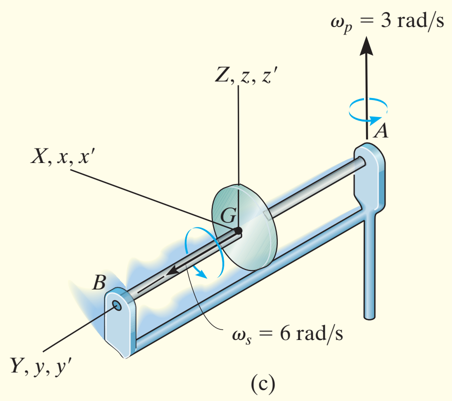
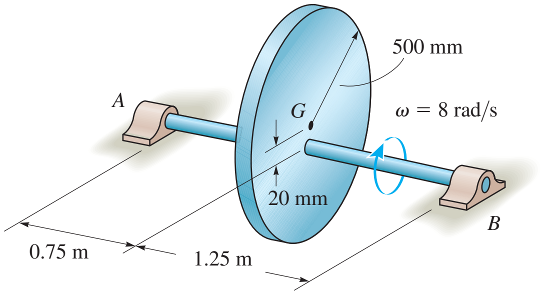


Fig. 21–14

PROBLEMS

*21–40. The 40-kg flywheel (disk) is mounted 20 mm off its true center at G. If the shaft is rotating at a constant speed $\omega = 8\,\text{rad/s}$, determine the minimum reactions exerted on the journal bearings at A and B during the motion.



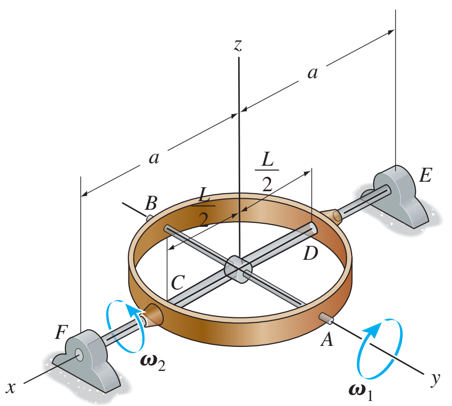
Prob. 21–40

21–41. Derive the scalar form of the rotational equation of motion about the x axis if $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$ and the moments and products of inertia of the body are *not constant* with respect to time.

21–42. Derive the scalar form of the rotational equation of motion about the x axis if $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$ and the moments and products of inertia of the body are *constant* with respect to time.

21–43. Derive the Euler equations of motion for $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$, i.e., Eqs. 21–26.

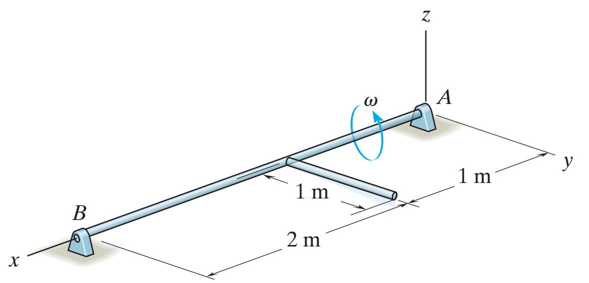
*21–44. Rod CD of mass m and length L is rotating with a constant angular rate of $\boldsymbol{\omega}_1$ about axle AB, while shaft EF rotates with a constant angular rate of $\boldsymbol{\omega}_2$. Determine the X, Y, and Z components of reaction at thrust bearing E and journal bearing F at the instant shown. Neglect the mass of the other members.



Prob. 21–44

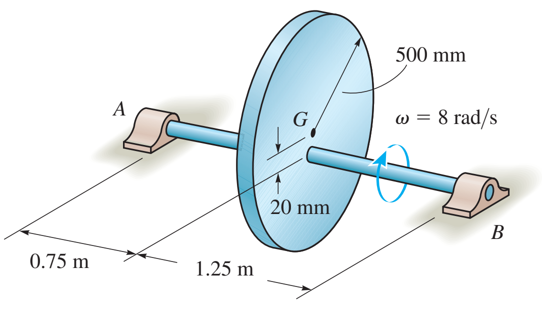
21–45. The assembly is supported by journal bearings at A and B, which develop only y and z force reactions on the shaft. If the shaft is rotating in the direction shown at $\boldsymbol{\omega} = \{2\mathbf{i}\}$ rad/s, determine the reactions at the bearings when the assembly is in the position shown. Also, what is the shaft's angular acceleration? The mass per unit length of each rod is 5 kg/m.

21–46. The assembly is supported by journal bearings at A and B, which develop only y and z force reactions on the shaft. If the shaft A is subjected to a couple moment $\mathbf{M} = \{40\mathbf{i}\}$ N·m, and at the instant shown the shaft has an angular velocity of $\boldsymbol{\omega} = \{2\mathbf{i}\}$ rad/s, determine the reactions at the bearings of the assembly at this instant. Also, what is the shaft's angular acceleration? The mass per unit length of each rod is 5 kg/m.



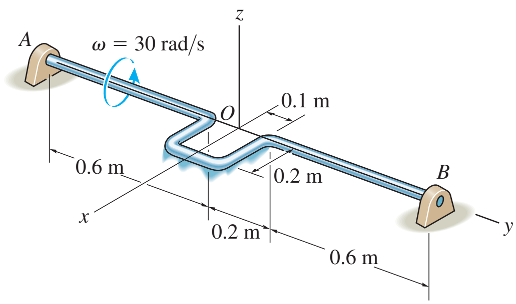
Probs. 21–45/46

21–47. The 40-kg flywheel (disk) is mounted 20 mm off its true center at G. If the shaft is rotating at a constant speed $\omega = 8\,\text{rad/s}$, determine the maximum reactions exerted on the journal bearings at A and B.



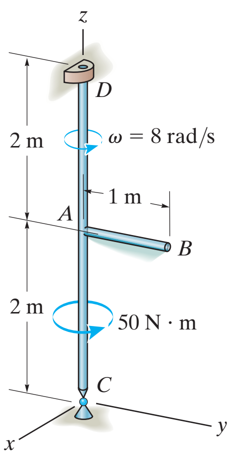
Prob. 21–47

***21–48.** The shaft is constructed from a rod which has a mass per unit of 2 kg/m. Determine the x, y, z components of reaction at the bearings A and B if at the instant shown the shaft spins freely and has an angular velocity of $\omega = 30$ rad/s. What is the angular acceleration of the shaft at this instant? Bearing A can support a component of force in the y direction, whereas bearing B cannot.



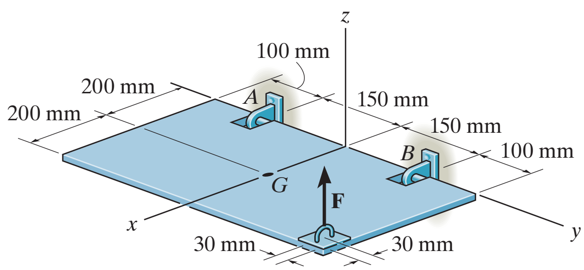
Prob. 21–48

21–49. The rod assembly is supported by a ball-and-socket joint at C and a journal bearing at D, which develops only x and y force reactions. The rods have a mass of 0.75 kg/m. Determine the angular acceleration of the rods and the components of reaction at the supports at the instant $\omega = 8$ rad/s as shown.



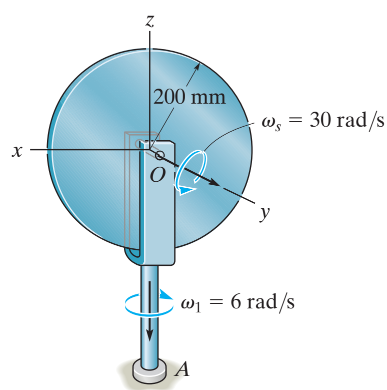
Prob. 21–49

21–50. The uniform hatch door, having a mass of 15 kg and a mass center at G, is supported in the horizontal plane by bearings at A and B. If a vertical force $F = 300$ N is applied to the door as shown, determine the components of reaction at the bearings and the angular acceleration of the door. The bearing at A will resist a component of force in the y direction, whereas the bearing at B will not. For the calculation, assume the door to be a thin plate and neglect the size of each bearing. The door is originally at rest.



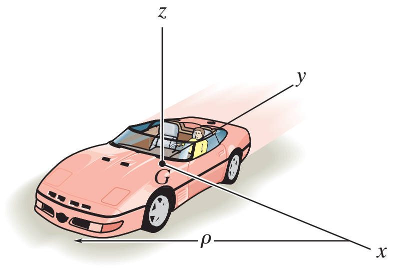
Prob. 21–50

21–51. The 20-kg disk is spinning on its axle at $\omega_s = 30$ rad/s, while the forked rod is turning at $\omega_1 = 6$ rad/s. Determine the x and z moment components the axle exerts on the disk during the motion.



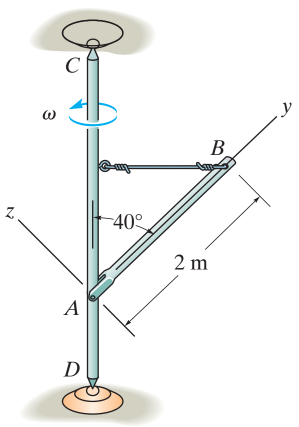
Prob. 21–51

***21–52.** The car travels around the curved road of radius ρ such that its mass center has a constant speed v_G. Write the equations of rotational motion with respect to the x, y, z axes. Assume that the car's six moments and products of inertia with respect to these axes are known.



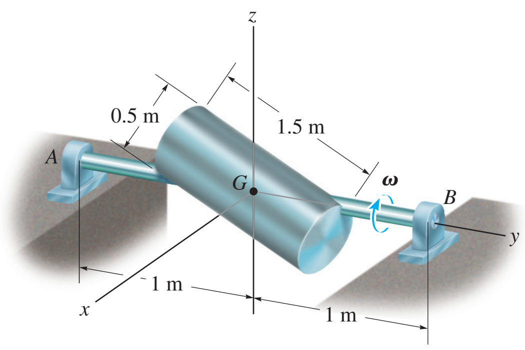
Prob. 21–52

21–53. The 4-kg slender rod AB is pinned at A and held at B by a cord. The axle CD is supported at its ends by ball-and-socket joints and is rotating with a constant angular velocity of 2 rad/s. Determine the tension developed in the cord and the magnitude of force developed at the pin A.



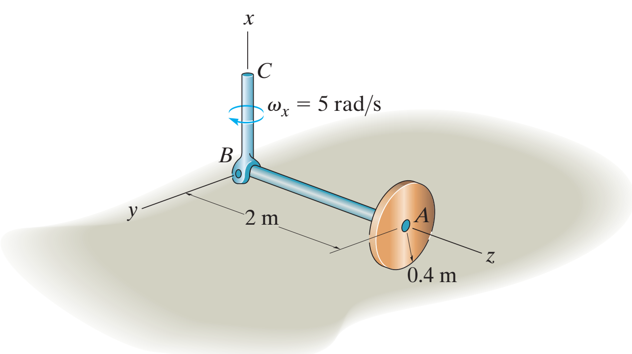
Prob. 21–53

21–54. The cylinder has a mass of 30 kg and is mounted on an axle which is supported by bearings at A and B. If the axle is subjected to a couple moment $\mathbf{M} = \{-30\mathbf{j}\}$ N·m, and at the instant shown has an angular velocity $\boldsymbol{\omega} = \{-40\mathbf{j}\}$ rad/s, determine the vertical components of force acting at the bearings at this instant.



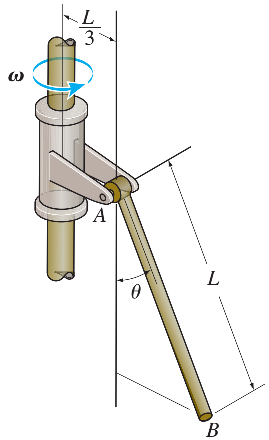
Prob. 21–54

21–55. The 10-kg disk turns around the shaft AB, while the shaft rotates about BC at a constant rate of $\omega_x = 5$ rad/s. If the disk does not slip, determine the normal and frictional force it exerts on the ground. Neglect the mass of shaft AB.



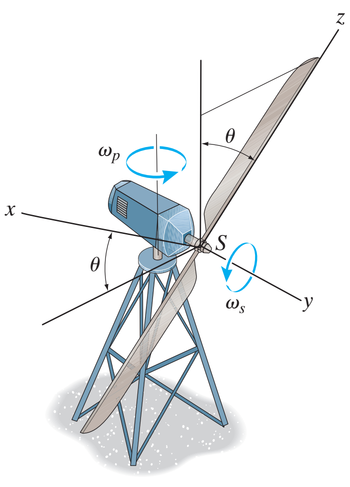
Prob. 21–55

***21–56.** The slender rod AB has a mass m and it is connected to the bracket by a smooth pin at A. The bracket is rigidly attached to the shaft. Determine the required constant angular velocity ω of the shaft, in order for the rod to make an angle θ with the vertical.



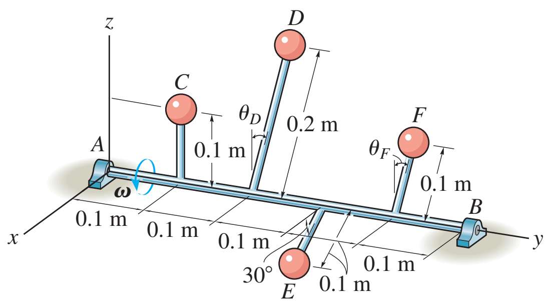
Prob. 21–56

21–57. The blades of a wind turbine spin about the shaft S with a constant angular speed of ω_s, while the frame precesses about the vertical axis with a constant angular speed of ω_p. Determine the x, y, and z components of moment that the shaft exerts on the blades as a function of θ. Consider each blade as a slender rod of mass m and length l.



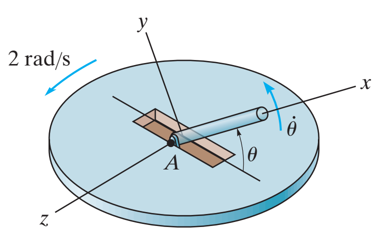
Prob. 21–57

21–58. Four spheres are connected to shaft AB. If $m_C = 1$ kg and $m_E = 2$ kg, determine the mass of spheres D and F and the angles of the rods, θ_D and θ_F, so that the shaft is dynamically balanced, that is, so that the bearings at A and B exert only vertical reactions on the shaft as it rotates. Neglect the mass of the rods.



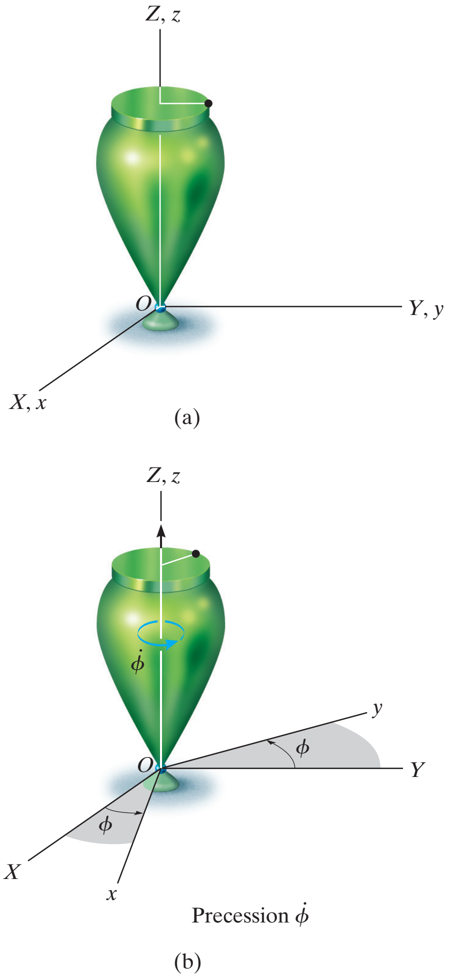
Prob. 21–58

21–59. The *thin rod* has a mass of 0.8 kg and a total length of 150 mm. It is rotating about its midpoint at a constant rate $\dot{\theta} = 6$ rad/s, while the table to which its axle A is fastened is rotating at 2 rad/s. Determine the x, y, z moment components which the axle exerts on the rod when the rod is in any position θ.



Prob. 21–59

21



*21.5 GYROSCOPIC MOTION

In this section we will develop the equations defining the motion of a body (top) which is symmetrical with respect to an axis and rotating about a fixed point. These equations also apply to the motion of a particularly interesting device, the gyroscope.

The body's motion will be analyzed using ***Euler angles*** ϕ, θ, ψ (phi, theta, psi). To show how they define the final position of a body, we will consider the top shown in Fig. 21–15*a*, which is in the final position shown in Fig. 21–15*d*. We will use stationary X, Y, Z axes and a second set of x, y, z axes fixed in the top. Starting with the X, Y, Z and x, y, z axes in coincidence, Fig. 21–15*a*, the final position of the top is determined using the following three steps:

1. Rotate the top about the Z (or z) axis through an angle ϕ $(0 \le \phi < 2\pi)$, Fig. 21–15*b*.
2. Rotate the top about the x axis through an angle θ $(0 \le \theta \le \pi)$, Fig. 21–15*c*.
3. Rotate the top about the z axis through an angle ψ $(0 \le \psi < 2\pi)$ to obtain the final position, Fig. 21–15*d*.

The sequence of these three angles, ϕ, θ, then ψ, must be maintained, since finite rotations are *not vectors* (see Fig. 20–1). Although this is the case, the differential rotations $d\boldsymbol{\phi}$, $d\boldsymbol{\theta}$, $d\boldsymbol{\psi}$ are vectors, and so the angular velocity $\boldsymbol{\omega}$ of the top can be expressed in terms of the time derivatives of the Euler angles. The angular-velocity components $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ are known as the ***precession***, ***nutation***, and ***spin***, respectively.

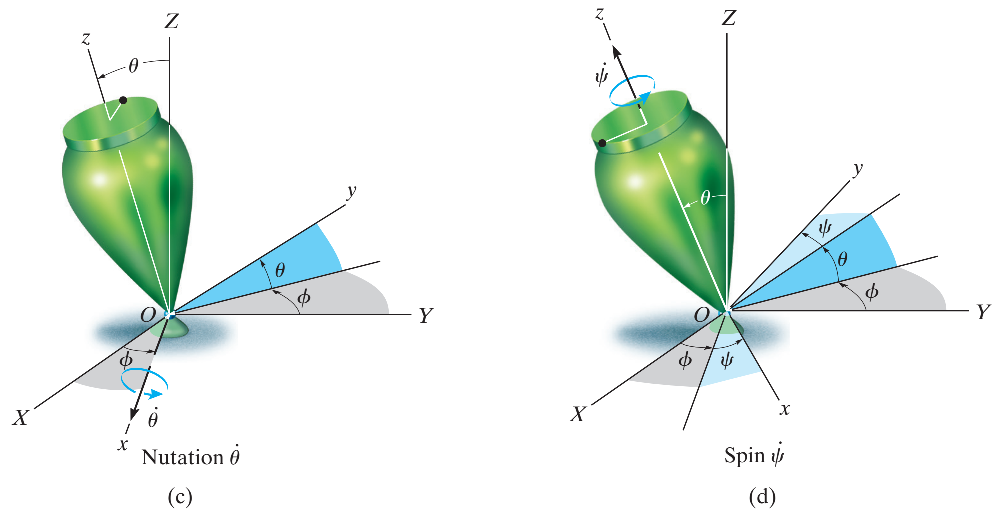


Fig. 21–15

Their positive directions are shown in Fig. 21–16. It is seen that these vectors are not all perpendicular to one another; however, $\boldsymbol{\omega}$ of the top can still be expressed in terms of these three components.

Since the body (top) is symmetric with respect to the z or spin axis, there is no need to attach the x, y, z axes to the top since the inertial properties of the top will remain constant with respect to this frame during the motion. Therefore $\boldsymbol{\Omega} = \boldsymbol{\omega}_p + \boldsymbol{\omega}_n$, Fig. 21–16, and so the angular velocity of the top is

$$\begin{aligned}\boldsymbol{\omega} &= \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}\\ &= \dot{\theta}\mathbf{i} + (\dot{\phi}\sin\theta)\mathbf{j} + (\dot{\phi}\cos\theta + \dot{\psi})\mathbf{k} \qquad (21\text{–}27)\end{aligned}$$

And the angular velocity of the axes is

$$\begin{aligned}\boldsymbol{\Omega} &= \Omega_x\mathbf{i} + \Omega_y\mathbf{j} + \Omega_z\mathbf{k}\\ &= \dot{\theta}\mathbf{i} + (\dot{\phi}\sin\theta)\mathbf{j} + (\dot{\phi}\cos\theta)\mathbf{k} \qquad (21\text{–}28)\end{aligned}$$

The x, y, z axes represent principal axes of inertia for the top, and so the moments of inertia will be represented as $I_{xx} = I_{yy} = I$ and $I_{zz} = I_z$. Since $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$, Eqs. 21–26 are used to establish the rotational equations of motion. Substituting into these equations the respective angular-velocity components defined by Eqs. 21–27 and 21–28, their corresponding time derivatives, and the moment of inertia components, yields

$$\begin{aligned}\Sigma M_x &= I(\ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta) + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi})\\ \Sigma M_y &= I(\ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta) - I_z\dot{\theta}(\dot{\phi}\cos\theta + \dot{\psi}) \qquad (21\text{–}29)\\ \Sigma M_z &= I_z(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta)\end{aligned}$$

Each moment summation applies only at the fixed point O or the center of mass G. Since the equations represent a coupled set of nonlinear second-order differential equations, in general a closed-form solution can not be obtained. Instead, the Euler angles ϕ, θ, and ψ may be obtained graphically as functions of time using numerical analysis and computer techniques.

A special case, however, does exist for which simplification of Eqs. 21–29 is possible. Commonly referred to as ***steady precession***, it occurs when the nutation angle θ, precession $\dot{\phi}$, and spin $\dot{\psi}$ all remain *constant*. Equations 21–29 then reduce to the form

$$\boxed{\Sigma M_x = -I\dot{\phi}^2\sin\theta\cos\theta + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi})} \qquad (21\text{–}30)$$

$$\Sigma M_y = 0$$

$$\Sigma M_z = 0$$

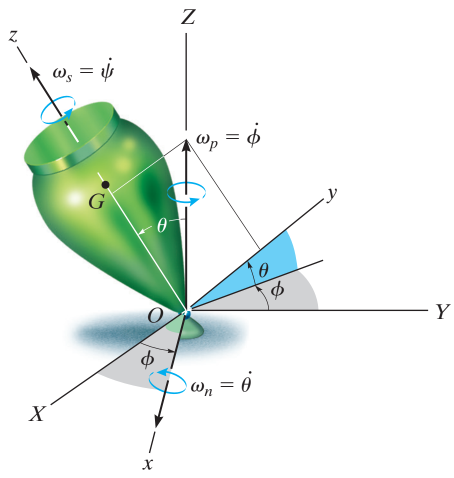


Fig. 21–16

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Equation 21–30 can be further simplified by noting that, from Eq. 21–27, $\omega_z = \dot{\phi}\cos\theta + \dot{\psi}$, so that

$$\Sigma M_x = -I\dot{\phi}^2 \sin\theta\cos\theta + I_z\dot{\phi}\,(\sin\theta)\omega_z$$

or

$$\boxed{\Sigma M_x = \dot{\phi}\sin\theta(I_z\omega_z - I\dot{\phi}\cos\theta)} \qquad (21\text{–}31)$$

It is interesting to note what effects the spin $\dot{\psi}$ has on the moment about the x axis. To show this, consider the spinning rotor in Fig. 21–17. Here $\theta = 90°$, in which case Eq. 21–30 reduces to the form

$$\Sigma M_x = I_z\dot{\phi}\dot{\psi}$$

or

$$\boxed{\Sigma M_x = I_z\Omega_y\omega_z} \qquad (21\text{–}32)$$

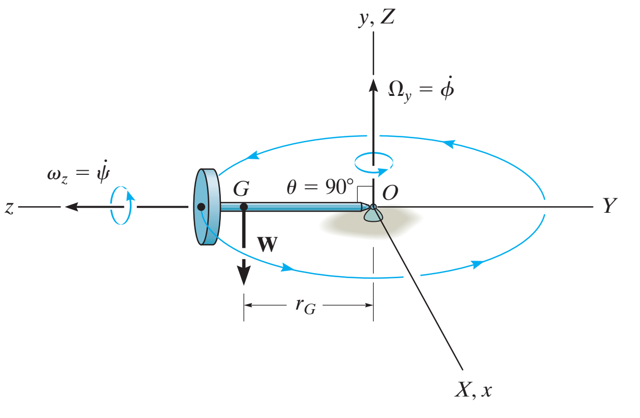


Fig. 21–17

Instinctively, one would expect the rotor to fall down under the influence of gravity! However, this is not the case at all, provided the product $I_z\Omega_y\omega_z$ is correctly chosen to counterbalance the moment $\Sigma M_x = Wr_G$ of the rotor's weight about O. This unusual phenomenon of rigid-body motion is often referred to as the ***gyroscopic effect***.

Perhaps a more intriguing demonstration of the gyroscopic effect comes from studying the action of a ***gyroscope***, frequently referred to as a ***gyro***. A gyro is a rotor which spins at a very high rate about its axis of symmetry. This spin is considerably greater than its precession about the vertical axis. Hence, for all practical purposes, the angular momentum of the gyro can be assumed directed along its axis of spin. Thus, for the gyro rotor shown in Fig. 21–18, $\omega_z \gg \Omega_y$, and the magnitude of the angular momentum about point O, as determined from Eqs. 21–11, reduces to the form $H_O = I_z\omega_z$. Since both the magnitude and direction of $\mathbf{H}_O$ are constant as observed from x, y, z, direct application of Eq. 21–22 yields

$$\Sigma\mathbf{M}_x = \mathbf{\Omega}_y \times \mathbf{H}_O \qquad (21\text{–}33)$$

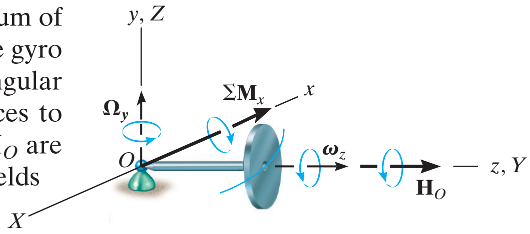
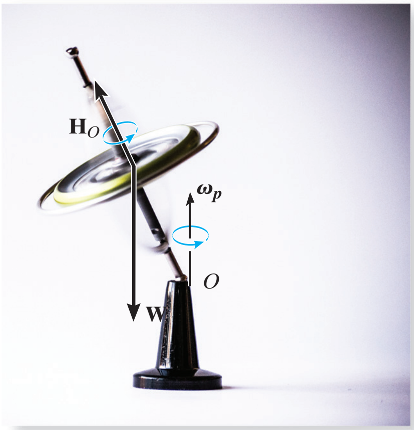


Fig. 21–18

Using the right-hand rule applied to the cross product, it can be seen that $\mathbf{\Omega}_y$ always swings $\mathbf{H}_O$ (or $\boldsymbol{\omega}_z$) toward the sense of $\Sigma\mathbf{M}_x$. In effect, the *change in direction* of the gyro's angular momentum, $d\mathbf{H}_O$, is equivalent to the angular impulse caused by the gyro's weight about O, i.e., $d\mathbf{H}_O = \Sigma\mathbf{M}_x\,dt$, Eq. 21–20. Also, since $H_O = I_z\omega_z$ and $\Sigma\mathbf{M}_x$, $\mathbf{\Omega}_y$, and $\mathbf{H}_O$ are mutually perpendicular, Eq. 21–33 reduces to Eq. 21–32.

When a gyro is mounted in gimbal rings, Fig. 21–19, it becomes *free* of external moments applied to its base. Thus, in theory, its angular momentum $\mathbf{H}$ will never precess but, instead, maintain its same fixed orientation along the axis of spin when the base is rotated. This type of gyroscope is called a ***free gyro*** and is useful as a gyrocompass when the spin axis of the gyro is directed north. In reality, however, the gimbal mechanism is never completely free of friction, so such a device is useful only for the local navigation of ships and aircraft. The gyroscopic effect is also useful as a means of stabilizing both the rolling motion of ships at sea and the trajectories of missiles and projectiles. Furthermore, this effect is of significant importance in the design of shafts and bearings for rotors which are subjected to forced precessions, as discussed in Example 21.6.



The spinning of the gyro within the frame of this toy gyroscope produces angular momentum $\mathbf{H}_O$, which is changing direction as the frame precesses $\boldsymbol{\omega}_p$ about the vertical axis. The gyroscope will not fall down since the moment of its weight $\mathbf{W}$ about the support is balanced by the change in the direction of $\mathbf{H}_O$.

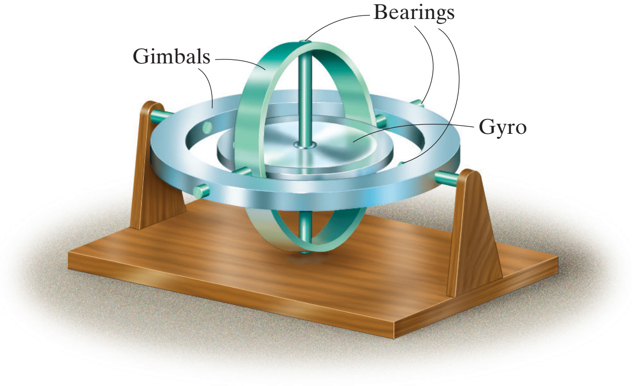
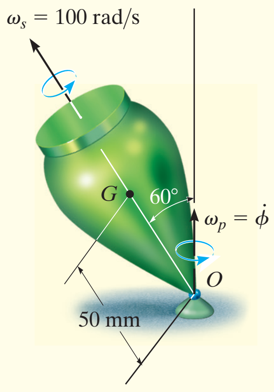


Fig. 21–19

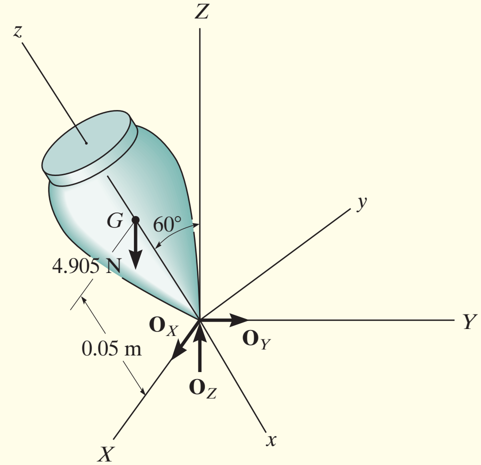
EXAMPLE 21.7



(a)

Fig. 21–20

The top shown in Fig. 21–20*a* has a mass of 0.5 kg and is precessing about the vertical axis at a constant angle of $\theta = 60°$. If it spins with an angular velocity $\omega_s = 100$ rad/s, determine its precession ω_p. Assume that the axial and transverse moments of inertia of the top are $0.45(10^{-3})\ \text{kg}\cdot\text{m}^2$ and $1.20(10^{-3})\ \text{kg}\cdot\text{m}^2$, respectively, measured with respect to the fixed point O.



(b)

SOLUTION

Equation 21–30 will be used for the solution since the motion is *steady precession*. As shown on the free-body diagram, Fig. 21–20*b*, the coordinate axes are established in the usual manner, that is, with the positive z axis in the direction of spin, the positive Z axis in the direction of precession, and the positive x axis in the direction of the moment ΣM_x (refer to Fig. 21–16). Thus,

$$\Sigma M_x = -I\dot{\phi}^2 \sin\theta\cos\theta + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi})$$

$$4.905\ \text{N}(0.05\ \text{m})\sin 60° = -[1.20(10^{-3})\ \text{kg}\cdot\text{m}^2\ \dot{\phi}^2]\sin 60°\cos 60° + [0.45(10^{-3})\ \text{kg}\cdot\text{m}^2]\dot{\phi}\sin 60°(\dot{\phi}\cos 60° + 100\ \text{rad/s})$$

or

$$\dot{\phi}^2 - 120.0\dot{\phi} + 654.0 = 0 \qquad (1)$$

Solving this quadratic equation for the precession gives

$$\dot{\phi} = 114\ \text{rad/s} \quad \text{(high precession)}$$ *Ans.*

and

$$\dot{\phi} = 5.72\ \text{rad/s} \quad \text{(low precession)}$$ *Ans.*

NOTE: In reality, low precession of the top would generally be observed, since high precession would require a larger kinetic energy.

EXAMPLE 21.8

The 1-kg disk shown in Fig. 21–21*a* spins about its axis with a constant angular velocity $\omega_D = 70$ rad/s. The block at *B* has a mass of 2 kg, and by adjusting its position *s* one can change the precession of the disk about its supporting pivot at *O* while the shaft remains horizontal. Determine the position *s* that will enable the disk to have a constant precession $\omega_p = 0.5$ rad/s about the pivot. Neglect the weight of the shaft.

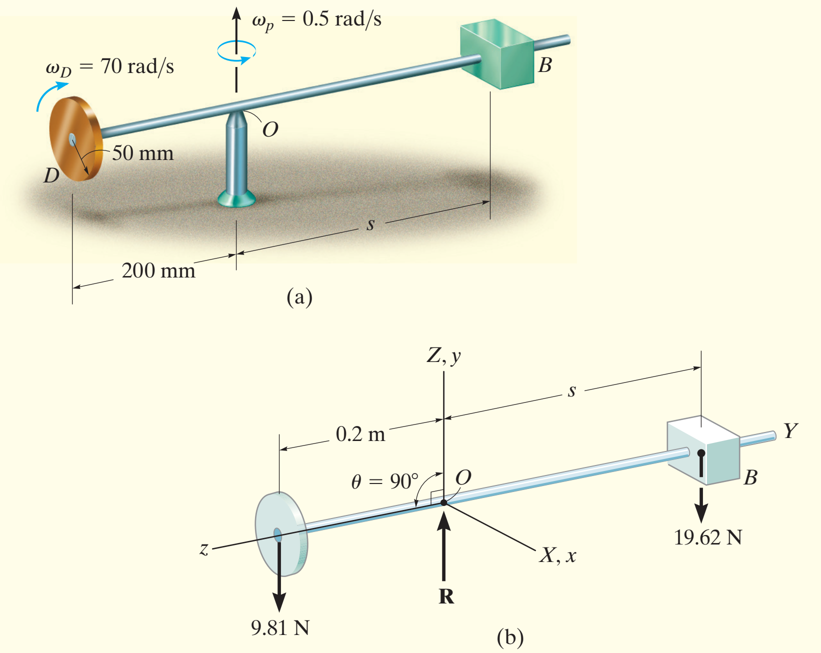


Fig. 21–21

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SOLUTION

The free-body diagram of the assembly is shown in Fig. 21–21*b*. The origin for both the *x*, *y*, *z* and *X*, *Y*, *Z* coordinate systems is located at the fixed point *O*. In the conventional sense, the *Z* axis is chosen along the axis of precession, and the *z* axis is along the axis of spin, so that $\theta = 90°$. Since the precession is *steady*, Eq. 21–32 can be used for the solution.

$$\Sigma M_x = I_z \Omega_y \omega_z$$

Substituting the required data gives

$$(9.81 \text{ N})\,(0.2 \text{ m}) - (19.62 \text{ N})s = \left[\tfrac{1}{2}(1 \text{ kg})(0.05 \text{ m})^2\right]0.5 \text{ rad/s}(-70 \text{ rad/s})$$

$$s = 0.102 \text{ m} = 102 \text{ mm} \qquad \textit{Ans.}$$

21.6 TORQUE-FREE MOTION

When the only external force acting on a body is caused by gravity, the general motion of the body is referred to as ***torque-free motion***. This type of motion is characteristic of planets, artificial satellites, and projectiles—provided air friction is neglected.

In order to describe the characteristics of this motion, the distribution of the body's mass will be assumed *axisymmetric*. The satellite shown in Fig. 21–22 is an example of such a body, where the z axis represents an axis of symmetry. The origin of the x, y, z coordinates is located at the mass center G, such that $I_{zz} = I_z$ and $I_{xx} = I_{yy} = I$. Since gravity is the only external force present, the summation of moments about the mass center is zero. From Eq. 21–21, this requires the angular momentum of the body to be constant, i.e.,

$$\mathbf{H}_G = \text{constant}$$

At the instant considered, it will be assumed that the inertial frame of reference is oriented so that the positive Z axis is directed along $\mathbf{H}_G$ and the y axis lies in the plane formed by the z and Z axes, Fig. 21–22. The Euler angle formed between Z and z is θ, and therefore, with this choice of axes the angular momentum can be expressed as

$$\mathbf{H}_G = H_G \sin\theta\,\mathbf{j} + H_G \cos\theta\,\mathbf{k}$$

Furthermore, using Eqs. 21–11, we have

$$\mathbf{H}_G = I\omega_x\mathbf{i} + I\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

Equating the respective $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ components of the above two equations yields

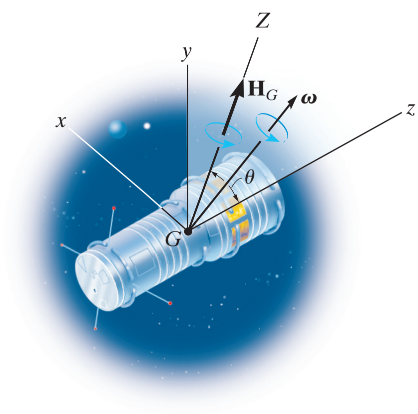


Fig. 21–22

$$\omega_x = 0 \quad \omega_y = \frac{H_G \sin\theta}{I} \quad \omega_z = \frac{H_G \cos\theta}{I_z} \tag{21–34}$$

or

$$\boldsymbol{\omega} = \frac{H_G \sin\theta}{I}\mathbf{j} + \frac{H_G \cos\theta}{I_z}\mathbf{k} \tag{21–35}$$

In a similar manner, equating the respective **i**, **j**, **k** components of Eq. 21–27 to those of Eq. 21–34, we obtain

$$\begin{aligned}\dot{\theta} &= 0\\ \dot{\phi}\sin\theta &= \frac{H_G \sin\theta}{I}\\ \dot{\phi}\cos\theta + \dot{\psi} &= \frac{H_G\cos\theta}{I_z}\end{aligned}$$

Solving, we get

$$\begin{aligned}\theta &= \text{constant}\\ \dot{\phi} &= \frac{H_G}{I}\\ \dot{\psi} &= \frac{I - I_z}{I I_z}H_G\cos\theta\end{aligned} \tag{21–36}$$

Thus, for torque-free motion of an axisymmetrical body, the angle θ formed between the angular-momentum vector and the spin of the body remains constant. Furthermore, the angular momentum $\mathbf{H}_G$, precession $\dot{\phi}$, and spin $\dot{\psi}$ for the body remain constant at all times during the motion.

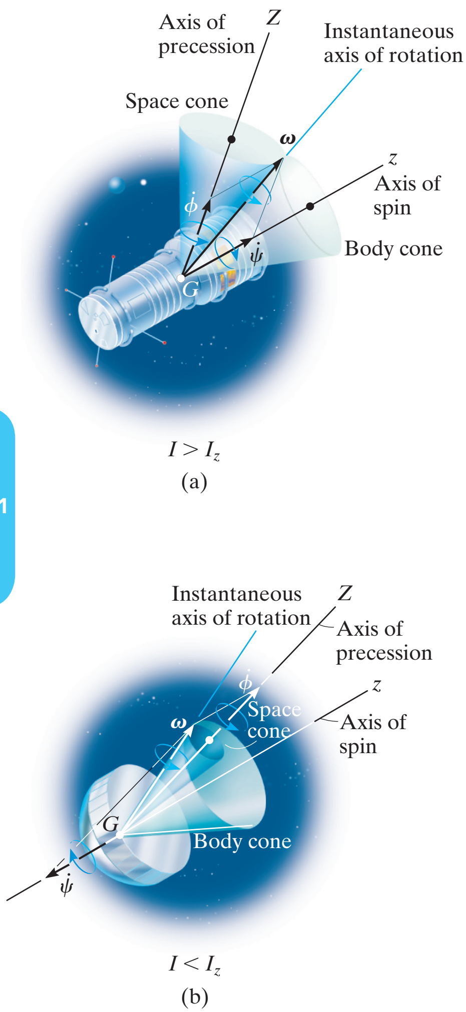


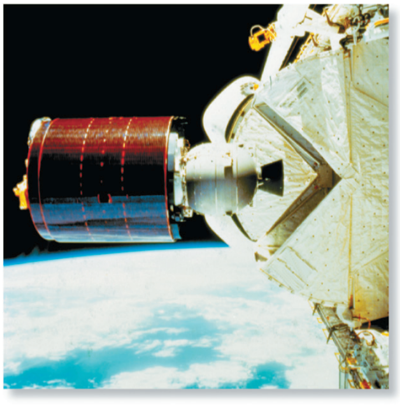
Fig. 21–23

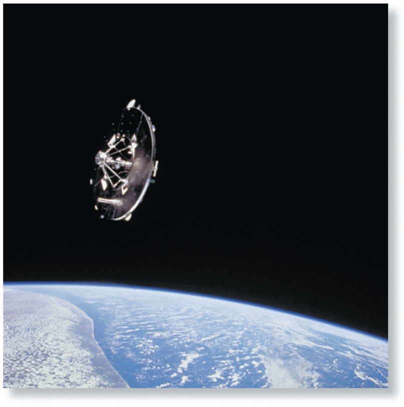
Eliminating H_G from the second and third of Eqs. 21–36 yields the following relation between the spin and precession:

$$\dot{\psi} = \frac{I - I_z}{I_z}\dot{\phi}\cos\theta \tag{21–37}$$

These two components of angular motion can be studied by using the body and space cone models introduced in Sec. 20.1. The *space cone*, which is oriented along the axis of precession, is fixed from rotating, since the precession has a fixed direction, while the surface of the *body cone* rolls on the space cone's surface. Try to imagine this satellite motion in Fig. 21–23*a*. The interior angle of each cone is chosen such that the angular velocity $\boldsymbol{\omega}$ of the satellite is directed along the line of contact of the two cones. This line represents the instantaneous axis of rotation for the body cone, and hence the angular velocity of both the body cone and the body must be directed along this line. Since the spin is a function of the moments of inertia I and I_z of the body, Eq. 21–36, the cone model in Fig. 21–23*a* is satisfactory for describing the motion, provided $I > I_z$. Torque-free motion which meets these requirements is called ***regular precession***. If $I < I_z$, the spin is negative and the precession positive. This motion is represented by the satellite motion shown in Fig. 21–23*b* ($I < I_z$). The cone model can again be used to represent the motion; however, to preserve the correct vector addition of spin and precession to obtain the angular velocity $\boldsymbol{\omega}$, the inside surface of the body cone must roll on the outside surface of the (fixed) space cone. This motion is referred to as ***retrograde precession***.

Satellites are often given a spin before they are launched. If their angular momentum is not collinear with the axis of spin, they will exhibit precession. For the satellite on the left, regular precession will occur since $I > I_z$, and for the one on the right, retrograde precession will occur since $I < I_z$.





EXAMPLE 21.9

The motion of a football is observed using a slow-motion projector. From the film, the spin of the football is seen to be directed 30° from the horizontal, as shown in Fig. 21–24*a*. Also, the football is precessing about the vertical axis at a rate $\dot{\phi} = 3$ rad/s. If the ratio of the axial to transverse moments of inertia of the football is $\frac{1}{3}$, measured with respect to the center of mass, determine the magnitude of the football's spin and its angular velocity. Neglect the effect of air resistance.

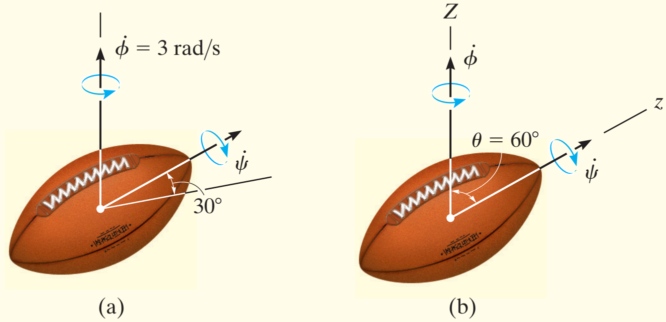


Fig. 21–24

SOLUTION

Since the weight of the football is the only force acting, the motion is torque free. In the conventional sense, if the z axis is established along the axis of spin and the Z axis along the precession axis, as shown in Fig. 21–24*b*, then the angle $\theta = 60°$. Applying Eq. 21–37, the spin is

$$\dot{\psi} = \frac{I - I_z}{I_z}\dot{\phi}\cos\theta = \frac{I - \frac{1}{3}I}{\frac{1}{3}I}(3)\cos 60°$$

$$= 3 \text{ rad/s} \qquad \textit{Ans.}$$

Using Eqs. 21–34, where $H_G = \dot{\phi}I$ (Eq. 21–36), we have

$$\omega_x = 0$$

$$\omega_y = \frac{H_G \sin\theta}{I} = \frac{3I\sin 60°}{I} = 2.60 \text{ rad/s}$$

$$\omega_z = \frac{H_G\cos\theta}{I_z} = \frac{3I\cos 60°}{\frac{1}{3}I} = 4.50 \text{ rad/s}$$

Thus,

$$\omega = \sqrt{(\omega_x)^2 + (\omega_y)^2 + (\omega_z)^2}$$

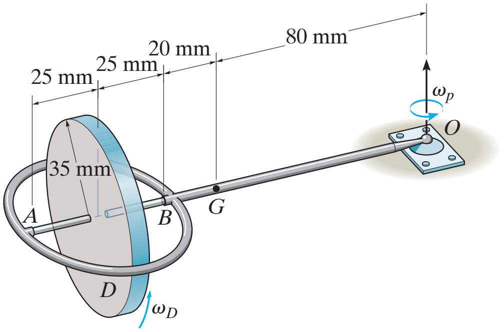
$$= \sqrt{(0)^2 + (2.60)^2 + (4.50)^2}$$

$$= 5.20 \text{ rad/s} \qquad \textit{Ans.}$$

PROBLEMS

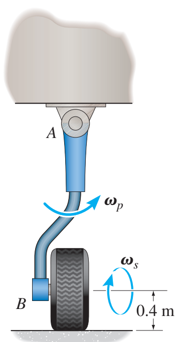
***21–60.** A thin rod is initially coincident with the Z axis when it is given three rotations defined by the Euler angles $\phi = 30°$, $\theta = 45°$, and $\psi = 60°$. If these rotations are given in the order stated, determine the coordinate direction angles α, β, γ of the axis of the rod with respect to the X, Y, and Z axes. Are these directions the same for any order of the rotations? Why?

21–61. The gyroscope consists of a uniform 450-g disk D which is attached to the axle AB of negligible mass. The supporting frame has a mass of 180 g and a center of mass at G. If the disk is rotating about the axle at $\omega_D = 90$ rad/s, determine the constant angular velocity ω_p at which the frame precesses about the pivot point O. The frame moves in the horizontal plane.



Prob. 21–61

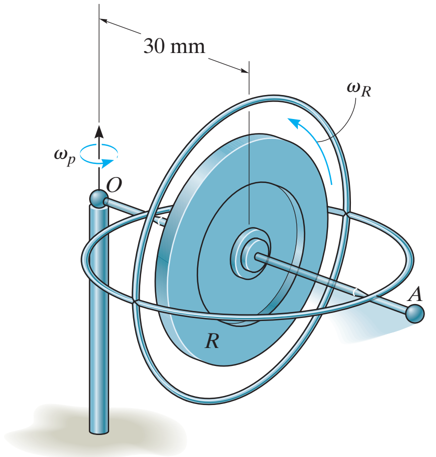
21–62. At the moment of takeoff, the landing gear of an airplane is retracted with a constant angular velocity of $\omega_p = 2$ rad/s, while the wheel continues to spin. If the plane takes off with a speed of $v = 320$ km/h, determine the torque at A due to the gyroscopic effect. The wheel has a mass of 50 kg, and the radius of gyration about its spinning axis is $k = 300$ mm.



Prob. 21–62

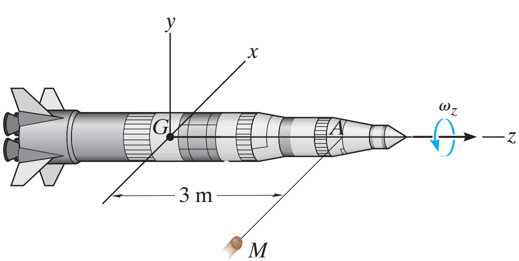
21–63. Show that the angular velocity of a body, in terms of Euler angles ϕ, θ, and ψ, can be expressed as $\boldsymbol{\omega} = (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\mathbf{i} + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\mathbf{j} + (\dot{\phi}\cos\theta + \dot{\psi})\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are directed along the x, y, z axes as shown in Fig. 21–15d.

***21–64.** The toy gyroscope consists of a rotor R which is attached to the frame of negligible mass. If it is observed that the frame is precessing about the pivot point O at $\omega_p = 2$ rad/s, determine the angular velocity ω_R of the rotor. The stem OA moves in the horizontal plane. The rotor has a mass of 200 g and a radius of gyration $k_{OA} = 20$ mm about OA.



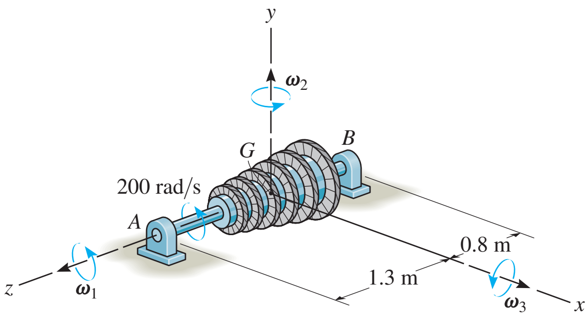
Prob. 21–64

21–65. The rocket has a mass of 4 Mg and radii of gyration $k_z = 0.85$ m and $k_x = k_y = 2.3$ m. It is initially spinning about the z axis at $\omega_z = 0.05$ rad/s when a meteoroid M strikes it at A and creates an impulse $\mathbf{I} = \{300\mathbf{i}\}$ N·s. Determine the axis of precession after the impact.



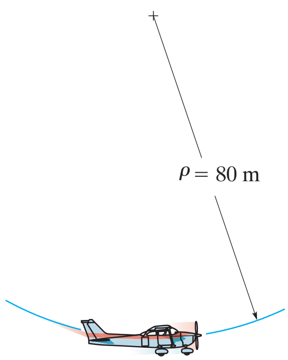
Prob. 21–65

21–66. The turbine on a ship has a mass of 400 kg and is mounted on bearings A and B as shown. Its center of mass is at G, its radius of gyration is $k_z = 0.3$ m, and $k_x = k_y = 0.5$ m. If it is spinning at 200 rad/s, determine the vertical reactions at the bearings when the ship undergoes each of the following motions: (a) rolling, $\omega_1 = 0.2$ rad/s, (b) turning, $\omega_2 = 0.8$ rad/s, (c) pitching, $\omega_3 = 1.4$ rad/s.



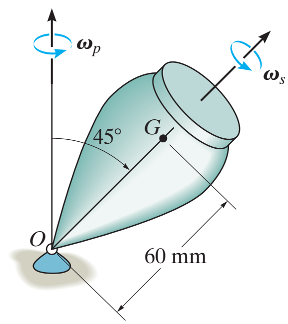
Prob. 21–66

21–67. The propeller on a single-engine airplane has a mass of 15 kg and a centroidal radius of gyration of 0.3 m calculated about the axis of spin. When viewed from the front of the airplane, the propeller is turning clockwise at 350 rad/s about the spin axis. If the airplane enters a vertical curve having a radius of 80 m and is traveling at 200 km/h, determine the gyroscopic bending moment which the propeller exerts on the bearings of the engine when the airplane is in its lowest position.



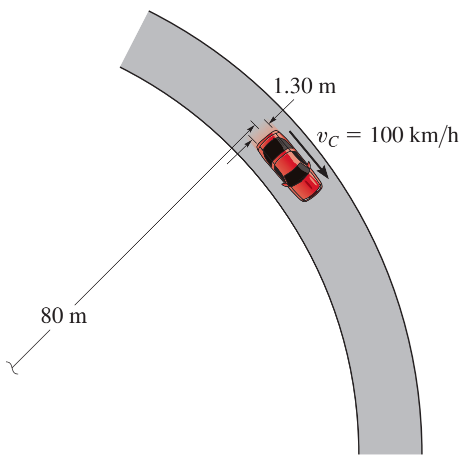
Prob. 21–67

*__21–68.__ The top has a mass of 90 g, a center of mass at G, and a radius of gyration $k = 18$ mm about its axis of symmetry. About any transverse axis acting through point O the radius of gyration is $k_t = 35$ mm. If the top is connected to a ball-and-socket joint at O and the precession is $\omega_p = 0.5$ rad/s, determine the spin $\boldsymbol{\omega}_s$.



Prob. 21–68

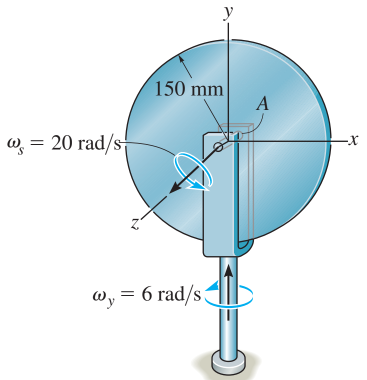
21–69. The car travels at a constant speed of $v_C = 100$ km/h around the horizontal curve having a radius of 80 m. If each wheel has a mass of 16 kg, a radius of gyration $k_G = 300$ mm about its spinning axis, and a radius of 400 mm, determine the difference between the normal forces of the rear wheels, caused by the gyroscopic effect. The distance between the wheels is 1.30 m.



Prob. 21–69

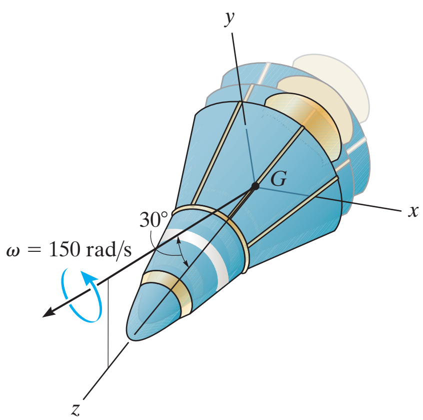
21

21–70. The 20-kg disk is spinning about its center at $\omega_s = 20$ rad/s while the supporting axle is rotating at $\omega_y = 6$ rad/s. Determine the gyroscopic moment caused by the force reactions which the pin A exerts on the disk due to the motion.



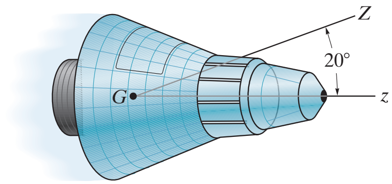
Prob. 21–70

21–71. The space capsule has a mass of 2 Mg, center of mass at G, and radii of gyration about its axis of symmetry (z axis) and its transverse axes (x or y axis) of $k_z = 2.75$ m and $k_x = k_y = 5.5$ m, respectively. If the capsule has the angular velocity shown, determine its precession $\dot{\phi}$ and spin $\dot{\psi}$. Indicate whether the precession is regular or retrograde. Also, draw the space cone and body cone for the motion.



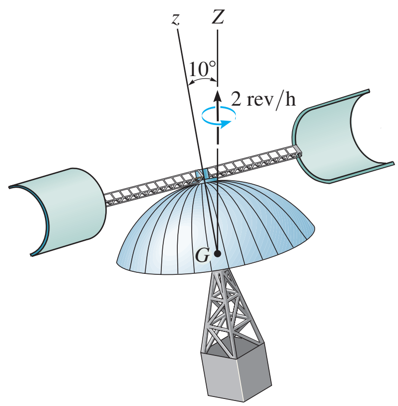
Prob. 21–71

***21–72.** The radius of gyration about an axis passing through the axis of symmetry of the 1.6-Mg space capsule is $k_z = 1.2$ m and about any transverse axis passing through the center of mass G, $k_t = 1.8$ m. If the capsule has a known steady-state precession of two revolutions per hour about the Z axis, determine the rate of spin about the z axis.



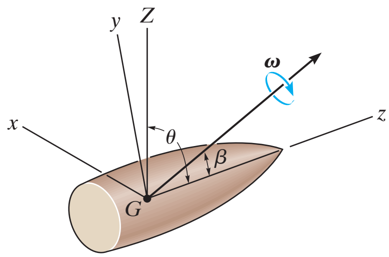
Prob. 21–72

21–73. The radius of gyration about an axis passing through the axis of symmetry of the 2.5-Mg satellite is $k_z = 2.3$ m, and about any transverse axis passing through the center of mass G, $k_t = 3.4$ m. If the satellite has a steady-state precession of two revolutions per hour about the Z axis, determine the rate of spin about the z axis.



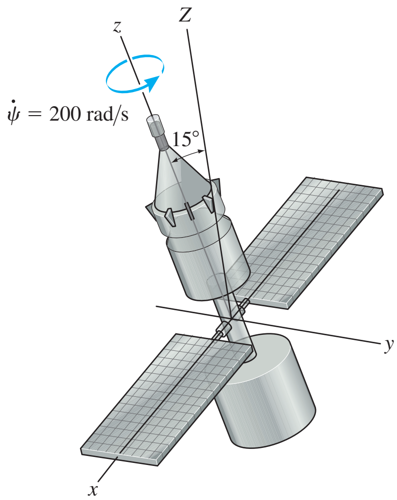
Prob. 21–73

21–74. The projectile shown is subjected to torque-free motion. The transverse and axial moments of inertia are I and I_z, respectively. If θ represents the angle between the precessional axis Z and the axis of symmetry z, and β is the angle between the angular velocity $\boldsymbol{\omega}$ and the z axis, show that β and θ are related by the equation $\tan\theta = (I/I_z)\tan\beta$.



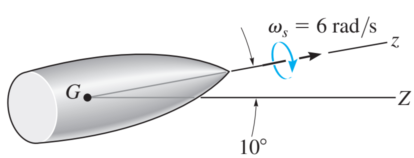
Prob. 21–74

21–75. The satellite has a mass of 100 kg and radii of gyration about its axis of symmetry (z axis) and its transverse axes (x or y axis) of $k_z = 300$ mm and $k_x = k_y = 900$ mm, respectively. If the satellite spins about the z axis at a constant rate of $\dot{\psi} = 200$ rad/s, and precesses about the Z axis, determine the precession $\dot{\phi}$ and the magnitude of its angular momentum $\mathbf{H}_G$.



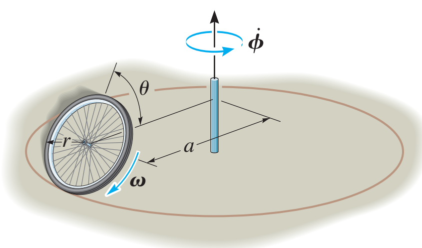
Prob. 21–75

*21–76. The projectile has a mass of 0.9 kg and axial and transverse radii of gyration of $k_z = 20$ mm and $k_t = 25$ mm, respectively. If it is spinning at $\omega_s = 6$ rad/s when it leaves the barrel of a gun, determine its angular momentum. Precession occurs about the Z axis.



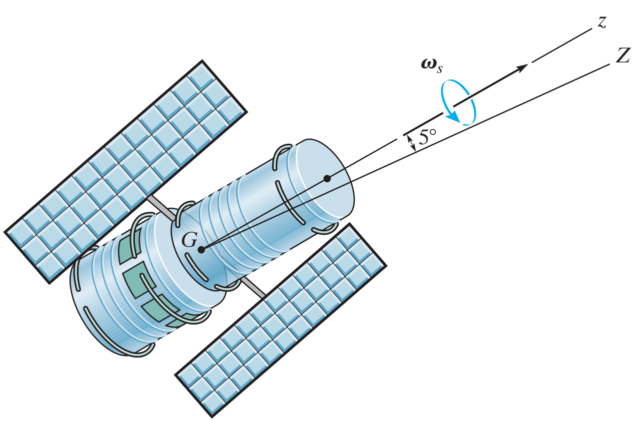
Prob. 21–76

21–77. A wheel of mass m and radius r rolls with constant spin $\boldsymbol{\omega}$ about a circular path having a radius a. If the angle of inclination is θ, determine the rate of precession. Treat the wheel as a thin ring. No slipping occurs.



Prob. 21–77

21–78. The satellite has a mass of 1.8 Mg, and about axes passing through the mass center G the axial and transverse radii of gyration are $k_z = 0.8$ m and $k_t = 1.2$ m, respectively. If it is spinning at $\omega_s = 6$ rad/s when it is launched, determine its angular momentum. Precession occurs about the Z axis.



Prob. 21–78

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CHAPTER REVIEW

Moments and Products of Inertia

A body has six components of inertia for any specified x, y, z axes. Three of these are moments of inertia about each of the axes, I_{xx}, I_{yy}, I_{zz}, and three are products of inertia, each defined from two orthogonal planes, I_{xy}, I_{yz}, I_{xz}. If either one or both of these planes are planes of symmetry, then the product of inertia with respect to these planes will be zero.

$$I_{xx} = \int_m r_x^2\, dm = \int_m (y^2 + z^2)\, dm \qquad I_{xy} = I_{yx} = \int_m xy\, dm$$

$$I_{yy} = \int_m r_y^2\, dm = \int_m (x^2 + z^2)\, dm \qquad I_{yz} = I_{zy} = \int_m yz\, dm$$

$$I_{zz} = \int_m r_z^2\, dm = \int_m (x^2 + y^2)\, dm \qquad I_{xz} = I_{zx} = \int_m xz\, dm$$

The moments and products of inertia can be determined by direct integration or by using tabulated values. If these quantities are to be determined with respect to axes or planes that do not pass through the mass center, then the parallel-axis and parallel-plane theorems must be used to calculate their values.

Provided the six components of inertia are known, then the moment of inertia about any axis can be determined using the inertia transformation equation.

$$I_{Oa} = I_{xx}u_x^2 + I_{yy}u_y^2 + I_{zz}u_z^2 - 2I_{xy}u_xu_y - 2I_{yz}u_yu_z - 2I_{zx}u_zu_x$$

Principal Moments of Inertia

At any point on or off the body, the x, y, z axes can be oriented so that the products of inertia will be zero. The resulting moments of inertia are called the principal moments of inertia. In general, one will be a maximum and the other a minimum.

$$\begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

Principle of Impulse and Momentum

The angular momentum for a body can be determined about any arbitrary point A.

Once the linear and angular momentum for the body have been formulated, then the principle of impulse and momentum can be used to solve problems that involve force, velocity, and time.

$$m(\mathbf{v}_G)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{F}\, dt = m(\mathbf{v}_G)_2$$

$$(\mathbf{H}_O)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{M}_O\, dt = (\mathbf{H}_O)_2$$

where

$$H_x = I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z$$

$$H_y = -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z$$

$$H_z = -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z$$

$$\mathbf{H}_O = \int_m \boldsymbol{\rho}_O \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_O)\, dm$$

Fixed Point O

$$\mathbf{H}_G = \int_m \boldsymbol{\rho}_G \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_G)\, dm$$

Center of Mass

$$\mathbf{H}_A = \boldsymbol{\rho}_{G/A} \times m\mathbf{v}_G + \mathbf{H}_G$$

Arbitrary Point

Principle of Work and Energy

The kinetic energy for a body is usually determined relative to a fixed point or the body's mass center.

$$T = \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 + \tfrac{1}{2}I_z\omega_z^2$$

Fixed Point

$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 + \tfrac{1}{2}I_z\omega_z^2$$

Center of Mass

These formulations can be used with the principle of work and energy to solve problems that involve force, velocity, and displacement.

$$T_1 + \Sigma U_{1-2} = T_2$$

Equations of Motion

There are three scalar equations of translational motion for a rigid body that moves in three dimensions.

$$\Sigma F_x = m(a_G)_x$$
$$\Sigma F_y = m(a_G)_y$$
$$\Sigma F_z = m(a_G)_z$$

The three scalar equations of rotational motion depend upon the motion of the x, y, z reference. Most often, these axes are oriented so that they are principal axes of inertia. If the axes are fixed in and move with the body so that $\mathbf{\Omega} = \boldsymbol{\omega}$, then the equations are referred to as the Euler equations of motion.

$$\Sigma M_x = I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z$$
$$\Sigma M_y = I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x$$
$$\Sigma M_z = I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y$$
$$\mathbf{\Omega} = \boldsymbol{\omega}$$

$$\Sigma M_x = I_x\dot{\omega}_x - I_y\Omega_z\omega_y + I_z\Omega_y\omega_z$$
$$\Sigma M_y = I_y\dot{\omega}_y - I_z\Omega_x\omega_z + I_x\Omega_z\omega_x$$
$$\Sigma M_z = I_z\dot{\omega}_z - I_x\Omega_y\omega_x + I_y\Omega_x\omega_y$$
$$\mathbf{\Omega} \neq \boldsymbol{\omega}$$

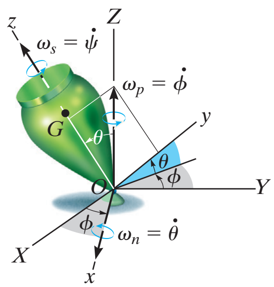
A free-body diagram should always accompany the application of the equations of motion.

Gyroscopic Motion

The angular motion of a gyroscope is best described using the three Euler angles ϕ, θ, and ψ. The angular velocity components are called the precession $\dot{\phi}$, the nutation $\dot{\theta}$, and the spin $\dot{\psi}$.

If $\dot{\theta} = 0$ and $\dot{\phi}$ and $\dot{\psi}$ are constant, then the motion is referred to as steady precession.

It is the spin of a gyro rotor that is responsible for holding a rotor from falling downward, and instead causing it to precess about a vertical axis. This phenomenon is called the gyroscopic effect.



$$\Sigma M_x = -I\dot{\phi}^2 \sin\theta\cos\theta + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi})$$

$$\Sigma M_y = 0, \Sigma M_z = 0$$

Torque-Free Motion

A body that is only subjected to a gravitational force will have no moments on it about its mass center, and so the motion is referred to as torque-free motion. The angular momentum for the body about its mass center will remain constant and so the body can have both a spin and a precession. For a symmetric body the motion depends upon the magnitude of the moment of inertia about the spin axis, I_z, versus that about a perpendicular axis, I.

$$\theta = \text{constant}$$

$$\dot{\phi} = \frac{H_G}{I}$$

$$\dot{\psi} = \frac{I - I_z}{I I_z}H_G\cos\theta$$

CHAPTER 22

The springs supporting this rail car truck and wheels absorb the impact loads of the track and the induced vibration of the car. Both effects should be considered in design.

VIBRATIONS

CHAPTER OBJECTIVES

- To discuss undamped one-degree-of-freedom vibration of a rigid body using the equation of motion and energy methods.
- To study the analysis of undamped forced vibration and viscous damped forced vibration.

22.1 UNDAMPED FREE VIBRATION

A ***vibration*** is the oscillating motion of a body or system of connected bodies displaced from a position of equilibrium. In general, there are two types of vibration, free and forced. ***Free vibration*** occurs when the motion is maintained by gravitational or elastic restoring forces, such as the swinging motion of a pendulum or the vibration of an elastic rod. ***Forced vibration*** is caused by an external periodic or intermittent force applied to the system. Both of these types of vibration can either be damped or undamped. ***Undamped*** vibrations exclude frictional effects in the analysis. Since in reality both internal and external frictional forces are present, the motion of all vibrating bodies is actually ***damped***.

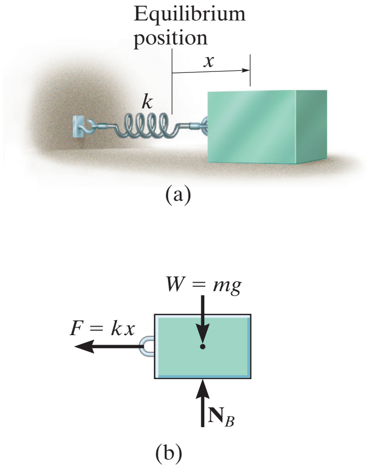


Fig. 22–1

The simplest type of vibrating motion is undamped free vibration, which can be represented by the block and spring model shown in Fig. 22–1*a*. Provided the supporting surface is smooth, vibrating motion occurs when the block is released from a displaced position x. The spring then pulls on the block, causing it to attain a velocity, and then move out of equilibrium when $x = 0$. Compression of the spring will arrest the motion, at position $-x$, and the back and forth oscillation is repeated.

The time-dependent path of motion of the block can be determined by applying the equation of motion to the block when it is in the displaced position x. The free-body diagram is shown in Fig. 22–1*b*. The elastic restoring force $F = kx$ is always directed toward the equilibrium position, whereas the acceleration **a** is assumed to act in the direction of *positive displacement*. Since $a = d^2x/dt^2 = \ddot{x}$, we have

$$\overset{+}{\rightarrow} \Sigma F_x = ma_x; \qquad -kx = m\ddot{x}$$

Because the acceleration is proportional to the block's displacement, the motion is called ***simple harmonic motion***. Rearranging the terms into a "standard form" gives

$$\ddot{x} + \omega_n^2 x = 0 \qquad (22\text{–}1)$$

The constant ω_n, generally reported in rad/s, is called the ***natural frequency***, and in this case

$$\omega_n = \sqrt{\frac{k}{m}} \qquad (22\text{–}2)$$

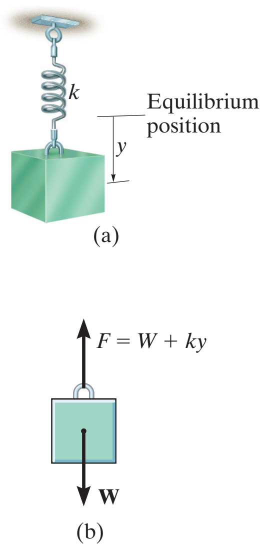


Fig. 22–2

Equation 22–1 can also be obtained by considering the block to be suspended so that the displacement y is measured from the block's *equilibrium position*, Fig. 22–2*a*. When the block is in equilibrium, the spring exerts an upward force of $F = W = mg$ on the block. Hence, when the block is displaced a distance y downward from this position, the magnitude of the spring force is $F = W + ky$, Fig. 22–2*b*. Applying the equation of motion gives

$$+\downarrow \Sigma F_y = ma_y; \qquad -W - ky + W = m\ddot{y}$$

or

$$\ddot{y} + \omega_n^2 y = 0$$

which is the same form as Eq. 22–1 and ω_n is defined by Eq. 22–2.

Equation 22–1 is a homogeneous, second-order, linear differential equation with constant coefficients. It can be shown that the general solution is

$$x = A \sin \omega_n t + B \cos \omega_n t \tag{22–3}$$

where A and B represent two constants of integration. The block's velocity and acceleration are determined by taking successive time derivatives, which yields

$$v = \dot{x} = A\omega_n \cos \omega_n t - B\omega_n \sin \omega_n t \tag{22–4}$$

$$a = \ddot{x} = -A\omega_n^2 \sin \omega_n t - B\omega_n^2 \cos \omega_n t \tag{22–5}$$

When Eqs. 22–3 and 22–5 are substituted into Eq. 22–1, the differential equation will be satisfied, showing that Eq. 22–3 is indeed the solution to Eq. 22–1.

The integration constants in Eq. 22–3 are generally determined from the initial conditions of the problem. For example, suppose that the block in Fig. 22–1*a* has been displaced a distance x_1 to the right from its equilibrium position and given an initial (positive) velocity $\mathbf{v}_1$ directed to the right. Substituting $x = x_1$ when $t = 0$ into Eq. 22–3 yields $B = x_1$. And since $v = v_1$ when $t = 0$, using Eq. 22–4 we obtain $A = v_1/\omega_n$. If these values are substituted into Eq. 22–3, the equation describing the motion becomes

$$x = \frac{v_1}{\omega_n} \sin \omega_n t + x_1 \cos \omega_n t \tag{22–6}$$

Equation 22–3 may also be expressed in terms of simple sinusoidal motion. To show this, let

$$A = C \cos \phi \tag{22–7}$$

and

$$B = C \sin \phi \tag{22–8}$$

where C and ϕ are new constants to be determined in place of A and B. Substituting into Eq. 22–3 yields

$$x = C \cos \phi \sin \omega_n t + C \sin \phi \cos \omega_n t$$

And since $\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi$, then

$$x = C \sin(\omega_n t + \phi) \tag{22–9}$$

If this equation is plotted on an x versus $\omega_n t$ axis, the graph shown in Fig. 22–3 is obtained. The maximum displacement of the block from its

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equilibrium position is the ***amplitude*** of vibration. From either the figure or Eq. 22–9 the amplitude is C. The angle ϕ is called the ***phase angle*** since it represents the amount by which the curve is displaced from the origin when $t = 0$. We can relate these two constants to A and B using Eqs. 22–7 and 22–8. Squaring and adding these equations, the amplitude becomes

$$C = \sqrt{A^2 + B^2} \tag{22–10}$$

If Eq. 22–8 is divided by Eq. 22–7, the phase angle is then

$$\phi = \tan^{-1}\frac{B}{A} \tag{22–11}$$

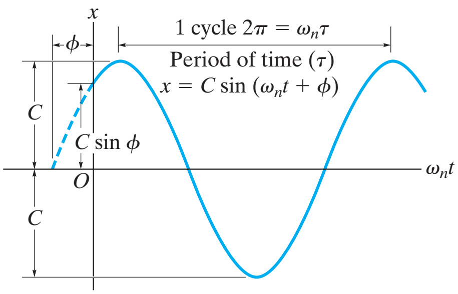

Fig. 22–3

Note that the sine curve, Eq. 22–9, completes one cycle in time $t = \tau$ (tau) when $\omega_n\tau = 2\pi$, or

$$\tau = \frac{2\pi}{\omega_n} \tag{22–12}$$

This time interval is called the ***period***, Fig. 22–3. Using Eq. 22–2, the period can also be represented as

$$\tau = 2\pi\sqrt{\frac{m}{k}} \tag{22–13}$$

Finally, the ***frequency*** f is defined as the number of cycles completed per unit of time, which is the reciprocal of the period; that is,

$$f = \frac{1}{\tau} = \frac{\omega_n}{2\pi} \tag{22–14}$$

or

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{22–15}$$

The unit for the frequency is the ***hertz*** (Hz), where 1 Hz = 1 cycle/s = 2π rad/s.

When a body or system of connected bodies is given an initial displacement from its equilibrium position and released, it will vibrate with the ***natural frequency***, ω_n. Provided the system has a single degree of freedom, that is, it requires only one coordinate to specify completely the position of the system at any time, then the vibrating motion will have the same characteristics as the simple harmonic motion of the block and spring model. Consequently, the motion is described by a differential equation of the same "standard form" as Eq. 22–1, i.e.,

$$\ddot{x} + \omega_n^2 x = 0 \tag{22–16}$$

Hence, if the natural frequency ω_n is known, the period of vibration τ, frequency f, and other vibrating characteristics can be established using Eqs. 22–3 through 22–15.

22

IMPORTANT POINTS

- Free vibration occurs when the motion is maintained by gravitational or elastic restoring forces.
- The amplitude is the maximum displacement of the body.
- The period is the time required to complete one cycle.
- The frequency is the number of cycles completed per unit of time, where 1 Hz = 1 cycle/s.
- Only one position coordinate is needed to describe the location of a one-degree-of-freedom system.

PROCEDURE FOR ANALYSIS

As in the case of the block and spring, the natural frequency ω_n of a body or system of connected bodies having a single degree of freedom can be determined using the following procedure.

Free-Body Diagram.

- Draw the free-body diagram of the body when the body is displaced a *small amount* from its equilibrium position.
- Locate the body with respect to its equilibrium position by using an appropriate position coordinate q. The acceleration of the body's mass center $\mathbf{a}_G$ or the body's angular acceleration $\boldsymbol{\alpha}$ should have an assumed sense of direction which is in the *positive direction* of the position coordinate.
- If the rotational equation of motion $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$ is to be used, then consider drawing the kinetic diagram since it graphically accounts for the components $m(\mathbf{a}_G)_x$, $m(\mathbf{a}_G)_y$, and $I_G\boldsymbol{\alpha}$, and thereby makes it convenient for visualizing the terms needed in the moment sum $\Sigma(\mathcal{M}_k)_P$.

Equation of Motion.

- Apply the equation of motion to relate the elastic or gravitational *restoring* forces and couple moments acting on the body to the body's accelerated motion.

Kinematics.

- Using kinematics, express the body's accelerated motion in terms of the second time derivative of the position coordinate, $\ddot{q}$.
- Substitute the result into the equation of motion and determine ω_n by rearranging the terms so that the resulting equation is in the "standard form," $\ddot{q} + \omega_n^2 q = 0$.

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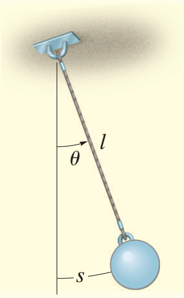
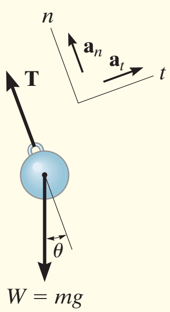
EXAMPLE 22.1



(a)



(b)

Fig. 22–4

Determine the period of oscillation for the simple pendulum shown in Fig. 22–4*a*. The bob has a mass m and is attached to a cord of length l. Neglect the size of the bob.

SOLUTION

Free-Body Diagram. Motion of the system will be described in terms of the position coordinate $(q =)\ \theta$, Fig. 22–4*b*. When the bob is displaced by a small angle θ, the *restoring force* acting on the bob is caused by the tangential component of its weight, $mg \sin\theta$. We assume $\mathbf{a}_t$ acts in the direction of *increasing* s (or θ).

Equation of Motion. Applying the equation of motion in the *tangential direction*, since it involves the restoring force, yields

$$+\nearrow \Sigma F_t = ma_t; \qquad -mg\sin\theta = ma_t \qquad (1)$$

Kinematics. $a_t = d^2s/dt^2 = \ddot{s}$. Also, s can be related to θ by the equation $s = l\theta$, so that $a_t = l\ddot{\theta}$. Hence, Eq. 1 becomes,

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \qquad (2)$$

The solution of this equation involves an elliptic integral; however, for *small displacements*, $\sin\theta \approx \theta$, and so

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \qquad (3)$$

Comparing this equation with Eq. 22–16 ($\ddot{x} + \omega_n^2 x = 0$), it is seen that $\omega_n = \sqrt{g/l}$. From Eq. 22–12, the period of time required for the bob to make one complete swing is therefore

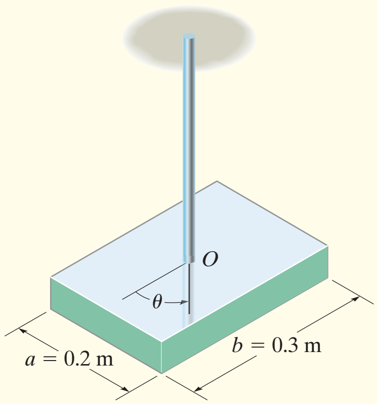
$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{l}{g}} \qquad \textit{Ans.}$$

This interesting result, originally discovered by Galileo Galilei through experiment, indicates that the period depends only on the length of the cord and not on the mass of the pendulum bob or the angle θ.

NOTE: The solution of Eq. 3 is given by Eq. 22–3, where $\omega_n = \sqrt{g/l}$ and θ is substituted for x. Like the block and spring, the constants A and B in this problem can be determined if, for example, one knows the displacement and velocity of the bob at a given instant.

EXAMPLE 22.2

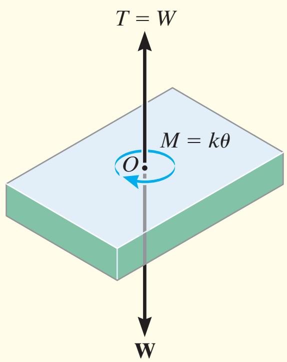
The 10-kg rectangular plate shown in Fig. 22–5*a* is suspended at its center from a rod having a torsional stiffness $k = 1.5\ \text{N}\cdot\text{m/rad}$. Determine the natural period of vibration of the plate when it is given a small angular displacement θ in the plane of the plate.



(a)

SOLUTION

Free-Body Diagram. Fig. 22–5*b*. Since the plate is displaced in its own plane, the torsional *restoring* moment created by the rod is $M = k\theta$. This moment acts in the direction opposite to the angular displacement θ. The angular acceleration $\ddot{\theta}$ acts in the direction of *positive* θ.



(b)

Fig. 22–5

Equation of Motion.

$\Sigma M_O = I_O\alpha;$ $$-k\theta = I_O\ddot{\theta}$$

or

$$\ddot{\theta} + \frac{k}{I_O}\theta = 0$$

Since this equation is in the "standard form," the natural frequency is $\omega_n = \sqrt{k/I_O}$.

From the table on the inside back cover, the moment of inertia of the plate about an axis coincident with the rod is $I_O = \frac{1}{12}m(a^2 + b^2)$. Hence,

$$I_O = \frac{1}{12}(10\text{ kg})\left[(0.2\text{ m})^2 + (0.3\text{ m})^2\right] = 0.1083\text{ kg}\cdot\text{m}^2$$

And so the natural period of vibration is

$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{I_O}{k}} = 2\pi\sqrt{\frac{0.1083}{1.5}} = 1.69\text{ s} \qquad \textit{Ans.}$$

EXAMPLE 22.3

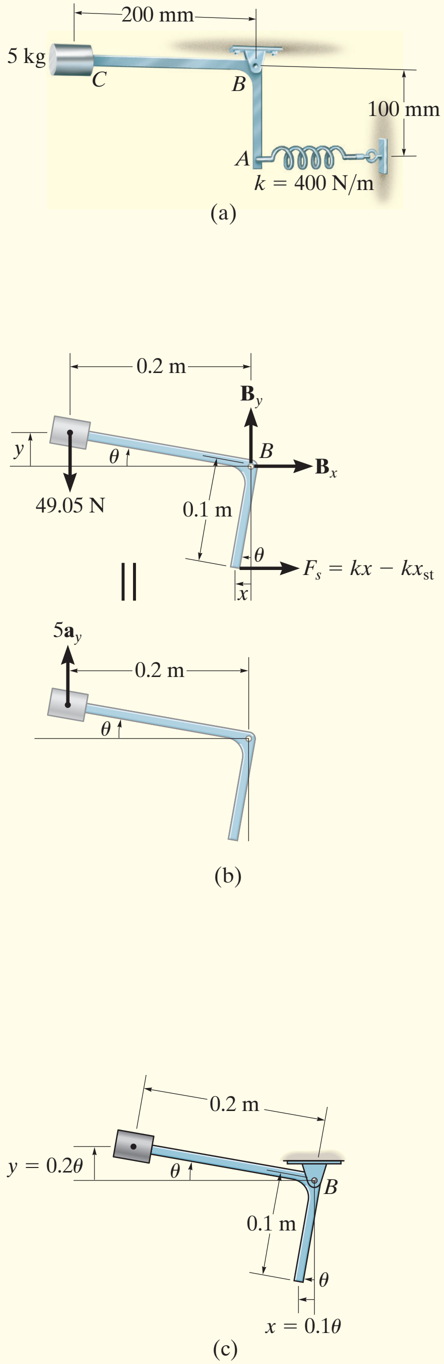


Fig. 22–6

The bent rod shown in Fig. 22–6*a* has a negligible mass and supports a 5-kg collar at its end. If the rod is in the equilibrium position shown, determine the natural period of vibration for the system.

SOLUTION

Free-Body and Kinetic Diagrams. Fig. 22–6*b*. Here the rod is displaced by a small angle θ from the equilibrium position. Since the spring is subjected to an initial compression of x_{st} for equilibrium, then when the displacement $x > x_{st}$ the spring exerts a force of $F_s = kx - kx_{st}$ on the rod. To obtain the "standard form," Eq. 22–16, $5\mathbf{a}_y$ must act *upward*, which is in accordance with positive θ displacement.

Equation of Motion. Moments will be summed about point B to eliminate the reactions $\mathbf{B}_x$ and $\mathbf{B}_y$. Since θ is small,

$$\circlearrowleft +\Sigma M_B = \Sigma(\mathcal{M}_k)_B;$$

$$kx(0.1\text{ m}) - kx_{st}(0.1\text{ m}) + 49.05\text{ N}(0.2\text{ m}) = -(5\text{ kg})a_y(0.2\text{ m})$$

The second term on the left side, $-kx_{st}(0.1\text{ m})$, represents the moment created by the spring force which is necessary to hold the collar in *equilibrium*, i.e., at $x = 0$. Since this moment is equal and opposite to the moment 49.05 N(0.2 m) created by the weight of the collar, these two terms cancel, and so

$$kx(0.1) = -5a_y(0.2) \qquad (1)$$

Kinematics. The deformation of the spring and the position of the collar can be related to the angle θ, Fig. 22–6*c*. Since θ is small, $x = (0.1\text{ m})\theta$ and $y = (0.2\text{ m})\theta$. Therefore, $a_y = \ddot{y} = 0.2\ddot{\theta}$. Substituting into Eq. 1 yields

$$400(0.1\theta)\,0.1 = -5(0.2\ddot{\theta})0.2$$

Rewriting this equation in the "standard form" gives

$$\ddot{\theta} + 20\theta = 0$$

Compared with $\ddot{x} + \omega_n^2 x = 0$ (Eq. 22–16), we have

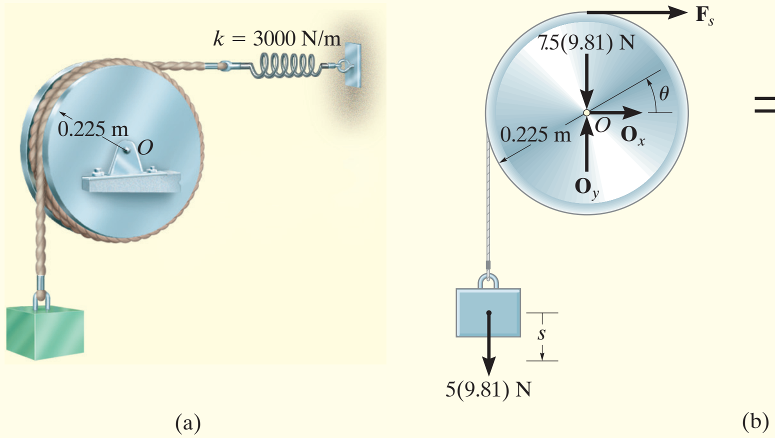
$$\omega_n^2 = 20 \quad \omega_n = 4.47\text{ rad/s}$$

The natural period of vibration is therefore

$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{4.47} = 1.40\text{ s} \qquad \textit{Ans.}$$

EXAMPLE 22.4

A 5-kg block is suspended from a cord that passes over a 7.5-kg disk, as shown in Fig. 22–7*a*. The spring has a stiffness $k = 3000\ \text{N/m}$. Determine the natural period of vibration for the system.



SOLUTION

Free-Body and Kinetic Diagrams. Fig. 22–7*b*. The *system* consists of the disk, which undergoes a rotation defined by the angle θ, and the block, which translates by an amount s. The vector $I_O\,\ddot{\boldsymbol{\theta}}$ acts in the direction of *positive* θ, and consequently $5\mathbf{a}_b$ acts downward in the direction of *positive* s.

Equation of Motion. Summing moments about point O to eliminate the reactions $\mathbf{O}_x$ and $\mathbf{O}_y$, realizing that $I_O = \frac{1}{2}mr^2$, yields

$$\circlearrowright + \Sigma M_O = \Sigma(\mathcal{M}_k)_O;$$

$$5(9.81)\ \text{N}(0.225\ \text{m}) - F_s(0.225\ \text{m})$$

$$= \frac{1}{2}(7.5\ \text{kg})(0.225\ \text{m})^2\,\ddot{\theta} + (5\ \text{kg})a_b(0.225\ \text{m}) \qquad (1)$$

Kinematics. As shown on the kinematic diagram in Fig. 22–7*c*, a small positive displacement θ of the disk causes the block to lower by an amount $s = 0.225\theta$; hence, $a_b = \ddot{s} = 0.225\ddot{\theta}$. When $\theta = 0^\circ$, the spring force required for *equilibrium* of the disk is 5(9.81) N, acting to the right. For position θ, the spring force is $F_s = (3000\ \text{N/m})(0.225\theta\ \text{m}) + 5(9.81)\ \text{N}$. Substituting these results into Eq. 1 and simplifying yields

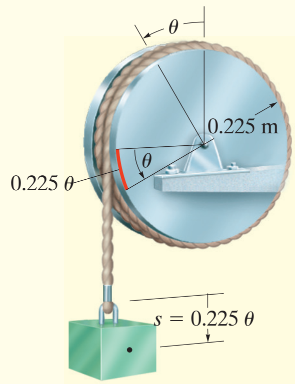
$$\ddot{\theta} + 343\theta = 0$$

Hence,

$$\omega_n^2 = 343 \qquad \omega_n = 18.52\ \text{rad/s}$$

Therefore, the natural period of vibration is

$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{18.52} = 0.339\ \text{s} \qquad \textit{Ans.}$$



(c)

Fig. 22–7

PROBLEMS

22–1. A pendulum has a 0.4-m-long cord and is given a tangential velocity of 0.2 m/s toward the vertical from a position $\theta = 0.3$ rad. Determine the equation which describes the angular motion.

22–2. A spring is stretched 175 mm by an 8-kg block. If the block is displaced 100 mm downward from its equilibrium position and given a downward velocity of 1.50 m/s, determine the differential equation which describes the motion. Assume that positive displacement is downward. Also, determine the position of the block when $t = 0.22$ s.

22–3. A spring has a stiffness of 800 N/m. If a 2-kg block is attached to the spring, pushed 50 mm above its equilibrium position, and released from rest, determine the equation that describes the block's motion. Assume that positive displacement is downward.

***22–4.** A spring is stretched 200 mm by a 15-kg block. If the block is displaced 100 mm downward from its equilibrium position and given a downward velocity of 0.75 m/s, determine the equation which describes the motion. What is the phase angle? Assume that positive displacement is downward.

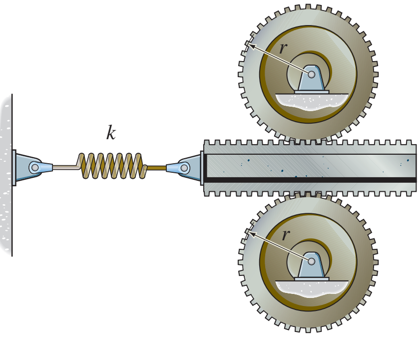
22–5. When a 2-kg block is suspended from a spring, the spring is stretched a distance of 40 mm. Determine the frequency and the period of vibration for a 0.5-kg block attached to the same spring.

22–6. When a 3-kg block is suspended from a spring, the spring is stretched a distance of 60 mm. Determine the natural frequency and the period of vibration if a 0.2-kg block is attached to the same spring rather than a 3-kg block.

22–7. An 8-kg block is suspended from a spring having a stiffness $k = 80$ N/m. If the block is given an upward velocity of 0.4 m/s when it is 90 mm above its equilibrium position, determine the equation which describes the motion and the maximum upward displacement of the block measured from the equilibrium position. Assume that positive displacement is downward.

***22–8.** A 2-kg block is suspended from a spring having a stiffness of 800 N/m. If the block is given an upward velocity of 2 m/s when it is displaced downward a distance of 150 mm from its equilibrium position, determine the equation which describes the motion. What is the amplitude of the motion? Assume that positive displacement is downward.

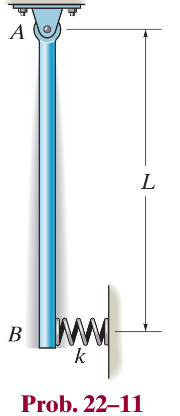
22–9. The two identical gears each have a mass of m and a radius of gyration about their center of mass of k_0. They are in mesh with the gear rack, which has a mass of M and is attached to a spring having a stiffness k. If the gear rack is displaced slightly horizontally, determine the natural period of oscillation.



Prob. 22–9

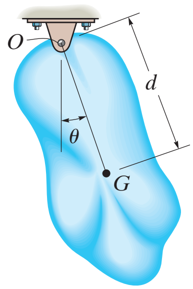
22–10. A 3-kg block is suspended from a spring having a stiffness of $k = 200$ N/m. If the block is pushed 50 mm upward from its equilibrium position and then released from rest, determine the equation that describes the motion. What are the amplitude and the natural frequency of the vibration? Assume that positive displacement is downward.

22–11. The uniform rod of mass m is supported by a pin at A and a spring at B. If B is given a small sideward displacement and released, determine the natural period of vibration.



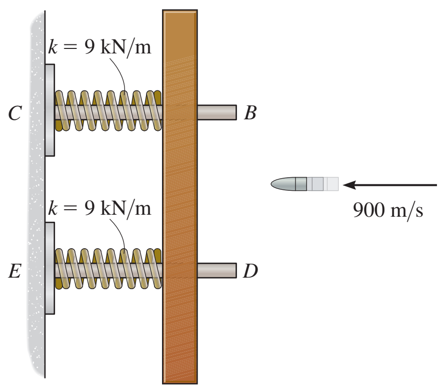
Prob. 22–11

***22–12.** The body of arbitrary shape has a mass m, mass center at G, and a radius of gyration about G of k_G. If it is displaced a slight amount θ from its equilibrium position and released, determine the natural period of vibration.



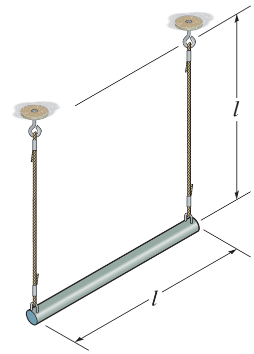
Prob. 22–12

22–13. The 3-kg target slides freely along the smooth horizontal guides BC and DE, which are "nested" in springs that each have a stiffness of $k = 9$ kN/m. If a 60-g bullet is fired with a velocity of 900 m/s and embeds into the target, determine the amplitude and frequency of oscillation of the target.



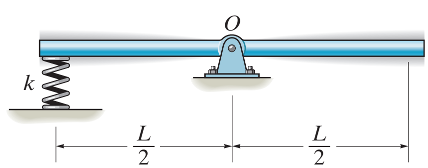
Prob. 22–13

22–14. The rod of mass m is supported by two cords, each having a length l. If the rod is given a slight rotation about a vertical axis through its center and released, determine the period of oscillation.



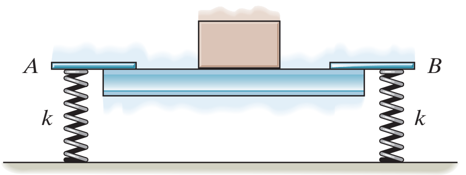
Prob. 22–14

22–15. Determine the natural period of vibration of the uniform bar of mass m when it is displaced downward slightly and released.



Prob. 22–15

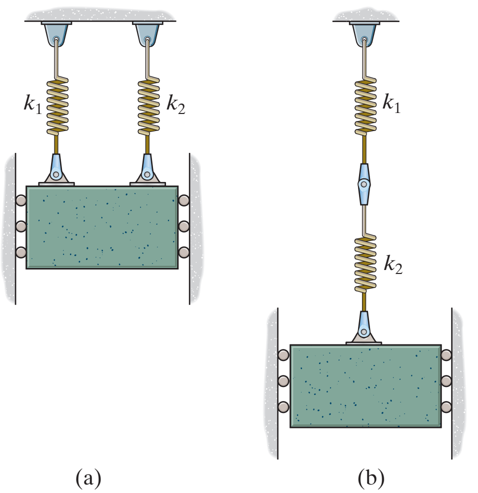
***22–16.** The uniform beam is supported at its ends by two springs A and B, each having the same stiffness k. When nothing is supported on the beam, it has a period of vertical vibration of 0.83 s. If a 50-kg mass is placed at its center, the period of vertical vibration is 1.52 s. Compute the stiffness of each spring and the mass of the beam.



Prob. 22–16

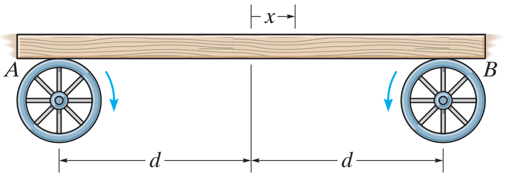
22–17. A block of mass m is suspended from two springs having a stiffness of k_1 and k_2, arranged a) parallel to each other, and b) as a series. Determine the equivalent stiffness of a single spring with the same oscillation characteristics and the period of oscillation for each case.

22–18. The 15-kg block is suspended from two springs having a different stiffness and arranged a) parallel to each other, and b) as a series. If the natural periods of oscillation of the parallel system and series system are observed to be 0.5 s and 1.5 s, respectively, determine the spring stiffnesses k_1 and k_2.



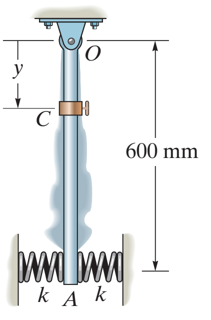
Probs. 22–17/18

22–19. A uniform board is supported on two wheels which rotate in opposite directions at a constant angular speed. If the coefficient of kinetic friction between the wheels and board is μ, determine the frequency of vibration of the board if it is displaced slightly, a distance x from the midpoint between the wheels, and released.



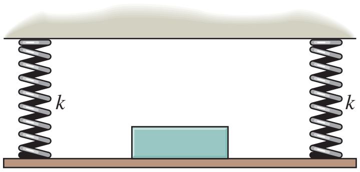
Prob. 22–19

***22–20.** The slender rod has a mass of 0.2 kg and is supported at O by a pin and at its end A by two springs, each having a stiffness $k = 4$ N/m. The period of vibration of the rod can be set by fixing the 0.5-kg collar C to the rod at an appropriate location along its length. If the springs are originally unstretched when the rod is vertical, determine the position y of the collar so that the natural period of vibration becomes $\tau = 1$ s. Neglect the size of the collar.



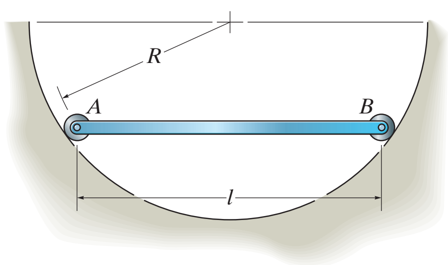
Prob. 22–20

22–21. A platform, having an unknown mass, is supported by *four* springs, each having the same stiffness k. When nothing is on the platform, the period of vertical vibration is measured as 2.35 s; whereas if a 3-kg block is supported on the platform, the period of vertical vibration is 5.23 s. Determine the mass of a block placed on the (empty) platform which causes the platform to vibrate vertically with a period of 5.62 s. What is the stiffness k of each of the springs?



Prob. 22–21

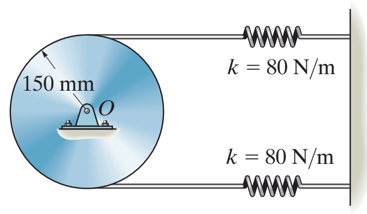
22–22. The bar has a length l and mass m. It is supported at its ends by rollers of negligible mass. If it is given a small displacement and released, determine the natural frequency of vibration.



Prob. 22–22

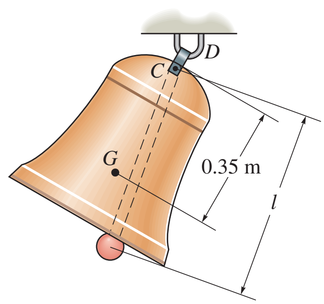
22–23. The 10-kg disk is pin connected at its mass center. Determine the natural period of vibration of the disk if the springs have sufficient tension in them to prevent the cord from slipping on the disk as it oscillates. *Hint*: Assume that the initial stretch in each spring is δ_0.

***22–24.** If the springs in Prob. 22–23 are originally unstretched, determine the frequency of vibration.



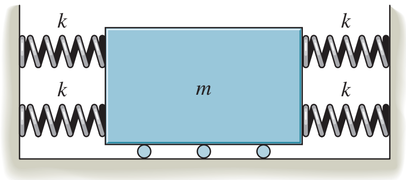
Probs. 22–23/24

22–25. The bell has a mass of 375 kg, a center of mass at G, and a radius of gyration about point D of $k_D = 0.4$ m. The tongue consists of a slender rod attached to the inside of the bell at C. If an 8-kg mass is attached to the end of the rod, determine the length l of the rod so that the bell will "ring silent," i.e., so that the natural period of vibration of the tongue is the same as that of the bell. For the calculation, neglect the small distance between C and D and neglect the mass of the rod.



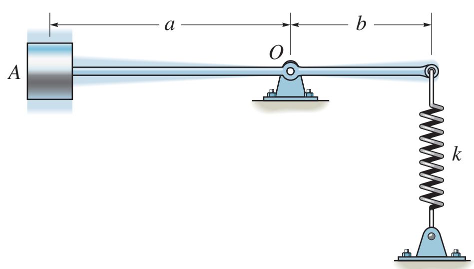
Prob. 22–25

22–26. Determine the frequency of vibration for the block. The springs are originally compressed Δ.



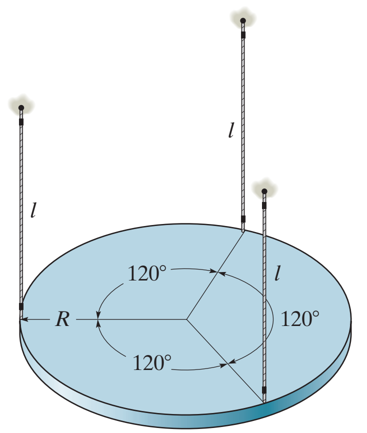
Prob. 22–26

22–27. The block has a mass m and is supported by a rigid bar of negligible mass. If the spring has a stiffness k, determine the natural period of vibration for the block.



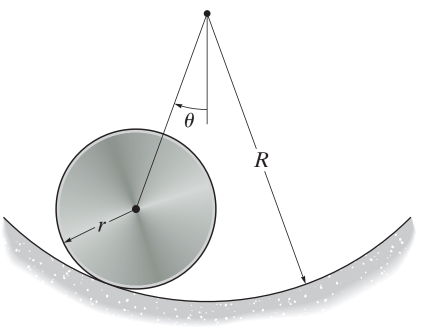
Prob. 22–27

***22–28.** The plate of mass m is supported by three symmetrically placed cords of length l as shown. If the plate is given a slight rotation about a vertical axis through its center and released, determine the natural period of oscillation.



Prob. 22–28

22–29. The cylinder of radius r and mass m is displaced a small amount on the curved surface. If it rolls without slipping, determine the frequency of oscillation when it is released.



Prob. 22–29

*22.2 ENERGY METHODS

The simple harmonic motion of a body, discussed in the previous section, is due only to gravitational and elastic restoring forces acting on the body. Since these forces are *conservative*, it is also possible to use the conservation of energy equation to obtain the body's natural frequency or period of vibration. To show how to do this, consider again the block and spring model in Fig. 22–8. When the block is in a general displaced position x from the equilibrium position, the kinetic energy is $T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$ and the potential energy is $V = \frac{1}{2}kx^2$. Since energy is conserved, it is necessary that

$$T + V = \text{constant}$$

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \text{constant} \qquad (22\text{–}17)$$

The equation describing the *accelerated motion* of the block can be obtained by *differentiating* this equation with respect to time; i.e.,

$$m\dot{x}\ddot{x} + kx\dot{x} = 0$$

$$\dot{x}(m\ddot{x} + kx) = 0$$

Since the velocity $\dot{x}$ is not *always* zero in a vibrating system, then

$$\ddot{x} + \omega_n^2 x = 0 \qquad \omega_n = \sqrt{k/m}$$

which is the same as Eq. 22–1.

If the conservation of energy equation is written for a *system of connected bodies*, the natural frequency or the equation of motion can also be determined by time differentiation. It is *not necessary* to dismember the system to account for the internal forces acting between the bodies because they do no work.

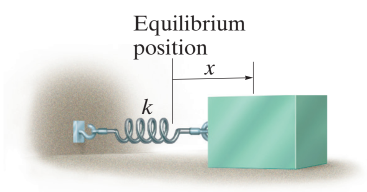


Fig. 22–8

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The suspension of a railroad car consists of a set of springs which are mounted between the frame of the car and the wheel truck. This will give the car a natural frequency of vibration.

PROCEDURE FOR ANALYSIS

The natural frequency ω_n of a body or system of connected bodies can be determined by applying the conservation of energy equation using the following procedure.

Energy Equation.

- Draw the body when it is displaced by a *small amount* from its equilibrium position and define the location of the body from its equilibrium position by an appropriate position coordinate q.
- Formulate the conservation of energy for the body, $T + V =$ constant, in terms of the position coordinate.
- In general, the kinetic energy must account for both the body's translational and rotational motion, $T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$, Eq. 18–2.
- The potential energy is the sum of the gravitational and elastic potential energies of the body, $V = V_g + V_e$, Eq. 18–17. In particular, V_g should be measured from a datum for which $q = 0$ (equilibrium position).

Time Derivative.

- Take the time derivative of the energy equation using the chain rule and factor out the common terms. The resulting differential equation represents the equation of motion for the system. The natural frequency ω_n is obtained after rearranging the terms in the "standard form," $\ddot{q} + \omega_n^2 q = 0$.

EXAMPLE 22.5

The thin hoop shown in Fig. 22–9*a* is supported by the peg at O. Determine the natural period of oscillation for small amplitudes of swing. The hoop has a mass m.

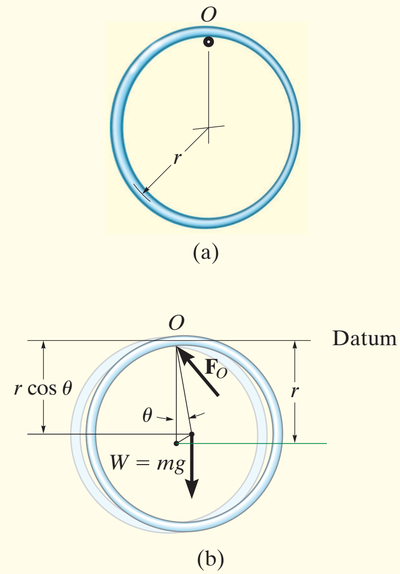

Fig. 22–9

SOLUTION

Energy Equation. A diagram of the hoop when it is displaced a small amount $(q =)\ \theta$ from the equilibrium position is shown in Fig. 22–9*b*. Using the table on the inside back cover and the parallel-axis theorem to determine I_O, the kinetic energy is

$$T = \tfrac{1}{2}I_O\omega_n^2 = \tfrac{1}{2}[mr^2 + mr^2]\dot{\theta}^2 = mr^2\dot{\theta}^2$$

If a horizontal datum is placed through point O, then in the displaced position, the potential energy is

$$V = -mg(r\cos\theta)$$

The total energy is

$$T + V = mr^2\dot{\theta}^2 - mgr\cos\theta$$

Time Derivative.

$$mr^2(2\dot{\theta})\ddot{\theta} + mgr(\sin\theta)\dot{\theta} = 0$$

$$mr\dot{\theta}(2r\ddot{\theta} + g\sin\theta) = 0$$

Since $\dot{\theta}$ is not always equal to zero, then the terms in parentheses give

$$\ddot{\theta} + \frac{g}{2r}\sin\theta = 0$$

For small angle θ, $\sin\theta \approx \theta$.

$$\ddot{\theta} + \frac{g}{2r}\theta = 0$$

$$\omega_n = \sqrt{\frac{g}{2r}}$$

so that

$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{2r}{g}} \qquad \textit{Ans.}$$

22

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EXAMPLE 22.6

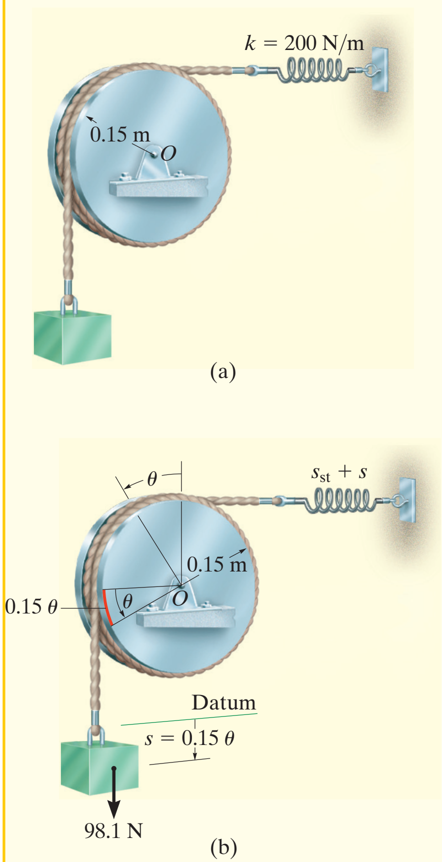


Fig. 22–10

A 10-kg block is suspended from a cord wrapped around a 5-kg disk, as shown in Fig. 22–10*a*. If the spring has a stiffness $k = 200\text{ N/m}$, determine the natural period of vibration for the system.

SOLUTION

Energy Equation. A diagram of the block and disk when they are displaced by respective amounts s and θ from the equilibrium position is shown in Fig. 22–10*b*. Since $s = (0.15\text{ m})\theta$, then $v_b \approx \dot{s} = (0.15\text{ m})\dot{\theta}$. Thus, the kinetic energy of the system is

$$T = \tfrac{1}{2}m_b v_b^2 + \tfrac{1}{2}I_O\omega_d^2$$

$$= \tfrac{1}{2}(10\text{ kg})[(0.15\text{ m})\dot{\theta}]^2 + \tfrac{1}{2}\left[\tfrac{1}{2}(5\text{ kg})(0.15\text{ m})^2\right](\dot{\theta})^2$$

$$= 0.1406(\dot{\theta})^2$$

Establishing the datum at the equilibrium position of the block and realizing that the spring stretches s_{st} for equilibrium, the potential energy is

$$V = \tfrac{1}{2}k(s_{st} + s)^2 - Ws$$

$$= \tfrac{1}{2}(200\text{ N/m})[s_{st} + (0.15\text{ m})\theta]^2 - 98.1\text{ N}[(0.15\text{ m})\theta]$$

The total energy for the system is therefore,

$$T + V = 0.1406(\dot{\theta})^2 + 100(s_{st} + 0.15\theta)^2 - 14.715\theta$$

Time Derivative.

$$0.28125(\dot{\theta})\ddot{\theta} + 200(s_{st} + 0.15\theta)0.15\dot{\theta} - 14.72\dot{\theta} = 0$$

Since $s_{st} = 98.1/200 = 0.4905\text{ m}$, the above equation reduces to the "standard form"

$$\ddot{\theta} + 16\theta = 0$$

so that

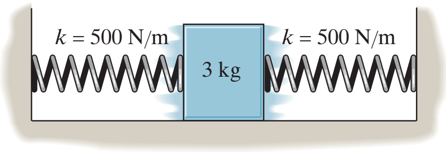
$$\omega_n = \sqrt{16} = 4\text{ rad/s}$$

Thus,

$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{4} = 1.57\text{ s} \qquad \textit{Ans.}$$

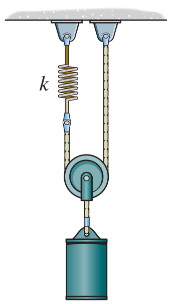
PROBLEMS

22–30. Determine the differential equation of motion of the 3-kg block when it is displaced slightly and released. The surface is smooth and the springs are originally unstretched.



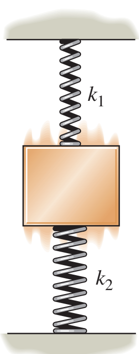
Prob. 22–30

22–31. Determine the frequency of oscillation of the cylinder of mass m when it is pulled down slightly and released. Neglect the mass of the small pulley.



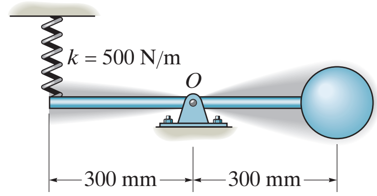
Prob. 22–31

***22–32.** Determine the differential equation of motion of the block of mass m when it is displaced slightly and released. Motion occurs in the vertical plane. The springs are attached to the block.



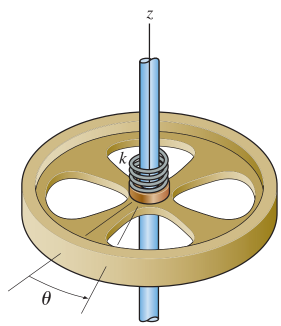
Prob. 22–32

22–33. Determine the natural period of vibration of the 3-kg sphere. Neglect the mass of the rod and the size of the sphere.



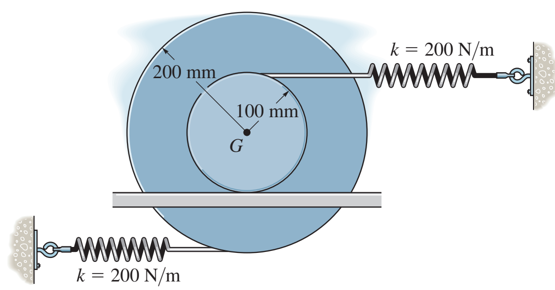
Prob. 22–33

22–34. A torsional spring of stiffness k is attached to a wheel that has a mass of M. If the wheel is given a small angular displacement of θ about the z axis, determine the natural period of oscillation. The wheel has a radius of gyration about the z axis of k_z.



Prob. 22–34

22–35. Determine the differential equation of motion of the 15-kg spool. Assume that it does not slip at the surface of contact as it oscillates. The radius of gyration of the spool about its center of mass is k_G = 125 mm. The springs are originally unstretched.



Prob. 22–35

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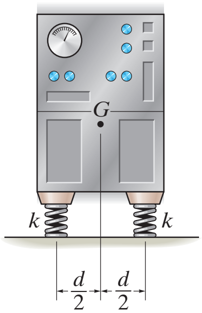
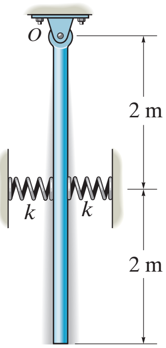
***22–36.** The machine has a mass m and is uniformly supported by *four* springs, each having a stiffness k. Determine the natural period of vertical vibration.



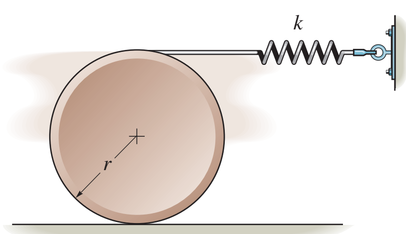
Prob. 22–36

22–37. If the lower end of the 6-kg slender rod is displaced a small amount and released from rest, determine the natural frequency of vibration. Each spring has a stiffness of $k = 200$ N/m and is unstretched when the rod is hanging vertically.



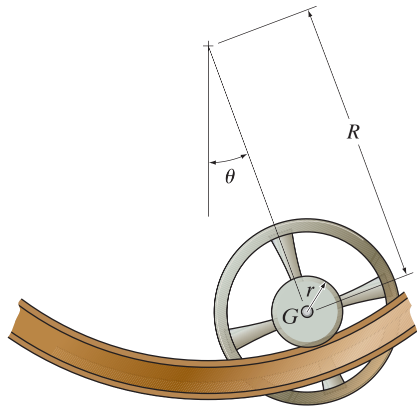
Prob. 22–37

22–38. Determine the natural period of vibration of the disk having a mass m and radius r. Assume the disk does not slip on the surface of contact as it oscillates.



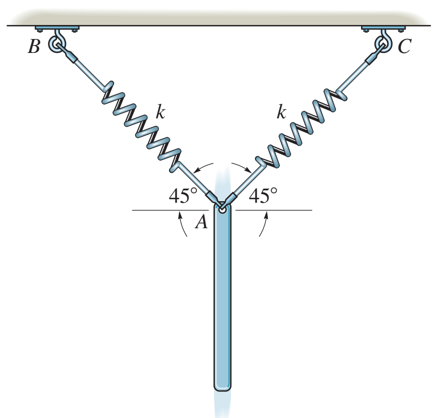
Prob. 22–38

22–39. If the wheel is given a small angular displacement of θ and released from rest, it is observed that it oscillates with a natural period of τ. Determine the wheel's radius of gyration about its center of mass G. The wheel has a mass of m and rolls on the rails without slipping.



Prob. 22–39

***22–40.** The bar has a mass of 8 kg and is suspended from two springs such that when it is in equilibrium, the springs make an angle of 45° with the horizontal as shown. Determine the natural period of vibration if the bar is pulled down a short distance and released. Each spring has a stiffness of $k = 40$ N/m.



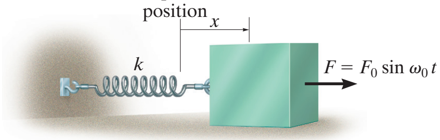
Prob. 22–40

22

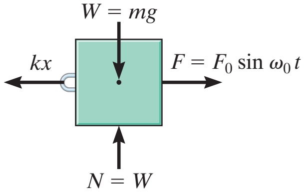
*22.3 UNDAMPED FORCED VIBRATION

Undamped forced vibration is considered to be one of the most important types of vibrating motion in engineering. Its principles can be used to describe the motion of many types of machines and structures.

Periodic Force. The block and spring shown in Fig. 22–11*a* provide a convenient model which represents the vibrational characteristics of a system subjected to a periodic force $F = F_0 \sin \omega_0 t$. This force has an amplitude of F_0 and a ***forcing frequency*** ω_0. The free-body diagram for the block when it is displaced a distance x is shown in Fig. 22–11*b*. Applying the equation of motion, we have



(a)



(b)

Fig. 22–11

$$\xrightarrow{+} \Sigma F_x = ma_x; \qquad F_0 \sin \omega_0 t - kx = m\ddot{x}$$

or

$$\ddot{x} + \frac{k}{m}x = \frac{F_0}{m}\sin \omega_0 t \tag{22–18}$$

This equation is a nonhomogeneous second-order differential equation. The general solution consists of a complementary solution, x_c, *plus* a particular solution, x_p.

The ***complementary solution*** is determined by setting the term on the right side of Eq. 22–18 equal to zero and solving the resulting homogeneous equation. The solution is defined by Eq. 22–9, i.e.,

$$x_c = C \sin(\omega_n t + \phi) \tag{22–19}$$

where ω_n is the natural frequency, $\omega_n = \sqrt{k/m}$, Eq. 22–2.

Since the motion is periodic, the ***particular solution*** of Eq. 22–18 can be determined by assuming a solution of the form

$$x_p = X \sin \omega_0 t \tag{22–20}$$

where X is a constant. Taking the second time derivative and substituting into Eq. 22–18 yields

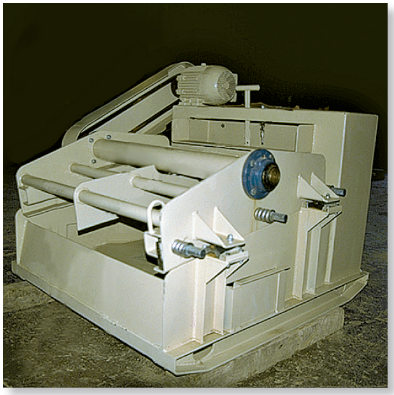
$$-X\omega_0^2 \sin \omega_0 t + \frac{k}{m}(X \sin \omega_0 t) = \frac{F_0}{m}\sin \omega_0 t$$

Factoring out $\sin \omega_0 t$ and solving for X gives

$$X = \frac{F_0/m}{(k/m) - \omega_0^2} = \frac{F_0/k}{1 - (\omega_0/\omega_n)^2} \tag{22–21}$$

Substituting into Eq. 22–20, we obtain

$$x_p = \frac{F_0/k}{1 - (\omega_0/\omega_n)^2}\sin \omega_0 t \tag{22–22}$$



Shaker tables provide forced vibration and are used to separate out granular materials.

22

The *general solution* is therefore the sum of two sine functions having different frequencies.

$$x = x_c + x_p = C \sin(\omega_n t + \phi) + \frac{F_0/k}{1 - (\omega_0/\omega_n)^2} \sin \omega_0 t \quad (22\text{–}23)$$

The *complementary solution* x_c defines the *free vibration*, which depends on the natural frequency $\omega_n = \sqrt{k/m}$ and the constants C and ϕ. The *particular solution* x_p describes the *forced vibration* of the block caused by the applied force $F = F_0 \sin \omega_0 t$. Since all vibrating systems are subject to *friction*, the free vibration, x_c, will in time dampen out. For this reason the free vibration is referred to as ***transient***, and the forced vibration is called ***steady-state***, since it is the only vibration that remains.

From Eq. 22–21 it is seen that the amplitude of forced or steady-state vibration depends on the ***frequency ratio*** ω_0/ω_n. If the ***magnification factor*** MF is defined as the ratio of the amplitude of steady-state vibration, X, to the static deflection, F_0/k, which would be produced by the amplitude of the periodic force F_0, then, from Eq. 22–21,

22

The soil compactor operates by forced vibration developed by an internal motor. It is important that the forcing frequency not be close to the natural frequency of vibration of the compactor, which can be determined when the motor is turned off; otherwise resonance will occur and the machine will become uncontrollable.

$$\text{MF} = \frac{X}{F_0/k} = \frac{1}{1 - (\omega_0/\omega_n)^2} \tag{22–24}$$

This equation is graphed in Fig. 22–12. Note that if the force or displacement is applied with a frequency close to the natural frequency of the system, i.e., $\omega_0/\omega_n \approx 1$, the amplitude of vibration of the block becomes extremely large. This occurs because the force **F** is applied to the block so that it always follows the motion of the block. This condition is called ***resonance***, and in practice, resonating vibrations can cause tremendous stress and rapid failure of parts.*

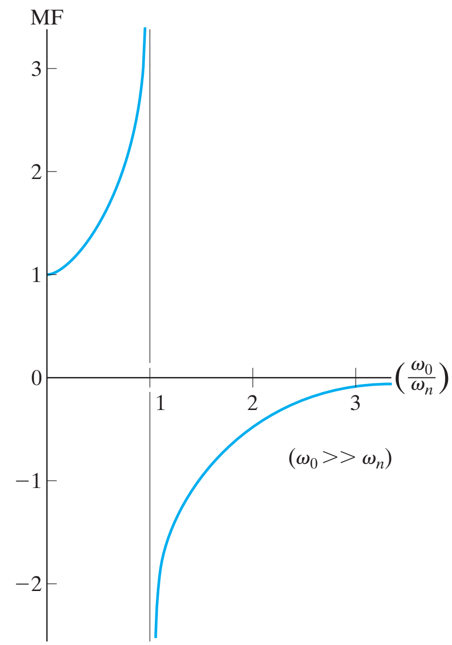


Fig. 22–12

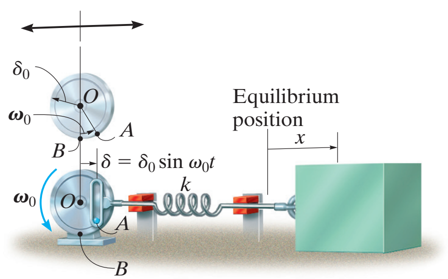
Periodic Support Displacement. Forced vibrations can also arise from the periodic excitation of the support of a system. For example, the model shown in Fig. 22–13*a* represents the periodic vibration of a block which is caused by harmonic movement $\delta = \delta_0 \sin \omega_0 t$ of the support. The free-body diagram for the block in this case is shown in Fig. 22–13*b*. If the displacement δ of the support is measured from the point of zero displacement, i.e., when the radial line OA coincides with OB, then the general deformation of the spring is $(x - \delta_0 \sin \omega_0 t)$. Applying the equation of motion,

$$\xrightarrow{+} \Sigma F_x = ma_x; \qquad -k(x - \delta_0 \sin \omega_0 t) = m\ddot{x}$$

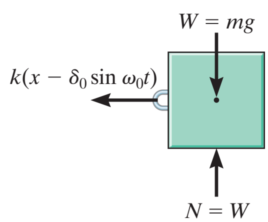
or

$$\ddot{x} + \frac{k}{m}x = \frac{k\delta_0}{m}\sin \omega_0 t \tag{22–25}$$

By comparison, this equation is identical to the form of Eq. 22–18, provided F_0 is replaced by $k\delta_0$. If this substitution is also made into Eqs. 22–21 to 22–23, the results are appropriate for describing the motion of the block when subjected to the support displacement $\delta = \delta_0 \sin \omega_0 t$.



(a)



(b)

Fig. 22–13

*A swing has a natural period of vibration, as determined in Example 22.1. If someone pushes on the swing only when it reaches its highest point, neglecting drag or wind resistance, resonance will occur since the natural and forcing frequencies are the same.

22

EXAMPLE 22.7

The instrument shown in Fig. 22–14 is rigidly attached to a platform P, which in turn is supported by *four* springs, each having a stiffness $k = 800\ \text{N/m}$. If the floor is subjected to a vertical displacement $\delta = 10 \sin(8t)$ mm, where t is in seconds, determine the amplitude of steady-state vibration. What is the frequency of the floor vibration required to cause resonance? The instrument and platform have a total mass of 20 kg.

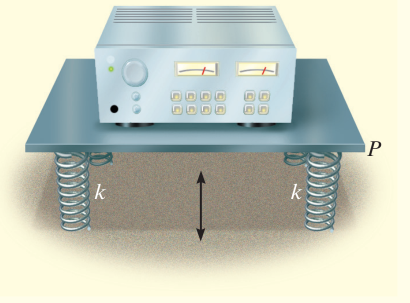


Fig. 22–14

SOLUTION

The natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4(800\ \text{N/m})}{20\ \text{kg}}} = 12.65\ \text{rad/s}$$

The amplitude of steady-state vibration is found using Eq. 22–21, with $k\delta_0$ replacing F_0.

$$X = \frac{\delta_0}{1 - (\omega_0/\omega_n)^2} = \frac{10}{1 - [(8\ \text{rad/s})/(12.65\ \text{rad/s})]^2} = 16.7\ \text{mm} \qquad \textit{Ans.}$$

Resonance will occur when the amplitude of vibration X caused by the floor displacement approaches infinity. This requires

$$\omega_0 = \omega_n = 12.6\ \text{rad/s} \qquad \textit{Ans.}$$

22

*22.4 VISCOUS DAMPED FREE VIBRATION

The vibration analysis considered thus far has not included the effects of friction or damping in the system, and as a result, the solutions obtained are only in close agreement with the actual motion. Since all vibrations die out in time, the presence of damping forces should be included in the analysis.

In many cases damping is attributed to the resistance created by the fluid, such as water, oil, or air, in which the system vibrates. Provided the body moves slowly through this fluid, then the resistance to motion is directly proportional to the body's speed. The type of force developed under these conditions is called a ***viscous damping force***. The magnitude of this force is expressed by an equation of the form

$$F = c\dot{x} \qquad (22\text{–}26)$$

where the constant c is called the ***coefficient of viscous damping*** and has units of $\text{N} \cdot \text{s/m}$.

The vibrating motion of a body or system having viscous damping can be characterized by the block and spring shown in Fig. 22–15*a*. The effect of damping is provided by the ***dashpot*** connected to the block on the right side. Damping occurs when the piston P moves to the right or left within the enclosed cylinder. The cylinder contains a fluid, and the motion of the piston is retarded since the fluid must flow around or through a small hole in the piston. The dashpot is assumed to have a coefficient of viscous damping c.

If the block is displaced a distance x from its equilibrium position, the resulting free-body diagram is shown in Fig. 22–15*b*. Both the spring and damping force oppose the forward motion of the block, and so the equation of motion becomes

$$\overset{+}{\rightarrow} \Sigma F_x = ma_x; \qquad -kx - c\dot{x} = m\ddot{x}$$

or

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad (22\text{–}27)$$

This linear, second-order, homogeneous, differential equation has a solution of the form

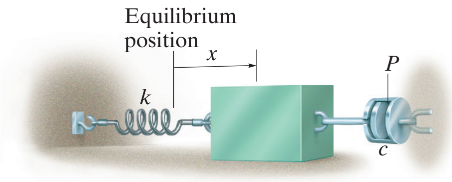
$$x = e^{\lambda t}$$

where e is the base of the natural logarithm and λ (lambda) is a constant. The value of λ can be obtained by substituting this solution and its time derivatives into Eq. 22–27, which yields

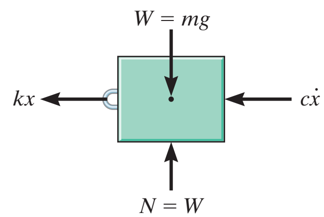
$$m\lambda^2 e^{\lambda t} + c\lambda e^{\lambda t} + k e^{\lambda t} = 0$$

or

$$e^{\lambda t}(m\lambda^2 + c\lambda + k) = 0$$



(a)



(b)

Fig. 22–15

Since $e^{\lambda t}$ can never be zero, a solution is possible provided

$$m\lambda^2 + c\lambda + k = 0$$

Hence, by the quadratic formula, the two values of λ are

$$\lambda_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$
$$\lambda_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \qquad (22\text{–}28)$$

The general solution of Eq. 22–27 is therefore a combination of exponentials which involves both of these roots. There are three possible combinations of λ_1 and λ_2 which must be considered. Before discussing these combinations, however, we will first define the ***critical damping coefficient*** c_c as the value of c which makes the radical in Eqs. 22–28 equal to zero; i.e.,

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0$$

22

or

$$c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n \qquad (22\text{–}29)$$

Overdamped System. When $c > c_c$, the roots λ_1 and λ_2 are both real. The general solution of Eq. 22–27 can then be written as

$$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \qquad (22\text{–}30)$$

Motion corresponding to this solution is nonvibrating. The effect of damping is so strong that when the block is displaced and released, it simply creeps back to its original position without oscillating. The system is said to be ***overdamped***.

Critically Damped System. If $c = c_c$, then $\lambda_1 = \lambda_2 = -c_c/2m = -\omega_n$. This situation is known as ***critical damping***, since it represents a condition where c has the smallest value necessary to cause the system to be nonvibrating. Using the methods of differential equations, it can be shown that the solution to Eq. 22–27 for critical damping is

$$x = (A + Bt)e^{-\omega_n t} \qquad (22\text{–}31)$$

Underdamped System. Most often $c < c_c$, in which case the system is referred to as ***underdamped***. In this case the roots λ_1 and λ_2 are complex numbers, and it can be shown that the general solution of Eq. 22–27 can be written as

$$x = D[e^{-(c/2m)t}\sin(\omega_d t + \phi)] \tag{22–32}$$

where D and ϕ are constants generally determined from the initial conditions of the problem. The constant ω_d is called the ***damped natural frequency*** of the system. It has a value of

$$\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = \omega_n\sqrt{1 - \left(\frac{c}{c_c}\right)^2} \tag{22–33}$$

where the ratio c/c_c is called the ***damping factor***.

The graph of Eq. 22–32 is shown in Fig. 22–16. The initial limit of motion, D, diminishes with each cycle of vibration, since motion is confined within the bounds of the exponential curve. Using the damped natural frequency ω_d, the period of damped vibration can be written as

$$\tau_d = \frac{2\pi}{\omega_d} \tag{22–34}$$

Since $\omega_d < \omega_n$, Eq. 22–33, the period of damped vibration, τ_d, will be greater than that of free vibration, $\tau = 2\pi/\omega_n$.

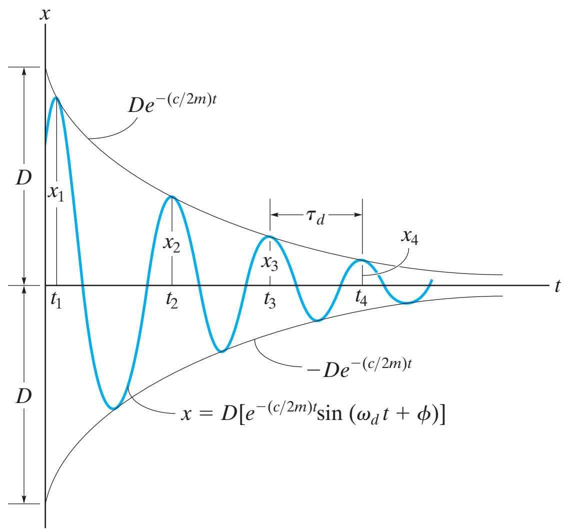


Fig. 22–16

*22.5 VISCOUS DAMPED FORCED VIBRATION

The most general case of single-degree-of-freedom vibrating motion occurs when the system includes the effects of forced motion and induced damping. The analysis of this particular type of vibration is of practical value when applied to systems having significant damping characteristics.

If a dashpot is attached to the block and spring shown in Fig. 22–11*a*, the differential equation which describes the motion becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega_0 t \qquad (22\text{–}35)$$

A similar equation can be written for a block and spring having a periodic support displacement, Fig. 22–13*a*, which includes the effects of damping. In that case, however, F_0 is replaced by $k\delta_0$. Since Eq. 22–35 is nonhomogeneous, the general solution is the sum of a complementary solution, x_c, and a particular solution, x_p. The complementary solution is determined by setting the right side of Eq. 22–35 equal to zero and solving the homogeneous equation, which is equivalent to Eq. 22–27. The solution is therefore given by Eq. 22–30, 22–31, or 22–32, depending on the values of λ_1 and λ_2. Because all systems are subjected to friction, this solution will dampen out with time. Only the particular solution, which describes the *steady-state vibration* of the system, will remain. Since the applied forcing function is harmonic, the steady-state motion will also be harmonic. Consequently, the particular solution will be of the form

$$X_P = X' \sin(\omega_0 t - \phi') \qquad (22\text{–}36)$$

The constants X' and ϕ' are determined by taking the first and second time derivatives and substituting them into Eq. 22–35, which after simplification yields

$$-X'm\omega_0^2 \sin(\omega_0 t - \phi') + X'c\omega_0 \cos(\omega_0 t - \phi') + X'k \sin(\omega_0 t - \phi') = F_0 \sin \omega_0 t$$

Since this equation holds for all time, the constant coefficients can be obtained by setting $\omega_0 t - \phi' = 0$ and $\omega_0 t - \phi' = \pi/2$, which causes the above equation to become

$$X'c\omega_0 = F_0 \sin \phi'$$

and

$$-X'm\omega_0^2 + X'k = F_0 \cos \phi'$$

The amplitude is obtained by squaring these equations, adding the results, and using the identity $\sin^2\phi' + \cos^2\phi' = 1$, which gives

$$X' = \frac{F_0}{\sqrt{(k - m\omega_0^2)^2 + c^2\omega_0^2}} \qquad (22\text{–}37)$$

22

Dividing the first equation by the second gives

$$\phi' = \tan^{-1}\left[\frac{c\omega_0}{k - m\omega_0^2}\right] \qquad (22\text{–}38)$$

Since $\omega_n = \sqrt{k/m}$ and $c_c = 2m\omega_n$, then the above equations can also be written as

$$X' = \frac{F_0/k}{\sqrt{[1 - (\omega_0/\omega_n)^2]^2 + [2(c/c_c)(\omega_0/\omega_n)]^2}}$$

$$\phi' = \tan^{-1}\left[\frac{2(c/c_c)(\omega_0/\omega_n)}{1 - (\omega_0/\omega_n)^2}\right] \qquad (22\text{–}39)$$

The angle ϕ' represents the phase difference between the applied force and the resulting steady-state vibration of the damped system.

The ***magnification factor*** MF has been defined in Sec. 22.3 as the ratio of the amplitude of deflection caused by the forced vibration to the deflection caused by a static force F_0. Thus,

$$\text{MF} = \frac{X'}{F_0/k} = \frac{1}{\sqrt{[1 - (\omega_0/\omega_n)^2]^2 + [2(c/c_c)(\omega_0/\omega_n)]^2}} \qquad (22\text{–}40)$$

The MF is plotted in Fig. 22–17 versus the frequency ratio ω_0/ω_n for various values of the damping factor c/c_c. It can be seen from this graph that the magnification of the amplitude increases as the damping factor decreases. Resonance obviously occurs only when the damping factor is zero and the frequency ratio equals 1.

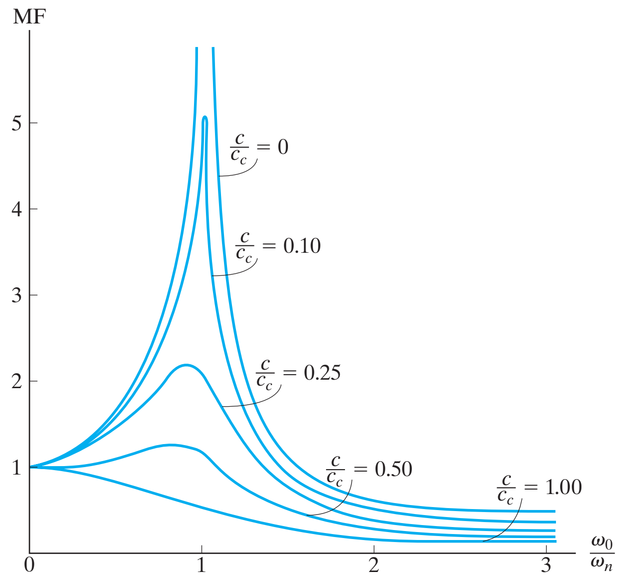


Fig. 22–17

22

EXAMPLE 22.8

The 30-kg electric motor shown in Fig. 22–18 is confined to move vertically, and is supported by *four* springs, each spring having a stiffness of 200 N/m. If the rotor is unbalanced such that its effect is equivalent to a 4-kg mass located 60 mm from the axis of rotation, determine the amplitude of vibration when the rotor is turning at $\omega_0 = 10$ rad/s. The damping factor is $c/c_c = 0.15$.

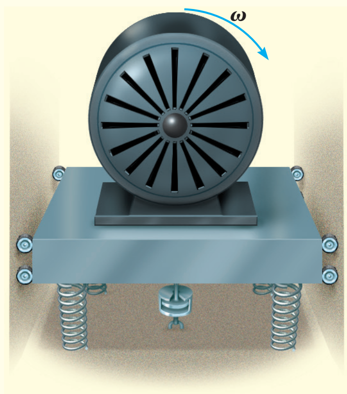


Fig. 22–18

SOLUTION

The periodic force which causes the motor to vibrate is the centrifugal force due to the unbalanced rotor. This force has a constant magnitude of

$$F_0 = ma_n = mr\omega_0^2 = 4\text{ kg}(0.06\text{ m})(10\text{ rad/s})^2 = 24\text{ N}$$

The stiffness of the entire system of four springs is $k = 4(200\text{ N/m}) = 800\text{ N/m}$. Therefore, the natural frequency of vibration is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{800\text{ N/m}}{30\text{ kg}}} = 5.164\text{ rad/s}$$

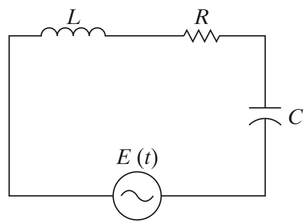
Since the damping factor is known, the steady-state amplitude can be determined from the first of Eqs. 22–39, i.e.,

$$X' = \frac{F_0/k}{\sqrt{[1 - (\omega_0/\omega_n)^2]^2 + [2(c/c_c)(\omega_0/\omega_n)]^2}}$$

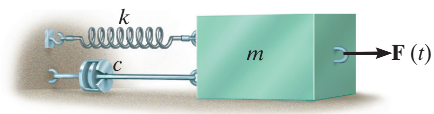
$$= \frac{24/800}{\sqrt{[1 - (10/5.164)^2]^2 + [2(0.15)(10/5.164)]^2}}$$

$$= 0.0107\text{ m} = 10.7\text{ mm} \qquad \textit{Ans.}$$

*22.6 ELECTRICAL CIRCUIT ANALOGS



(a)



(b)

Fig. 22–19

The characteristics of a vibrating mechanical system can be represented by an electric circuit. Consider the circuit shown in Fig. 22–19*a*, which consists of an inductor L, a resistor R, and a capacitor C. When a voltage $E(t)$ is applied, it causes a current of magnitude i to flow through the circuit. As the current flows past the inductor the voltage drop is $L(di/dt)$, when it flows across the resistor the drop is Ri, and when it arrives at the capacitor the drop is $(1/C)\int i\,dt$. Since current cannot flow past a capacitor, it is only possible to measure the charge q acting on the capacitor. The charge can, however, be related to the current by the equation $i = dq/dt$. Thus, the voltage drops which occur across the inductor, resistor, and capacitor become $L\,d^2q/dt^2$, $R\,dq/dt$, and q/C, respectively. According to Kirchhoff's voltage law, the applied voltage balances the sum of the voltage drops around the circuit. Therefore,

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t) \qquad (22\text{–}41)$$

Consider now the model of a single-degree-of-freedom mechanical system, Fig. 22–19*b*, which is subjected to both a general forcing function $F(t)$ and damping. The equation of motion for this system was established in the previous section and can be written as

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t) \qquad (22\text{–}42)$$

By comparison, it is seen that Eqs. 22–41 and 22–42 have the same form, and therefore mathematically the procedure of analyzing an electric circuit is the same as that of analyzing a vibrating mechanical system. The analogs between the two equations are given in Table 22–1.

This analogy has important application to experimental work, for it is much easier to simulate the vibration of a complex mechanical system using an electric circuit, which can be constructed on an analog computer, than to make an equivalent mechanical spring-and-dashpot model.

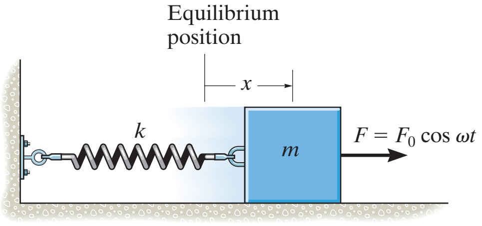
TABLE 22–1
Electrical–Mechanical Analogs

Electrical		Mechanical	
Electric charge	q	Displacement	x
Electric current	i	Velocity	dx/dt
Voltage	$E(t)$	Applied force	$F(t)$
Inductance	L	Mass	m
Resistance	R	Viscous damping coefficient	c
Reciprocal of capacitance	$1/C$	Spring stiffness	k

PROBLEMS

22–41. The barrel of a cannon has a mass of 700 kg, and after firing it recoils a distance of 0.64 m. If it returns to its original position by means of a single recuperator having a damping coefficient of 2 kN · s/m, determine the required stiffness of each of the two springs fixed to the base and attached to the barrel so that the barrel recuperates without vibration.

22–42. If the block-and-spring model is subjected to the periodic force $F = F_0 \cos \omega t$, show that the differential equation of motion is $\ddot{x} + (k/m)x = (F_0/m) \cos \omega t$, where x is measured from the equilibrium position of the block. What is the general solution of this equation?



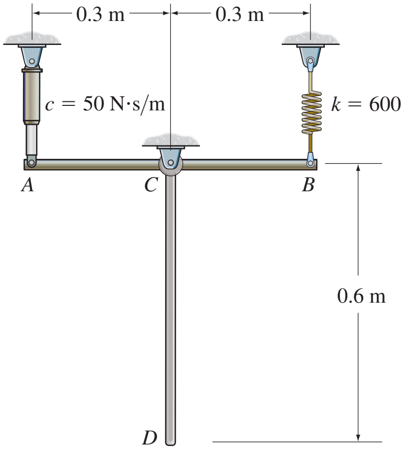
Prob. 22–42

22–43. A block which has a mass m is suspended from a spring having a stiffness k. If an impressed downward vertical force $F = F_0$ acts on the weight, determine the equation which describes the position of the block as a function of time.

***22–44.** Use a block-and-spring model like that shown in Fig. 22–13a, but suspended from a vertical position and subjected to a periodic support displacement $\delta = \delta_0 \sin \omega_0 t$, determine the equation of motion for the system, and obtain its general solution. Define the displacement y measured from the static equilibrium position of the block when $t = 0$.

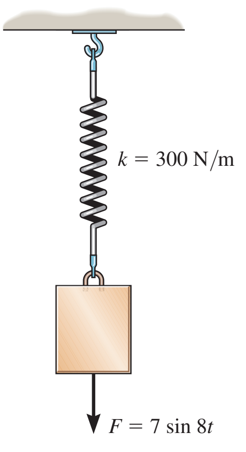
22–45. A 4-kg block is suspended from a spring that has a stiffness of $k = 600$ N/m. The block is drawn downward 50 mm from the equilibrium position and released from rest when $t = 0$. If the support moves with an impressed displacement of $\delta = (10 \sin 4t)$ mm, where t is in seconds, determine the equation that describes the vertical motion of the block. Assume positive displacement is downward.

22–46. If the dashpot has a damping coefficient of $c = 50$ N · s/m, and the spring has a stiffness of $k = 600$ N/m, show that the system is underdamped, and then find the pendulum's period of oscillation. The uniform rods have a mass per unit length of 10 kg/m.



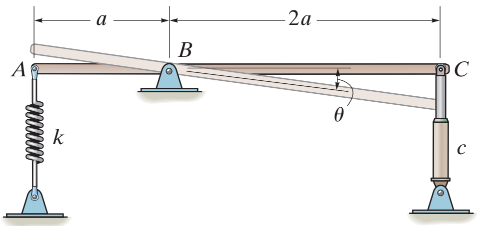
Prob. 22–46

22–47. A 5-kg block is suspended from a spring having a stiffness of 300 N/m. If the block is acted upon by a vertical force $F = (7 \sin 8t)$ N, where t is in seconds, determine the equation which describes the motion of the block when it is pulled down 100 mm from the equilibrium position and released from rest at $t = 0$. Assume that positive displacement is downward.



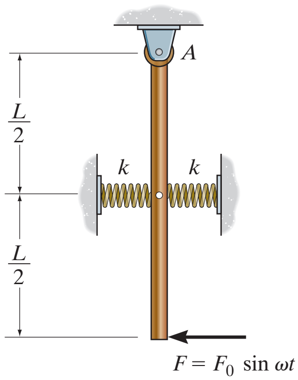
Prob. 22–47

***22–48.** Find the differential equation for small oscillations in terms of θ for the uniform rod of mass m. Also show that if $c < \sqrt{mk}/2$, then the system remains underdamped. The rod is in a horizontal position when it is in equilibrium.



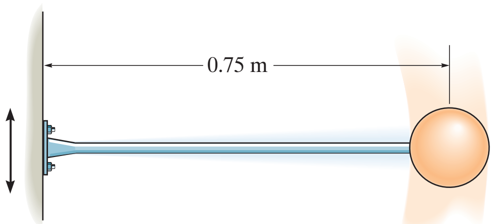
Prob. 22–48

22–49. The uniform rod has a mass of m. If it is acted upon by a periodic force of $F = F_0 \sin \omega t$, determine the amplitude of the steady-state vibration.



Prob. 22–49

22–50. The light elastic rod supports a 4-kg sphere. When an 18-N vertical force is applied to the sphere, the rod deflects 14 mm. If the wall oscillates with harmonic frequency of 2 Hz and has an amplitude of 15 mm, determine the amplitude of vibration for the sphere.

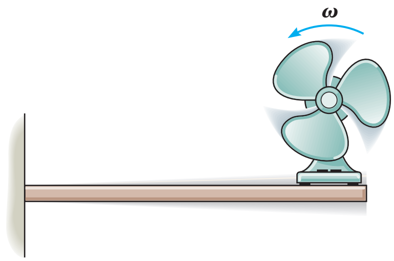


Prob. 22–50

22–51. The fan has a mass of 25 kg and is fixed to the end of a horizontal beam that has a negligible mass. The fan blade is mounted eccentrically on the shaft such that it is equivalent to an unbalanced 3.5-kg mass located 100 mm from the axis of rotation. If the static deflection of the beam is 50 mm as a result of the weight of the fan, determine the angular velocity of the fan blade at which resonance will occur. *Hint*: See the first part of Example 22.8.

***22–52.** In Prob. 22–51, determine the amplitude of steady-state vibration of the fan if its angular velocity is 10 rad/s.

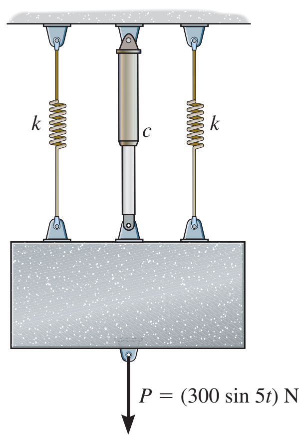
22–53. What will be the amplitude of steady-state vibration of the fan in Prob. 22–51 if the angular velocity of the fan blade is 18 rad/s? *Hint*: See the first part of Example 22.8.



Probs. 22–51/52/53

22

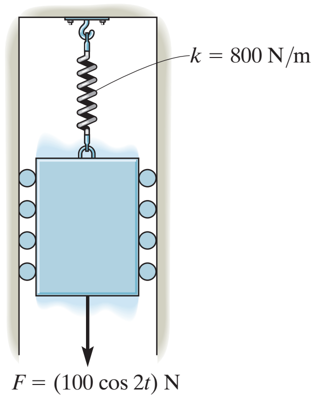
22–54. If the 30-kg block is subjected to a periodic force of $P = (300 \sin 5t)$ N, $k = 1500$ N/m, and $c = 300$ N · s/m, determine the equation that describes the steady-state vibration as a function of time.



Prob. 22–54

22–55. Using a block-and-spring model, like that shown in Fig. 22–13a, but suspended from a vertical position and subjected to a periodic support displacement of $\delta = \delta_0 \cos \omega_0 t$, determine the equation of motion for the system, and obtain its general solution. Define the displacement y measured from the static equilibrium position of the block when $t = 0$.

***22–56.** The 40-kg block is attached to a spring having a stiffness of 800 N/m. A force $F = (100 \cos 2t)$ N, where t is in seconds is applied to the block. Determine the maximum speed of the block for the steady-state vibration.

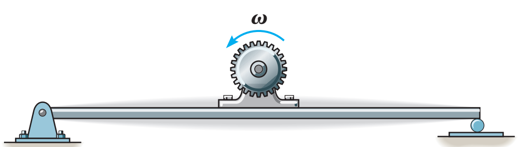


Prob. 22–56

22–57. The electric motor turns an eccentric flywheel which is equivalent to an unbalanced 0.125-kg mass located 250 mm from the axis of rotation. If the static deflection of the beam is 25 mm because of the mass of the motor, determine the angular velocity of the flywheel at which resonance will occur. The motor has a mass of 75 kg. Neglect the mass of the beam.

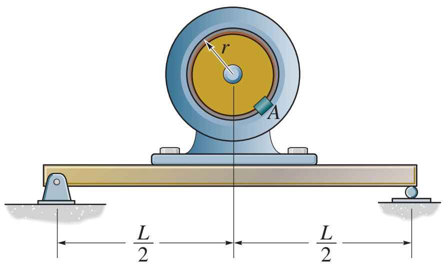
22–58. What will be the amplitude of steady-state vibration of the motor in Prob. 22–57 if the angular velocity of the flywheel is 20 rad/s?

22–59. Determine the angular velocity of the flywheel in Prob. 22–57 which will produce an amplitude of vibration of 6 mm.



Probs. 22–57/58/59

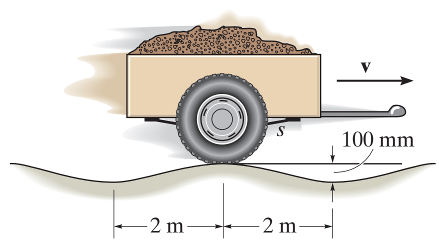
***22–60.** The motor of mass M is supported by a simply supported beam of negligible mass. If block A of mass m is clipped onto the rotor, which is turning at constant angular velocity of ω, determine the amplitude of the steady-state vibration. *Hint*: When the beam is subjected to a concentrated force of P at its mid-span, it deflects $\delta = PL^3/48EI$ at this point. Here E is Young's modulus of elasticity, a property of the material, and I is the moment of inertia of the beam's cross-sectional area.



Prob. 22–60

22–61. The 450-kg trailer is pulled with a constant speed over the surface of a bumpy road, which may be approximated by a cosine curve having an amplitude of 50 mm and wave length of 4 m. If the two springs s which support the trailer each have a stiffness of 800 N/m, determine the speed v which will cause the greatest vibration (resonance) of the trailer. Neglect the weight of the wheels.

22–62. Determine the amplitude of vibration of the trailer in Prob. 22–61 if the speed $v = 15$ km/h.

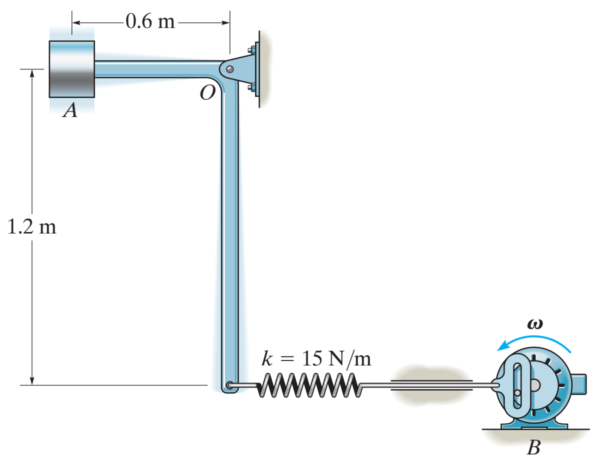


Probs. 22–61/62

22–63. A block having a mass of 7 kg is suspended from a spring that has a stiffness $k = 600$ N/m. If the block is given an upward velocity of 0.6 m/s from its equilibrium position at $t = 0$, determine its position as a function of time. Assume that positive displacement of the block is downward and that motion takes place in a medium which furnishes a damping force $F = (50|v|)$ N, where v is in m/s.

***22–64.** A block having a mass of 0.8 kg is suspended from a spring having a stiffness of 120 N/m. If a dashpot provides a damping force of 2.5 N when the speed of the block is 0.2 m/s, determine the period of free vibration.

22–65. The small block at A has a mass of 4 kg and is mounted on the bent rod having negligible mass. If the rotor at B causes a harmonic movement $\delta_B = (0.1 \cos 15t)$ m, where t is in seconds, determine the steady-state amplitude of vibration of the block.

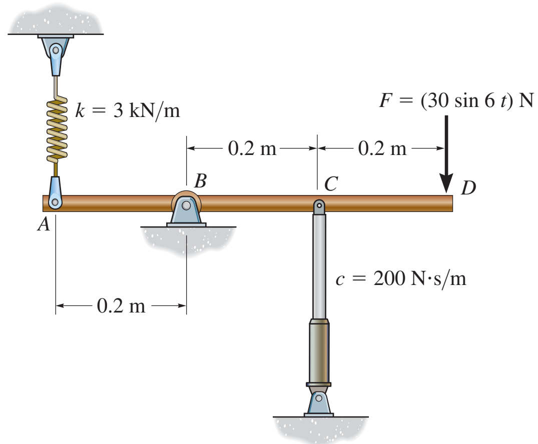


Prob. 22–65

22–66. A 3.5-kg block is suspended from a spring having a stiffness of $k = 1250$ N/m. The support to which the spring is attached is given simple harmonic motion which can be expressed as $\delta = (0.045 \sin 2t)$ m, where t is in seconds. If the damping factor is $c/c_c = 0.8$, determine the phase angle ϕ of forced vibration.

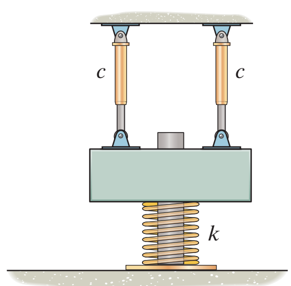
22–67. Determine the magnification factor of the block, spring, and dashpot combination in Prob. 22–66.

***22–68.** If the 12-kg rod is subjected to a periodic force of $F = (30 \sin 6t)$ N, where t is in seconds, determine the steady-state vibration amplitude θ_{max} of the rod about the pin B. Assume θ is small.



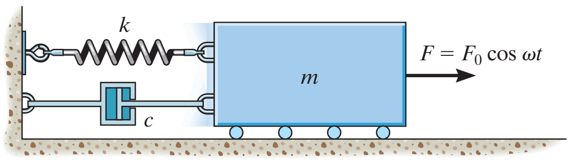
Prob. 22–68

22–69. Two identical dashpots are arranged parallel to each other, as shown. Show that if the damping coefficient $c < \sqrt{mk}$, then the block of mass m will vibrate as an underdamped system.



Prob. 22–69

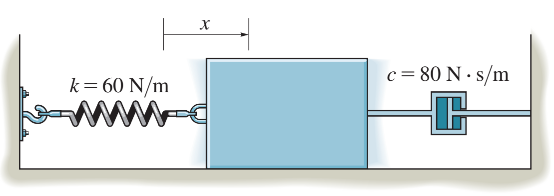
22–70. Draw the electrical circuit that is equivalent to the mechanical system shown. Determine the differential equation which describes the charge q in the circuit.



Prob. 22–70

22–71. The damping factor, c/c_c, may be determined experimentally by measuring the successive amplitudes of vibrating motion of a system. If two of these maximum displacements can be approximated by x_1 and x_2, as shown in Fig. 22–16, show that $\ln (x_1/x_2) = 2\pi(c/c_c)/\sqrt{1-(c/c_c)^2}$. The quantity $\ln (x_1/x_2)$ is called the *logarithmic decrement*.

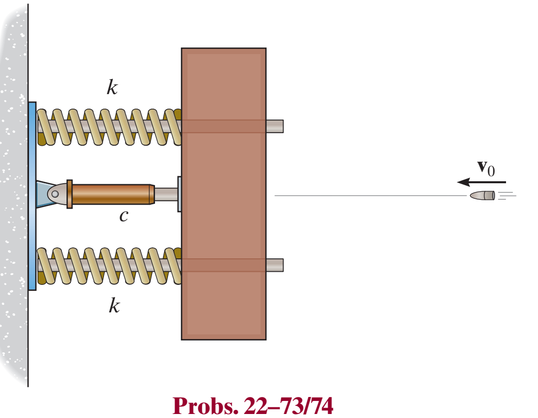
***22–72.** The 10-kg block-spring-damper system is damped. If the block is displaced to $x = 50$ mm and released from rest, determine the time required for it to return to the position $x = 2$ mm.



Prob. 22–72

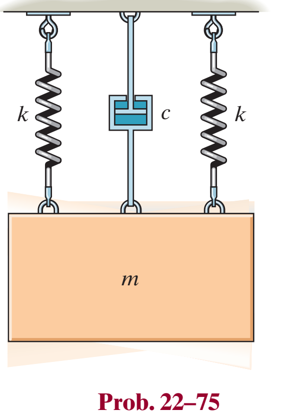
22–73. A bullet of mass m has a velocity $\mathbf{v}_0$ just before it strikes the target of mass M. If the bullet embeds in the target, and the vibration is to be critically damped, determine the dashpot's damping coefficient, and the springs' maximum compression. The target is free to move along the two horizontal guides that are "nested" in the springs.

22–74. A bullet of mass m has a velocity $\mathbf{v}_0$ just before it strikes the target of mass M. If the bullet embeds in the target, and the dashpot's damping coefficient is $0 < c \ll c_c$, determine the springs' maximum compression. The target is free to move along the two horizontal guides that are "nested" in the springs.



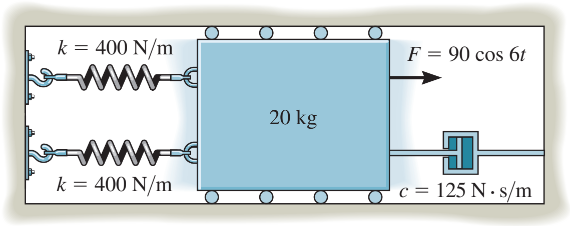
Probs. 22–73/74

22–75. Draw the electrical circuit that is equivalent to the mechanical system shown. What is the differential equation which describes the charge q in the circuit?



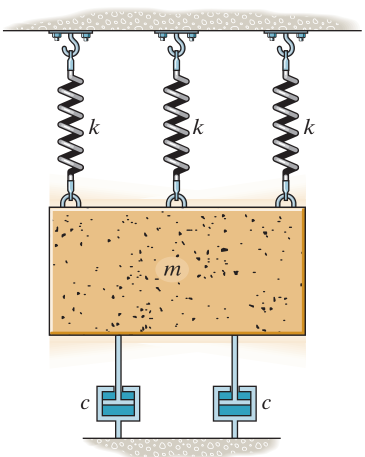
Prob. 22–75

***22–76.** The 20-kg block is subjected to the action of the harmonic force $F = (90 \cos 6t)$ N, where t is in seconds. Write the equation which describes the steady-state motion.



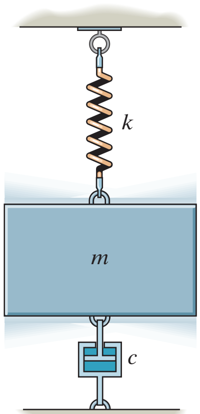
Prob. 22–76

22–77. Determine the differential equation of motion for the damped vibratory system shown. What type of motion occurs? Take $k = 100$ N/m, $c = 200$ N·s/m, $m = 25$ kg.



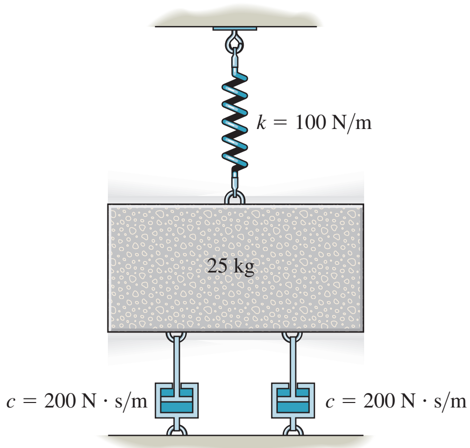
Prob. 22–77

22–78. Draw the electrical circuit that is equivalent to the mechanical system shown. What is the differential equation which describes the charge q in the circuit?



Prob. 22–78

22–79. Determine the differential equation of motion for the damped vibratory system shown. What type of motion occurs?



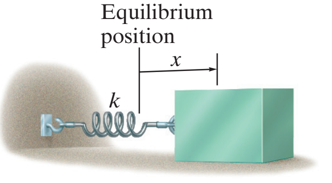
Prob. 22–79

CHAPTER REVIEW

Undamped Free Vibration

A body has free vibration when gravitational or elastic restoring forces cause the motion. This motion is undamped when friction forces are neglected. The periodic motion of an undamped, freely vibrating body can be studied by displacing the body from the equilibrium position and then applying the equation of motion along the path.

For a one-degree-of-freedom system, the resulting differential equation can be written in terms of its natural frequency ω_n.



$$\ddot{x} + \omega_n^2 x = 0 \qquad \tau = \frac{2\pi}{\omega_n} \qquad f = \frac{1}{\tau} = \frac{\omega_n}{2\pi}$$

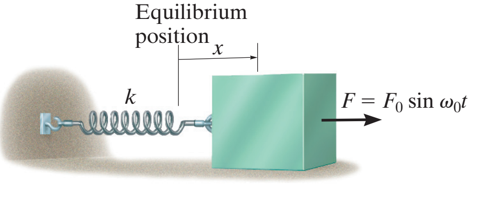
Energy Methods

Provided the restoring forces acting on the body are gravitational and elastic, then the conservation of energy theorem can also be used to determine the simple harmonic motion. To do this, the body is displaced a small amount from its equilibrium position, and an expression for its kinetic and potential energy is written. The time derivative of this equation can then be rearranged in the standard form $\ddot{x} + \omega_n^2 x = 0$.

Undamped Forced Vibration

When the equation of motion is applied to a body which is subjected to a periodic force, or the support has a displacement with a frequency ω_0, then the solution of the differential equation consists of a complementary solution and a particular solution. The complementary solution is caused by the free vibration and can be neglected. The particular solution is caused by the forced vibration.

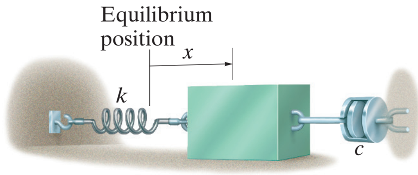
Resonance will occur if the natural frequency of vibration ω_n is equal to the forcing frequency ω_0. This should be avoided, since the motion will tend to become unbounded.



$$x_p = \frac{F_0/k}{1 - (\omega_0/\omega_n)^2} \sin \omega_0 t$$

Viscous Damped Free Vibration

A viscous damping force is caused by fluid drag on the system as it vibrates. If the motion is slow, this drag force will be proportional to the velocity, that is, $F = c\dot{x}$, where c is the coefficient of viscous damping. By comparing its value to the critical damping coefficient $c_c = 2m\omega_n$, we can specify the type of vibration that occurs. If $c > c_c$, it is an overdamped system; if $c = c_c$, it is a critically damped system; and if $c < c_c$, it is an underdamped system.



Viscous Damped Forced Vibration

The most general type of vibration for a one-degree-of-freedom system occurs when the system is damped and subjected to periodic forced motion. The solution provides insight as to how the damping factor, c/c_c, and the frequency ratio, ω_0/ω_n, influence the vibration.

Resonance is avoided provided $c/c_c \neq 0$ and $\omega_0/\omega_n \neq 1$.

Electrical Circuit Analogs

The vibrating motion of a complex mechanical system can be studied by modeling it as an electrical circuit. This is possible since the differential equations that govern the behavior of each system are the same.

APPENDIX A

MATHEMATICAL EXPRESSIONS

Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Hyperbolic Functions

$\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$,

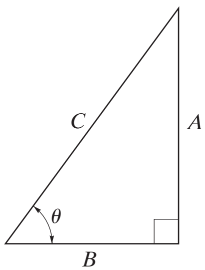
$\tanh x = \dfrac{\sinh x}{\cosh x}$

Trigonometric Identities

$\sin\theta = \dfrac{A}{C}$, $\csc\theta = \dfrac{C}{A}$

$\cos\theta = \dfrac{B}{C}$, $\sec\theta = \dfrac{C}{B}$

$\tan\theta = \dfrac{A}{B}$, $\cot\theta = \dfrac{B}{A}$



$\sin^2\theta + \cos^2\theta = 1$

$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos\theta = \pm\sqrt{\dfrac{1 + \cos 2\theta}{2}}$, $\sin\theta = \pm\sqrt{\dfrac{1 - \cos 2\theta}{2}}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$1 + \tan^2\theta = \sec^2\theta \quad 1 + \cot^2\theta = \csc^2\theta$

Power-Series Expansions

$\sin x = x - \dfrac{x^3}{3!} + \cdots \qquad \sinh x = x + \dfrac{x^3}{3!} + \cdots$

$\cos x = 1 - \dfrac{x^2}{2!} + \cdots \qquad \cosh x = 1 + \dfrac{x^2}{2!} + \cdots$

Derivatives

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}(\cot u) = -\csc^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\sec u) = \tan u \sec u \frac{du}{dx}$$

$$\frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx}$$

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

$$\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$$

$$\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx) + C$$

$$\int \frac{dx}{a+bx^2} = \frac{1}{2\sqrt{-ba}}\ln\left[\frac{a+x\sqrt{-ab}}{a-x\sqrt{-ab}}\right] + C,\ ab < 0$$

$$\int \frac{x\,dx}{a+bx^2} = \frac{1}{2b}\ln(bx^2+a) + C$$

$$\int \frac{x^2\,dx}{a+bx^2} = \frac{x}{b} - \frac{a}{b\sqrt{ab}}\tan^{-1}\frac{x\sqrt{ab}}{a} + C, ab > 0$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left[\frac{a+x}{a-x}\right] + C, a^2 > x^2$$

$$\int \sqrt{a+bx}\,dx = \frac{2}{3b}\sqrt{(a+bx)^3} + C$$

$$\int x\sqrt{a+bx}\,dx = \frac{-2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2} + C$$

$$\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3} + C$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right] + C,\quad a > 0$$

$$\int x\sqrt{x^2 \pm a^2}\,dx = \frac{1}{3}\sqrt{(x^2 \pm a^2)^3} + C$$

$$\int x^2\sqrt{a^2-x^2}\,dx = -\frac{x}{4}\sqrt{(a^2-x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right) + C, a > 0$$

$$\int \sqrt{x^2 \pm a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2 \pm a^2} \pm a^2\ln\left(x + \sqrt{x^2 \pm a^2}\right)\right] + C$$

$$\int x\sqrt{a^2-x^2}\,dx = -\frac{1}{3}\sqrt{(a^2-x^2)^3} + C$$

$$\int x^2\sqrt{x^2 \pm a^2}\,dx = \frac{x}{4}\sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8}x\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\ln\left(x + \sqrt{x^2 \pm a^2}\right) + C$$

$$\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b} + C$$

$$\int \frac{x\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2} + C$$

$$\int \frac{dx}{\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{c}}\ln\left[\sqrt{a+bx+cx^2} + x\sqrt{c} + \frac{b}{2\sqrt{c}}\right] + C,\ c > 0$$

$$= \frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{-2cx-b}{\sqrt{b^2-4ac}}\right) + C, c < 0$$

$$\int \sin x\,dx = -\cos x + C$$

A

$$\int \cos x \, dx = \sin x + C$$

$$\int x \cos(ax) \, dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax) + C$$

$$\int x^2 \cos(ax) \, dx = \frac{2x}{a^2}\cos(ax) + \frac{a^2x^2 - 2}{a^3}\sin(ax) + C$$

$$\int e^{ax} \, dx = \frac{1}{a}e^{ax} + C$$

$$\int xe^{ax} \, dx = \frac{e^{ax}}{a^2}(ax - 1) + C$$

$$\int \sinh x \, dx = \cosh x + C$$

$$\int \cosh x \, dx = \sinh x + C$$

APPENDIX B

VECTOR ANALYSIS

The following discussion provides a brief review of vector analysis. A more detailed treatment of these topics is given in *Engineering Mechanics: Statics*.

Vector. A vector, $\mathbf{A}$, is a quantity which has magnitude and direction, and adds according to the parallelogram law. As shown in Fig. B–1, $\mathbf{A} = \mathbf{B} + \mathbf{C}$, where $\mathbf{A}$ is the *resultant vector* and $\mathbf{B}$ and $\mathbf{C}$ are *component vectors*.

Unit Vector. A unit vector, $\mathbf{u}_A$, has a magnitude of one "dimensionless" unit and acts in the same direction as $\mathbf{A}$. It is determined by dividing $\mathbf{A}$ by its magnitude A, i.e.,

$$\mathbf{u}_A = \frac{\mathbf{A}}{A} \qquad \text{(B–1)}$$

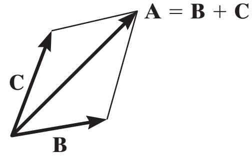


Fig. B–1

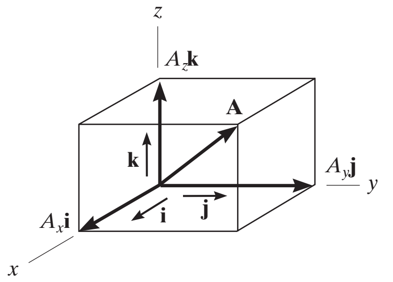


Fig. B–2

Cartesian Vector Notation. The directions of the positive x, y, z axes are defined by the Cartesian unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, respectively.

As shown in Fig. B–2, vector $\mathbf{A}$ is formulated by the addition of its x, y, z components

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \tag{B–2}$$

The *magnitude* of $\mathbf{A}$ is determined from

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{B–3}$$

The *direction* of $\mathbf{A}$ is defined in terms of its *coordinate direction angles*, α, β, γ, measured from the *tail* of $\mathbf{A}$ to the *positive x, y, z* axes, Fig. B–3. These angles are determined from *direction cosines* which represent the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components of the unit vector $\mathbf{u}_A$; i.e., from Eqs. B–1 and B–2

$$\mathbf{u}_A = \frac{A_x}{A}\mathbf{i} + \frac{A_y}{A}\mathbf{j} + \frac{A_z}{A}\mathbf{k} \tag{B–4}$$

so that the direction cosines are

$$\cos\alpha = \frac{A_x}{A} \qquad \cos\beta = \frac{A_y}{A} \qquad \cos\gamma = \frac{A_z}{A} \tag{B–5}$$

Hence, $\mathbf{u}_A = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$, and using Eq. B–3, it is seen that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{B–6}$$

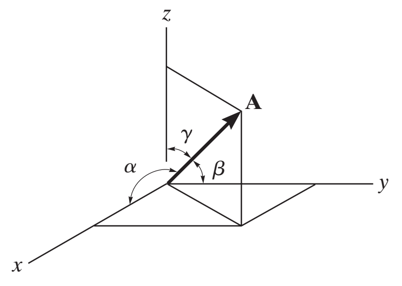


Fig. B–3

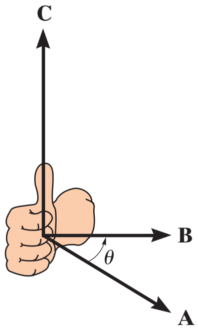


Fig. B–4

The Cross Product. The cross product of two vectors **A** and **B**, which yields the resultant vector **C**, is written as

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \tag{B–7}$$

and reads **C** equals **A** "cross" **B**. The *magnitude* of **C** is

$$C = AB \sin \theta \tag{B–8}$$

where θ is the angle between the tails of **A** and **B** ($0° \le \theta \le 180°$). The *direction* of **C** is determined by the right-hand rule, whereby the fingers of the right hand are curled *from* **A** *to* **B** and the thumb points in the direction of **C**, Fig. B–4. This vector is perpendicular to the plane containing vectors **A** and **B**.

The vector cross product is *not* commutative, i.e., $\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$. Rather,

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{B–9}$$

The distributive law is valid; i.e.,

$$\mathbf{A} \times (\mathbf{B} + \mathbf{D}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{D} \tag{B–10}$$

And the cross product may be multiplied by a scalar m in any manner; i.e.,

$$m(\mathbf{A} \times \mathbf{B}) = (m\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (m\mathbf{B}) = (\mathbf{A} \times \mathbf{B})m \tag{B–11}$$

B

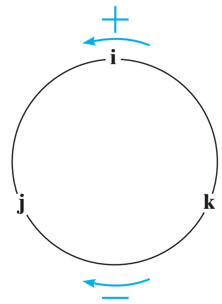


Fig. B–5

Equation B–7 can be used to find the cross product of any pair of Cartesian unit vectors. For example, to find $\mathbf{i} \times \mathbf{j}$, the magnitude is $(i)(j) \sin 90° = (1)(1)(1) = 1$, and its direction $+\mathbf{k}$ is determined from the right-hand rule, applied to $\mathbf{i} \times \mathbf{j}$, Fig. B–2. A simple scheme shown in Fig. B–5 may be helpful in obtaining this and other results when the need arises. If the circle is constructed as shown, then "crossing" two of the unit vectors in a *counterclockwise* fashion around the circle yields a *positive* third unit vector, e.g., $\mathbf{k} \times \mathbf{i} = \mathbf{j}$. Moving *clockwise*, a *negative* unit vector is obtained, e.g., $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

If $\mathbf{A}$ and $\mathbf{B}$ are expressed in Cartesian component form, then the cross product, Eq. B–7, may be evaluated by expanding the determinant

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{B–12}$$

which yields

$$\mathbf{C} = (A_yB_z - A_zB_y)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$$

Recall that the cross product is used in statics to define the moment of a force $\mathbf{F}$ about point O, in which case

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} \tag{B–13}$$

where $\mathbf{r}$ is a position vector directed from point O to *any point* on the line of action of $\mathbf{F}$.

The Dot Product. The dot product of two vectors **A** and **B**, which yields a scalar, is defined as

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta \qquad \text{(B–14)}$$

and is read as **A** "dot" **B**. The angle θ is formed between the *tails* of **A** and **B** ($0° \leq \theta \leq 180°$).

The dot product is commutative; i.e.,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \qquad \text{(B–15)}$$

The distributive law is valid; i.e.,

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{D}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{D} \qquad \text{(B–16)}$$

And scalar multiplication can be performed in any manner, i.e.,

$$m(\mathbf{A} \cdot \mathbf{B}) = (m\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (m\mathbf{B}) = (\mathbf{A} \cdot \mathbf{B})m \qquad \text{(B–17)}$$

Using Eq. B–14, the dot product between any two Cartesian unit vectors can be determined. For example, $\mathbf{i} \cdot \mathbf{i} = (1)(1) \cos 0° = 1$ and $\mathbf{i} \cdot \mathbf{j} = (1)(1) \cos 90° = 0$.

If **A** and **B** are expressed in Cartesian component form, then the dot product, Eq. B–14, can be determined from

$$\boxed{\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z} \qquad \text{(B–18)}$$

The dot product may be used to determine the angle θ formed between two vectors. From Eq. B–14,

$$\theta = \cos^{-1}\left(\frac{\mathbf{A} \cdot \mathbf{B}}{AB}\right) \qquad \text{(B–19)}$$

It is also possible to find the component of a vector in a given direction using the dot product. For example, the magnitude of the component (or projection) of vector **A** in the direction of **B**, Fig. B–6, is defined by $A \cos \theta$. From Eq. B–14, this magnitude is

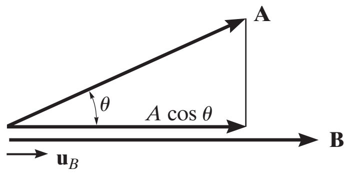


Fig. B–6

$$A \cos \theta = \mathbf{A} \cdot \frac{\mathbf{B}}{B} = \mathbf{A} \cdot \mathbf{u}_B \tag{B–20}$$

where $\mathbf{u}_B$ represents a unit vector acting in the direction of **B**, Fig. B–6.

Differentiation and Integration of Vector Functions. The rules for differentiation and integration of the sums and products of scalar functions also apply to vector functions. Consider, for example, the two vector functions $\mathbf{A}(s)$ and $\mathbf{B}(s)$. Provided these functions are smooth and continuous for all s, then

$$\frac{d}{ds}(\mathbf{A} + \mathbf{B}) = \frac{d\mathbf{A}}{ds} + \frac{d\mathbf{B}}{ds} \tag{B–21}$$

$$\int (\mathbf{A} + \mathbf{B})\, ds = \int \mathbf{A}\, ds + \int \mathbf{B}\, ds \tag{B–22}$$

For the cross product,

$$\frac{d}{ds}(\mathbf{A} \times \mathbf{B}) = \left(\frac{d\mathbf{A}}{ds} \times \mathbf{B}\right) + \left(\mathbf{A} \times \frac{d\mathbf{B}}{ds}\right) \tag{B–23}$$

Similarly, for the dot product,

$$\frac{d}{ds}(\mathbf{A} \cdot \mathbf{B}) = \frac{d\mathbf{A}}{ds} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{ds} \tag{B–24}$$

APPENDIX C

THE CHAIN RULE

The chain rule of calculus is used to determine the time derivative of a composite function. For example, if y is a function of x and x is a function of t, then we can find the derivative of y with respect to t as follows

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\dot{x}$$

In other words, to find $\dot{y}$ we find the change in y with respect to x, and then the change in x with respect to time. And so,

$$\boxed{\dot{y} = \frac{dy}{dx}\dot{x}} \tag{C–1}$$

If a second time derivative of this equation is to be determined, then the product rule $d(uv) = du\,v + u\,dv$ must be used along with the chain rule when taking the time derivatives because both dy/dx and $\dot{x}$ are functions of time. And so letting $u = dy/dx$ and $v = \dot{x}$, we have

$$\ddot{y} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\dot{x} + \frac{dy}{dx}\frac{d}{dt}(\dot{x})$$

Applying the chain rule to the first term on the right, we first find the change in (dy/dx) with respect to x, and then the change in x with respect to time.

$$\ddot{y} = \left(\frac{d^2y}{dx^2}\frac{dx}{dt}\right)\dot{x} + \left(\frac{dy}{dx}\right)\ddot{x}$$

or

$$\ddot{y} = \frac{d^2y}{dx^2}\dot{x}^2 + \frac{dy}{dx}\ddot{x} \qquad \text{(C–2)}$$

The following examples illustrate these ideas.

EXAMPLE C–1

If $y = x^3$ and $x = t^4$, find $\ddot{y}$, the second derivative of y with respect to time.

SOLUTION

Using the chain rule, Eq. C–1,

$$\dot{y} = 3x^2\dot{x}$$

To obtain the second time derivative we must use the product rule since x and $\dot{x}$ are both functions of time, and also, for $3x^2$ the chain rule must be applied. Thus, with $u = 3x^2$ and $v = \dot{x}$, we have

$$d(uv) = du\, v + u\, dv$$
$$\ddot{y} = [6x\dot{x}]\dot{x} + 3x^2[\ddot{x}]$$
$$= 3x[2\dot{x}^2 + x\ddot{x}]$$

Since $x = t^4$, then $\dot{x} = 4t^3$ and $\ddot{x} = 12t^2$ so that

$$\ddot{y} = 3(t^4)[2(4t^3)^2 + t^4(12t^2)]$$
$$= 132t^{10} \qquad \textit{Ans.}$$

We can also apply Eq. C–2.

$$\ddot{y} = (6x)(4t^3)^2 + (3x^2)(12t^2)$$

Setting $x = t^4$, we get the result

$$\ddot{y} = 132t^{10} \qquad \textit{Ans.}$$

Note that this result can also be obtained by combining the functions, then taking the time derivatives, that is,

$$y = x^3 = (t^4)^3 = t^{12}$$
$$\dot{y} = 12t^{11}$$
$$\ddot{y} = 132t^{10} \qquad \textit{Ans.}$$

C

EXAMPLE C–2

If $y = xe^x$, and $x = t^2$, find $\ddot{y}$.

SOLUTION

Since x and e^x are both functions of time, then the product and chain rules must be applied. Setting $u = x$ and $v = e^x$.

$$d(uv) = du\,v + u\,dv$$

$$\dot{y} = [\dot{x}]e^x + x[e^x\dot{x}]$$

The second time derivative also requires application of the product and chain rules. In this case the product rule applies to the three time variables in the last term, i.e., x, e^x, and $\dot{x}$. We have

$$\ddot{y} = \{[\ddot{x}]e^x + \dot{x}[e^x\dot{x}]\} + \{[\dot{x}]e^x\dot{x} + x[e^x\dot{x}]\dot{x} + xe^x[\ddot{x}]\}$$

$$= e^x[\ddot{x}(1 + x) + \dot{x}^2(2 + x)]$$

Since $x = t^2$ then $\dot{x} = 2t$, $\ddot{x} = 2$ so that in terms in t, we have

$$\ddot{y} = e^{t^2}[2(1 + t^2) + 4t^2(2 + t^2)]$$

EXAMPLE C–3

If the path in radial coordinates is given as $r = 5\theta^2$, where θ is a known function of time, find $\ddot{r}$.

SOLUTION

First, using the chain rule then the chain and product rules where $u = 10\theta$ and $v = \dot{\theta}$, we have

$$r = 5\theta^2$$

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta};\ \dot{r} = 10\theta\dot{\theta}$$

$$d(uv) = du\,v + u\,dv$$

$$\ddot{r} = 10[(\dot{\theta})\dot{\theta} + \theta(\ddot{\theta})]$$

$$= 10\dot{\theta}^2 + 10\theta\ddot{\theta}$$

Apply Eq. C–2 and show that you get the same result.

C

Fundamental Problems Solutions And Answers

Chapter 12

F12–1. $v = v_0 + a_c t$

$10 = 35 + a_c(15)$

$a_c = -1.67\ \text{m/s}^2 = 1.67\ \text{m/s}^2 \leftarrow$ *Ans.*

F12–2. $s = s_0 + v_0 t + \frac{1}{2}a_c t^2$

$0 = 0 + 15t + \frac{1}{2}(-9.81)t^2$

$t = 3.06\ \text{s}$ *Ans.*

F12–3. $\int_0^s ds = \int_0^t \left(4t - 3t^2\right) dt$

$s = \left(2t^2 - t^3\right)\text{m}$

$s = 2\left(4^2\right) - 4^3$

$= -32\ \text{m} = 32\ \text{m} \leftarrow$ *Ans.*

F12–4. $a = \frac{dv}{dt} = \frac{d}{dt}\left(0.5t^3 - 8t\right)$

$a = \left(1.5t^2 - 8\right)\ \text{m/s}^2$

When $t = 2$ s,

$a = 1.5\left(2^2\right) - 8 = -2\ \text{m/s}^2 = 2\ \text{m/s}^2 \leftarrow$ *Ans.*

F12–5. $v = \frac{ds}{dt} = \frac{d}{dt}(2t^2 - 8t + 6) = (4t - 8)\ \text{m/s}$

$v = 0 = (4t - 8)$

$t = 2\ \text{s}$ *Ans.*

$s|_{t=0} = 2\left(0^2\right) - 8(0) + 6 = 6\ \text{m}$

$s|_{t=2} = 2\left(2^2\right) - 8(2) + 6 = -2\ \text{m}$

$s|_{t=3} = 2\left(3^2\right) - 8(3) + 6 = 0\ \text{m}$

Plot these positions, then

$(\Delta s)_{\text{Tot}} = 8\ \text{m} + 2\ \text{m} = 10\ \text{m}$ *Ans.*

F12–6. $v\,dv = a\,ds$

$\int_{5\ \text{m/s}}^{v} v\,dv = \int_0^s (10 - 0.2s)ds$

$v = \left(\sqrt{20s - 0.2s^2 + 25}\right)\ \text{m/s}$

At $s = 10$ m,

$v = \sqrt{20(10) - 0.2(10^2) + 25}$

$= 14.3\ \text{m/s} \rightarrow$ *Ans.*

F12–7. $v = \int (4t^2 - 2)\,dt$

$v = \frac{4}{3}t^3 - 2t + C_1$

$s = \int \left(\frac{4}{3}t^3 - 2t + C_1\right) dt$

$s = \frac{1}{3}t^4 - t^2 + C_1 t + C_2$

$t = 0, s = -2, C_2 = -2$

$t = 2, s = -20, C_1 = -9.67$

$t = 4, s = 28.7\ \text{m}$ *Ans.*

F12–8. $a = v\frac{dv}{ds}$

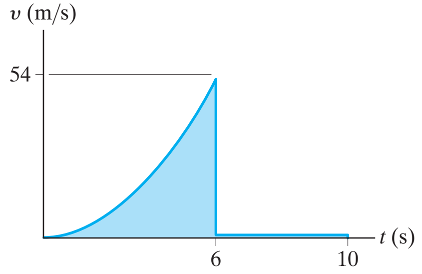
$= \left(20 - 0.05s^2\right)\left(-0.1s\right)$

At $s = 15$ m,

$a = -13.1\ \text{m/s}^2 = 13.1\ \text{m/s}^2 \leftarrow$ *Ans.*

F12–9. $v = \frac{ds}{dt} = \frac{d}{dt}\left(0.5t^3\right) = 1.5t^2$

$v = \frac{ds}{dt} = \frac{d}{dt}(108) = 0$ *Ans.*



F12–10. $0 \le t < 5$ s,

$v = \frac{ds}{dt} = \frac{d}{dt}(3t^2) = (6t)\ \text{m/s}$

$5\ \text{s} < t \le 10$ s,

$v = \frac{ds}{dt} = \frac{d}{dt}(30t - 75) = 30\ \text{m/s}$

$v = \frac{\Delta s}{\Delta t} = \frac{225\ \text{m} - 75\ \text{m}}{10\ \text{s} - 5\ \text{s}} = 30\ \text{m/s}$

$0 \le t < 5$ s,

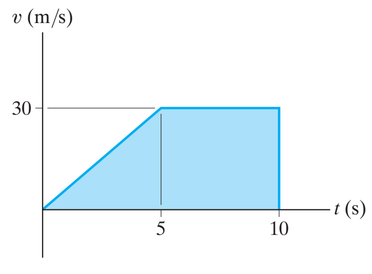
$a = \frac{dv}{dt} = \frac{d}{dt}(6t) = 6\ \text{m/s}^2$

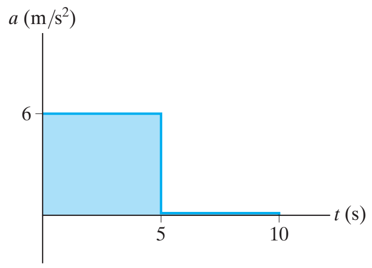
$5\ \text{s} < t \le 10$ s,

$a = \frac{dv}{dt} = \frac{d}{dt}(30) = 0$

$0 \le t < 5\ \text{s}, a = \Delta v/\Delta t = 6\ \text{m/s}^2$

$5\ \text{s} < t \le 10\ \text{s}, a = \Delta v/\Delta t = 0$

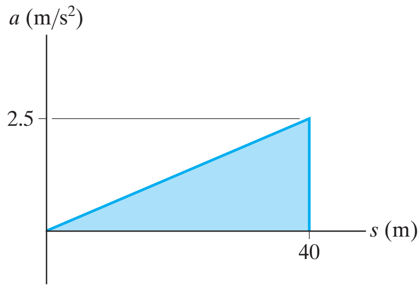




F12–11. $a\,ds = v\,dv$

$a = v\frac{dv}{ds} = 0.25s\frac{d}{ds}(0.25s) = 0.0625s$

$a\big|_{s=40\text{ m}} = 0.0625(40\text{ m}) = 2.5\text{ m/s}^2 \rightarrow$



F12–12. For $0 \le s \le 10$ m

$a = s$

$$\int_0^v v\,dv = \int_0^s s\,ds$$

$v = s$

at $s = 10$ m, $v = 10$ m/s

For $10\text{ m} \le s \le 15$

$a = 10$

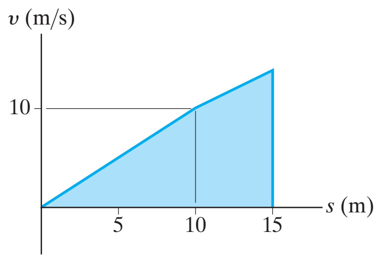
$$\int_{10}^v v\,dv = \int_{10}^s 10\,ds$$

$$\frac{1}{2}v^2 - 50 = 10s - 100$$

$$v = \sqrt{20s - 100}$$

at $s = 15$ m

$v = 14.1$ m/s *Ans.*



F12–13. $0 \le t < 5$ s,

$dv = a\,dt \quad \int_0^v dv = \int_0^t 20\,dt$

$v = (20t)$ m/s

$5\text{ s} < t \le t'$,

$(\overset{+}{\rightarrow}) \quad dv = a\,dt \quad \int_{100\text{ m/s}}^v dv = \int_{5\text{ s}}^t -10\,dt$

$v - 100 = (50 - 10t)$ m/s,

$0 = 150 - 10t'$

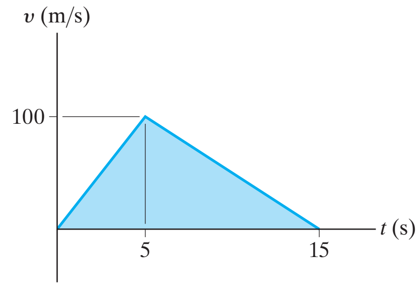
$t' = 15$ s

Also,

$\Delta v = 0 =$ Area under the $a-t$ graph

$0 = (20\text{ m/s}^2)(5\text{ s}) + [-(10\text{ m/s})(t' - 5)\text{ s}]$

$t' = 15$ s



F12–14. $0 \le t \le 5$ s,

$ds = v\,dt \qquad \int_0^s ds = \int_0^t 30t\,dt$

$s\big|_0^s = 15t^2\big|_0^t$

$s = (15t^2)$ m

$5\text{ s} < t \le 15$ s,

$(\overset{+}{\rightarrow})\ ds = v\,dt; \quad \int_{375\text{ m}}^s ds = \int_{5\text{ s}}^t (-15t + 225)dt$

$s = (-7.5t^2 + 225t - 562.5)$ m

$s = (-7.5)(15)^2 + 225(15) - 562.5$ m

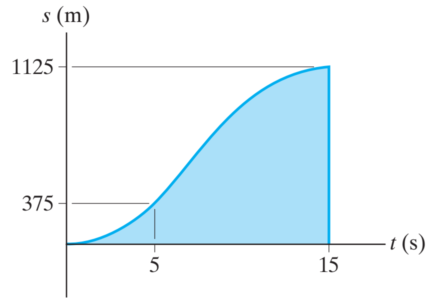
$= 1125$ m *Ans.*

Also,

$\Delta s =$ Area under the $v-t$ graph

$= \frac{1}{2}(150\text{ m/s})(15\text{ s})$

$= 1125$ m *Ans.*



F12–15. $\int_0^x dx = \int_0^t 32t\,dt$

$x = \left(16t^2\right)$ m (1)

$$\int_0^y dy = \int_0^t 8\,dt$$

$t = \frac{y}{8}$ (2)

Substituting Eq. (2) into Eq. (1), get

$y = 2\sqrt{x}$ *Ans.*

F12–16. $y = 0.75(8t) = 6t$

$v_x = \dot{x} = \frac{dx}{dt} = \frac{d}{dt}(8t) = 8 \text{ m/s} \rightarrow$

$v_y = \dot{y} = \frac{dy}{dt} = \frac{d}{dt}(6t) = 6 \text{ m/s} \uparrow$

$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(8 \text{ m/s})^2 + (6 \text{ m/s})^2}$

$= 10 \text{ m/s}$ *Ans.*

F12–17. $y = (4t^2) \text{ m}$

$v_x = \dot{x} = \frac{d}{dt}\left(4t^4\right) = \left(16t^3\right) \text{ m/s} \rightarrow$

$v_y = \dot{y} = \frac{d}{dt}\left(4t^2\right) = (8t) \text{ m/s} \uparrow$

When $t = 0.5$ s,

$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(2 \text{ m/s})^2 + (4 \text{ m/s})^2}$

$= 4.47 \text{ m/s}$ *Ans.*

$a_x = \dot{v}_x = \frac{d}{dt}\left(16t^3\right) = \left(48t^2\right) \text{ m/s}^2$

$a_y = \dot{v}_y = \frac{d}{dt}(8t) = 8 \text{ m/s}^2$

When $t = 0.5$ s,

$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(12 \text{ m/s}^2)^2 + (8 \text{ m/s}^2)^2}$

$= 14.4 \text{ m/s}^2$ *Ans.*

F12–18. $y = 0.5x$

$\dot{y} = 0.5\dot{x}$

$v_y = t^2$

When $t = 4$ s,

$v_x = 32 \text{ m/s} \qquad v_y = 16 \text{ m/s}$

$v = \sqrt{v_x^2 + v_y^2} = 35.8 \text{ m/s}$ *Ans.*

$a_x = \dot{v}_x = 4t$

$a_y = \dot{v}_y = 2t$

When $t = 4$ s,

$a_x = 16 \text{ m/s}^2 \qquad a_y = 8 \text{ m/s}^2$

$a = \sqrt{a_x^2 + a_y^2} = \sqrt{16^2 + 8^2} = 17.9 \text{ m/s}^2$ *Ans.*

F12–19. $v_y = \dot{y} = 0.5\,x\,\dot{x} = 0.5(8)(8) = 32 \text{ m/s}$

Thus,

$v = \sqrt{v_x^2 + v_y^2} = 33.0 \text{ m/s}$ *Ans.*

$a_y = \dot{v}_y = 0.5 \quad \dot{x}^2 + 0.5x\ddot{x}$

$= 0.5(8)^2 + 0.5(8)(4)$

$= 48 \text{ m/s}^2$

$a_x = 4 \text{ m/s}^2$

Thus,

$a = \sqrt{a_x^2 + a_y^2} = \sqrt{4^2 + 48^2} = 48.2 \text{ m/s}^2$ *Ans.*

F12–20. $\dot{y} = 0.1x\dot{x}$

$v_y = 0.1(5)(-3) = -1.5 \text{ m/s} = 1.5 \text{ m/s}\downarrow$ *Ans.*

$\ddot{y} = 0.1[\dot{x}\dot{x} + x\ddot{x}]$

$a_y = 0.1[(-3)^2 + 5(-1.5)] = 0.15 \text{ m/s}^2 \uparrow$ *Ans.*

F12–21. $(v_B)_y^2 = (v_A)_y^2 + 2a_y(y_B - y_A)$

$0^2 = (5 \text{ m/s})^2 + 2(-9.81 \text{ m/s}^2)(h - 0)$

$h = 1.27 \text{ m}$ *Ans.*

F12–22. $y_C = y_A + (v_A)_y t_{AC} + \frac{1}{2}a_y t_{AC}^2$

$0 = 0 + (5 \text{ m/s})t_{AC} + \frac{1}{2}(-9.81 \text{ m/s}^2)t_{AC}^2$

$t_{AC} = 1.0194 \text{ s}$

$(v_C)_y = (v_A)_y + a_y t_{AC}$

$(v_C)_y = 5 \text{ m/s} + (-9.81 \text{ m/s}^2)(1.0194 \text{ s})$

$= -5 \text{ m/s} = 5 \text{ m/s} \downarrow$

$v_C = \sqrt{(v_C)_x^2 + (v_C)_y^2}$

$= \sqrt{(8.660 \text{ m/s})^2 + (5 \text{ m/s})^2} = 10 \text{ m/s}$ *Ans.*

$R = x_A + (v_A)_x t_{AC} = 0 + (8.660 \text{ m/s})(1.0194 \text{ s})$

$= 8.83 \text{ m}$ *Ans.*

F12–23. $s = s_0 + v_0 t$

$10 = 0 + v_A \cos 30° t$

$s = s_0 + v_0 t + \frac{1}{2}a_c t^2$

$3 = 1.5 + v_A \sin 30° t + \frac{1}{2}(-9.81)t^2$

$t = 0.9334 \text{ s}, \quad v_A = 12.4 \text{ m/s}$ *Ans.*

F12–24. $s = s_0 + v_0 t$

$R\left(\frac{4}{5}\right) = 0 + 20\left(\frac{3}{5}\right)t$

$s = s_0 + v_0 t + \frac{1}{2}a_c t^2$

$-R\left(\frac{3}{5}\right) = 0 + 20\left(\frac{4}{5}\right)t + \frac{1}{2}(-9.81)t^2$

$t = 5.10 \text{ s}$

$R = 76.5 \text{ m}$ *Ans.*

F12–25. $x_B = x_A + (v_A)_x t_{AB}$

$3.6 \text{ m} = 0 + (0.8660\, v_A)t_{AB}$

$v_A t_{AB} = 4.157$ (1)

$y_B = y_A + (v_A)_y t_{AB} + \frac{1}{2}a_y t_{AB}^2$

$$(2.4 - 0.9)\text{ m} = 0 + 0.5v_A t_{AB} + \tfrac{1}{2}(-9.81\text{ m/s}^2)t_{AB}^2$$

Using Eq. (1),

$$1.5 = 0.5(4.157) - 4.905\,t_{AB}^2$$

$$t_{AB} = 0.3434\text{ s}$$

$$v_A = 12.1\text{ m/s}$$ *Ans.*

F12–26. $y_B = y_A + (v_A)_y t_{AB} + \frac{1}{2}a_y t_{AB}^2$

$$-150\text{ m} = 0 + (90\text{ m/s})t_{AB} + \tfrac{1}{2}(-9.81\text{ m/s}^2)t_{AB}^2$$

$$t_{AB} = 19.89\text{ s}$$

$$x_B = x_A + (v_A)_x t_{AB}$$

$$R = 0 + 120\text{ m/s}(19.89\text{ s}) = 2386.37\text{ m}$$

$$= 2.39\text{ km}$$ *Ans.*

F12–27. $a_t = \dot{v} = \frac{dv}{dt} = \frac{d}{dt}(0.0625t^2) = (0.125t)\text{ m/s}^2\Big|_{t=10\text{ s}}$

$$= 1.25\text{ m/s}^2$$

$$a_n = \frac{v^2}{\rho} = \frac{(0.0625t^2)^2}{40\text{ m}} = \left[97.656(10^{-6})t^4\right]\text{ m/s}^2\Big|_{t=10\text{ s}}$$

$$= 0.9766\text{ m/s}^2$$

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{(1.25\text{ m/s}^2)^2 + (0.9766\text{ m/s}^2)^2}$$

$$= 1.59\text{ m/s}^2$$ *Ans.*

F12–28. $v = 2s\Big|_{s=10} = 20\text{ m/s}$

$$a_n = \frac{v^2}{\rho} = \frac{(20\text{ m/s})^2}{50\text{ m}} = 8\text{ m/s}^2$$

$$a_t = v\frac{dv}{ds} = 4s\Big|_{s=10} = 40\text{ m/s}^2$$

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{\left(40\text{ m/s}^2\right)^2 + \left(8\text{ m/s}^2\right)^2}$$

$$= 40.8\text{ m/s}^2$$ *Ans.*

F12–29. $v_C^2 = v_A^2 + 2a_t(s_C - s_A)$

$$(15\text{ m/s})^2 = (25\text{ m/s})^2 + 2a_t(300\text{ m} - 0)$$

$$a_t = -0.6667\text{ m/s}^2$$

$$v_B^2 = v_A^2 + 2a_t(s_B - s_A)$$

$$v_B^2 = (25\text{ m/s})^2 + 2(-0.6667\text{ m/s}^2)(250\text{ m} - 0)$$

$$v_B = 17.08\text{ m/s}$$

$$(a_B)_n = \frac{v_B^2}{\rho} = \frac{(17.08\text{ m/s})^2}{300\text{ m}} = 0.9722\text{ m/s}^2$$

$$a_B = \sqrt{(a_B)_t^2 + (a_B)_n^2}$$

$$= \sqrt{(-0.6667\text{ m/s}^2)^2 + (0.9722\text{ m/s}^2)^2}$$

$$= 1.18\text{ m/s}^2$$ *Ans.*

F12–30. $\tan\theta = \frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{8}x^2\right) = \frac{1}{4}x$

$$\theta = \tan^{-1}\left(\tfrac{1}{4}x\right)\Big|_{x=3\text{ m}}$$

$$= \tan^{-1}\left(\tfrac{3}{4}\right) = 36.87° = 36.9°\ \measuredangle$$ *Ans.*

$$\rho = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{|d^2y/dx^2|} = \frac{\left[1 + \left(\frac{1}{4}x\right)^2\right]^{3/2}}{\left|\frac{1}{4}\right|}\Bigg|_{x=3\text{ m}}$$

$$= 7.8125\text{ m}$$

$$a_n = \frac{v^2}{\rho} = \frac{(6\text{ m/s})^2}{7.8125\text{ m}} = 4.608\text{ m/s}^2$$

$$a = \sqrt{(a_t)^2 + (a_n)^2} = \sqrt{(2\text{ m/s}^2)^2 + (4.608\text{ m/s}^2)^2}$$

$$= 5.02\text{ m/s}^2$$ *Ans.*

F12–31. $(a_B)_t = -0.001s = (-0.001)(300\text{ m})\left(\frac{\pi}{2}\text{ rad}\right)\text{ m/s}^2$

$$= -0.4712\text{ m/s}^2$$

$$v\,dv = a_t\,ds$$

$$\int_{25\text{ m/s}}^{v_B} v\,dv = \int_0^{150\pi\text{ m}} -0.001s\,ds$$

$$v_B = 20.07\text{ m/s}$$

$$(a_B)_n = \frac{v_B^2}{\rho} = \frac{(20.07\text{ m/s})^2}{300\text{ m}} = 1.343\text{ m/s}^2$$

$$a_B = \sqrt{(a_B)_t^2 + (a_B)_n^2}$$

$$= \sqrt{(-0.4712\text{ m/s}^2)^2 + (1.343\text{ m/s}^2)^2}$$

$$= 1.42\text{ m/s}^2$$ *Ans.*

F12–32. $a_t\,ds = v\,dv$

$$a_t = v\frac{dv}{ds} = (0.2s)(0.2) = (0.04s)\text{ m/s}^2$$

$$a_t = 0.04(50\text{ m}) = 2\text{ m/s}^2$$

$$v = 0.2(50\text{ m}) = 10\text{ m/s}$$

$$a_n = \frac{v^2}{\rho} = \frac{(10\text{ m/s})^2}{500\text{ m}} = 0.2\text{ m/s}^2$$
$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{(2\text{ m/s}^2)^2 + (0.2\text{ m/s}^2)^2}$$
$$= 2.01\text{ m/s}^2 \qquad \textit{Ans.}$$

F12–33. $v_r = \dot{r} = 0$
$$v_\theta = r\dot{\theta} = (120\dot{\theta})\text{ m/s}$$
$$v = \sqrt{v_r^2 + v_\theta^2}$$
$$16.5\text{ m/s} = \sqrt{0^2 + [(120\dot{\theta})\text{ m/s}]^2}$$
$$\dot{\theta} = 0.1375\text{ rad/s} \qquad \textit{Ans.}$$

F12–34. $r = 0.1t^3\big|_{t=1.5\text{ s}} = 0.3375\text{ m}$
$$\dot{r} = 0.3t^2\big|_{t=1.5\text{ s}} = 0.675\text{ m/s}$$
$$\ddot{r} = 0.6t\big|_{t=1.5\text{ s}} = 0.900\text{ m/s}^2$$
$$\theta = 4t^{3/2}\big|_{t=1.5\text{ s}} = 7.348\text{ rad}$$
$$\dot{\theta} = 6t^{1/2}\big|_{t=1.5\text{ s}} = 7.348\text{ rad/s}$$
$$\ddot{\theta} = 3t^{-1/2}\big|_{t=1.5\text{ s}} = 2.449\text{ rad/s}^2$$
$$v_r = \dot{r} = 0.675\text{ m/s}$$
$$v_\theta = r\dot{\theta} = (0.3375\text{ m})(7.348\text{ rad/s}) = 2.480\text{ m/s}$$
$$a_r = \ddot{r} - r\dot{\theta}^2$$
$$= (0.900\text{ m/s}^2) - (0.3375\text{ m})(7.348\text{ rad/s})^2$$
$$= -17.325\text{ m/s}^2$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = (0.3375\text{ m})(2.449\text{ rad/s}^2)$$
$$+ 2(0.675\text{ m/s})(7.348\text{ rad/s}) = 10.747\text{ m/s}^2$$
$$v = \sqrt{v_r^2 + v_\theta^2}$$
$$= \sqrt{(0.675\text{ m/s})^2 + (2.480\text{ m/s})^2}$$
$$= 2.57\text{ m/s} \qquad \textit{Ans.}$$
$$a = \sqrt{a_r^2 + a_\theta^2}$$
$$= \sqrt{(-17.325\text{ m/s}^2)^2 + (10.747\text{ m/s}^2)^2}$$
$$= 20.4\text{ m/s}^2 \qquad \textit{Ans.}$$

F12–35. $r = 0.6\theta$
$$\dot{r} = 0.6\dot{\theta}$$
$$\ddot{r} = 0.6\ddot{\theta}$$
At $\theta = \pi/4$ rad.
$$r = 0.6\left(\tfrac{\pi}{4}\right) = 0.15\pi\text{ m}$$
$$\dot{r} = 0.6(3\text{ rad/s}) = 1.8\text{ m/s}$$
$$\ddot{r} = 0.6(1\text{ rad/s}^2) = 0.6\text{ m/s}^2$$
$$a_r = \ddot{r} - r\dot{\theta}^2 = 0.6\text{ m/s}^2 - \left(0.15\pi\text{ m}\right)(3\text{ rad/s})^2$$
$$= -3.641\text{ m/s}^2$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$
$$= \left(0.15\pi\text{ m}\right)(1\text{ rad/s}^2) + 2(1.8\text{ m/s})(3\text{ rad/s})$$
$$= 11.27\text{ m/s}^2$$
$$a = \sqrt{a_r^2 + a_\theta^2}$$
$$= \sqrt{(-3.641\text{ m/s}^2)^2 + (11.27\text{ m/s}^2)^2}$$
$$= 11.8\text{ m/s}^2 \qquad \textit{Ans.}$$

F12–36. $r = e^\theta$
$$\dot{r} = e^\theta\dot{\theta}$$
$$\ddot{r} = e^\theta\ddot{\theta} + e^\theta\dot{\theta}^2$$
$$a_r = \ddot{r} - r\dot{\theta}^2 = (e^\theta\ddot{\theta} + e^\theta\dot{\theta}^2) - e^\theta\dot{\theta}^2 = e^{\pi/4}(4)$$
$$= 8.77\text{ m/s}^2 \qquad \textit{Ans.}$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = (e^\theta\ddot{\theta}) + (2(e^\theta\dot{\theta})\dot{\theta}) = e^\theta(\ddot{\theta} + 2\dot{\theta}^2)$$
$$= e^{\pi/4}(4 + 2(2)^2)$$
$$= 26.3\text{ m/s}^2 \qquad \textit{Ans.}$$

F12–37. $r = [0.2(1 + \cos\theta)]\text{ m}\big|_{\theta=30^\circ} = 0.3732\text{ m}$
$$\dot{r} = \left[-0.2\,(\sin\theta)\dot{\theta}\right]\text{ m/s}\big|_{\theta=30^\circ}$$
$$= -0.2\sin 30^\circ(3\text{ rad/s})$$
$$= -0.3\text{ m/s}$$
$$v_r = \dot{r} = -0.3\text{ m/s}$$
$$v_\theta = r\dot{\theta} = (0.3732\text{ m})(3\text{ rad/s}) = 1.120\text{ m/s}$$
$$v = \sqrt{v_r^2 + v_\theta^2} = \sqrt{(-0.3\text{ m/s})^2 + (1.120\text{ m/s})^2}$$
$$= 1.16\text{ m/s} \qquad \textit{Ans.}$$

F12–38. $30\text{ m} = r\sin\theta$
$$r = \left(\tfrac{30\text{ m}}{\sin\theta}\right) = (30\csc\theta)\text{ m}$$
$$r = (30\csc\theta)\big|_{\theta=45^\circ} = 42.426\text{ m}$$
$$\dot{r} = -30\csc\theta\,\text{ctn}\,\theta\,\dot{\theta}\,\big|_{\theta=45^\circ} = -\left(42.426\dot{\theta}\right)\text{ m/s}$$
$$v_r = \dot{r} = -(42.426\dot{\theta})\text{ m/s}$$
$$v_\theta = r\dot{\theta} = (42.426\dot{\theta})\text{ m/s}$$
$$v = \sqrt{v_r^2 + v_\theta^2}$$
$$2 = \sqrt{(-42.426\dot{\theta})^2 + (42.426\dot{\theta})^2}$$
$$\dot{\theta} = 0.0333\text{ rad/s} \qquad \textit{Ans.}$$

F12–39. $l_T = 3s_D + s_A$
$$0 = 3v_D + v_A$$
$$0 = 3v_D + 3\text{ m/s}$$
$$v_D = -1\text{ m/s} = 1\text{ m/s}\uparrow \qquad \textit{Ans.}$$

F12–40. $s_B + 2s_A + 2h = l$

$v_B + 2v_A = 0$

$6 + 2v_A = 0 \quad v_A = -3\text{ m/s} = 3\text{ m/s}\uparrow$ *Ans.*

F12–41. $3s_A + s_B = l$

$3v_A + v_B = 0$

$3v_A + 1.5 = 0 \quad v_A = -0.5\text{ m/s} = 0.5\text{ m/s}\uparrow$ *Ans.*

F12–42. $l_T = 4\,s_A + s_F$

$0 = 4\,v_A + v_F$

$0 = 4\,v_A + 3\text{ m/s}$

$v_A = -0.75\text{ m/s} = 0.75\text{ m/s}\uparrow$ *Ans.*

F12–43. $s_A + 2(s_A - a) + (s_A - s_P) = l$

$4s_A - s_P = l + 2a$

$4v_A - v_P = 0$

$4v_A - (-4) = 0$

$4v_A + 4 = 0 \qquad v_A = -1\text{ m/s} = 1\text{ m/s}\nearrow$ *Ans.*

F12–44. $s_C + s_B = l_{CED}$ (1)

$(s_A - s_C) + (s_B - s_C) + s_B = l_{ACDF}$

$s_A + 2s_B - 2s_C = l_{ACDF}$ (2)

Thus

$v_C + v_B = 0$

$v_A + 2v_B - 2v_C = 0$

Eliminating v_C,

$v_A + 4v_B = 0$

Thus,

$1.2\text{ m/s} + 4v_B = 0$

$v_B = -0.3\text{ m/s} = 0.3\text{ m/s}\uparrow$ *Ans.*

F12–45. $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$100\mathbf{i} = 80\mathbf{j} + \mathbf{v}_{B/A}$

$\mathbf{v}_{B/A} = 100\mathbf{i} - 80\mathbf{j}$

$\mathbf{v}_{B/A} = \sqrt{(v_{B/A})_x^2 + (v_{B/A})_y^2}$

$= \sqrt{(100\text{ km/h})^2 + (-80\text{ km/h})^2}$

$= 128\text{ km/h}$ *Ans.*

$\theta = \tan^{-1}\left[\dfrac{(v_{B/A})_y}{(v_{B/A})_x}\right] = \tan^{-1}\left(\dfrac{80\text{ km/h}}{100\text{ km/h}}\right) = 38.7^\circ$ ⦨ *Ans.*

F12–46. $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$(-400\mathbf{i} - 692.82\mathbf{j}) = (650\mathbf{i}) + \mathbf{v}_{B/A}$

$\mathbf{v}_{B/A} = \{-1050\mathbf{i} - 692.82\mathbf{j}\}\text{ km/h}$

$\mathbf{v}_{B/A} = \sqrt{(v_{B/A})_x^2 + (v_{B/A})_y^2}$

$= \sqrt{(1050\text{ km/h})^2 + (692.82\text{ km/h})^2}$

$= 1258\text{ km/h}$ *Ans.*

$\theta = \tan^{-1}\left[\dfrac{(v_{B/A})_y}{(v_{B/A})_x}\right] = \tan^{-1}\left(\dfrac{692.82\text{ km/h}}{1050\text{ km/h}}\right) = 33.4^\circ$ ⦩

Ans.

F12–47. $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$(5\mathbf{i} + 8.660\mathbf{j}) = (12.99\mathbf{i} + 7.5\mathbf{j}) + \mathbf{v}_{B/A}$

$\mathbf{v}_{B/A} = \{-7.990\mathbf{i} + 1.160\mathbf{j}\}\text{ m/s}$

$v_{B/A} = \sqrt{(-7.990\text{ m/s})^2 + (1.160\text{ m/s})^2}$

$= 8.074\text{ m/s}$

$d_{AB} = v_{B/A}t = (8.074\text{ m/s})(4\text{ s}) = 32.3\text{ m}$ *Ans.*

F12–48. $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$

$-20\cos 45^\circ\mathbf{i} + 20\sin 45^\circ\mathbf{j} = 65\mathbf{i} + \mathbf{v}_{A/B}$

$\mathbf{v}_{A/B} = -79.14\mathbf{i} + 14.14\mathbf{j}$

$\mathbf{v}_{A/B} = \sqrt{(-79.14)^2 + (14.14)^2}$

$= 80.4\text{ km/h}$ *Ans.*

$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$

$\dfrac{(20)^2}{0.1}\cos 45^\circ\mathbf{i} + \dfrac{(20)^2}{0.1}\sin 45^\circ\mathbf{j} = 1200\mathbf{i} + \mathbf{a}_{A/B}$

$\mathbf{a}_{A/B} = 1628\mathbf{i} + 2828\mathbf{j}$

$a_{A/B} = \sqrt{(1628)^2 + (2828)^2}$

$= 3.26(10^3)\text{ km/h}^2$ *Ans.*

Chapter 13

F13–1. $s = s_0 + v_0 t + \frac{1}{2}a_c t^2$

$6\text{ m} = 0 + 0 + \frac{1}{2}a(3\text{ s})^2$

$a = 1.333\text{ m/s}^2$

Draw FBD.

$\Sigma F_y = ma_y; \quad N_A - 20(9.81)\text{ N}\cos 30^\circ = 0$

$N_A = 169.91\text{ N}$

$\Sigma F_x = ma_x; \quad T - 20(9.81)\text{ N}\sin 30^\circ$

$- 0.3(169.91\text{ N}) = (20\text{ kg})(1.333\text{ m/s}^2)$

$T = 176\text{ N}$ *Ans.*

F13–2. $(F_f)_{max} = \mu_s N_A = 0.3(245.25\text{ N}) = 73.575\text{ N}$

Since $F = 100\text{ N} > (F_f)_{max}$ when $t = 0$, the crate will start to move immediately after **F** is applied.

Draw FBD.

$+\uparrow\Sigma F_y = ma_y; \quad N_A - 25(9.81)\text{ N} = 0$

$N_A = 245.25\text{ N}$

$\overset{+}{\rightarrow}\Sigma F_x = ma_x;$

$10t^2 + 100 - 0.25(245.25\text{ N}) = (25\text{ kg})a$

$a = (0.4t^2 + 1.5475)\text{ m/s}^2$

$dv = a\,dt$

$\int_0^v dv = \int_0^{4\text{ s}} (0.4t^2 + 1.5475)\,dt$

$v = 14.7\text{ m/s}\rightarrow$ *Ans.*

F13–3. Draw FBD.

$\xrightarrow{+} \Sigma F_x = ma_x;$

$\left(\frac{4}{5}\right)500\text{ N} - (500s)\text{ N} = (10\text{ kg})a$

$a = (40 - 50s)\text{ m/s}^2$

$v\,dv = a\,ds$

$$\int_0^v v\,dv = \int_0^{0.5\text{ m}} (40 - 50s)\,ds$$

$\frac{v^2}{2}\Big|_0^v = \left(40s - 25s^2\right)\Big|_0^{0.5\text{ m}}$

$v = 5.24\text{ m/s}$ *Ans.*

F13–4. Draw FBD.

$\xrightarrow{+} \Sigma F_x = ma_x;\quad 100(s + 1)\text{ N} = (2000\text{ kg})a$

$a = (0.05(s + 1))\text{ m/s}^2$

$v\,dv = a\,ds$

$$\int_0^v v\,dv = \int_0^{10\text{ m}} 0.05(s + 1)\,ds$$

$v = 2.45\text{ m/s}$

F13–5. $F_{sp} = k(l - l_0) = (200\text{ N/m})(0.5\text{ m} - 0.3\text{ m})$

$= 40\text{ N}$

$\theta = \tan^{-1}\left(\frac{0.3\text{ m}}{0.4\text{ m}}\right) = 36.86°$

Draw FBD.

$\xrightarrow{+} \Sigma F_x = ma_x;$

$100\text{ N} - (40\text{ N})\cos 36.86° = (25\text{ kg})a$

$a = 2.72\text{ m/s}^2$

F13–6. Consider no slipping between A and B.

Draw FBD.

$\xrightarrow{+} \Sigma F_x = ma_x;\ 30 = 35a;\ a = 0.857\text{ m/s}^2$

Check if slipping occurs between A and B.

Draw FBD.

$\xrightarrow{+} \Sigma F_x = ma_x;\ 30 - F = 10\,(0.857);$

$F = 21.43\text{ N} < 0.4(10)(9.81) = 39.24\text{ N}$

$a_A = a_B = 0.857\text{ m/s}^2$ *Ans.*

F13–7. Draw FBD.

$\Sigma F_n = m\,\frac{v^2}{\rho};\ (0.3)m(9.81) = m\frac{v^2}{2}$

$v = 2.43\text{ m/s}$ *Ans.*

F13–8. Draw FBD.

$+\downarrow\Sigma F_n = ma_n;\ m(9.81) = m\left(\frac{v^2}{75}\right)$

$v = 27.1\text{ m/s}$ *Ans.*

F13–9. Draw FBD.

$+\downarrow\Sigma F_n = ma_n;\ 70(9.81) + N_p = 70\left(\frac{(36)^2}{120}\right)$

$N_p = 69.3\text{ N}$ *Ans.*

F13–10. Draw FBD.

$\xleftarrow{+} \Sigma F_n = ma_n;$

$N_c \sin 30° + 0.2\,N_c \cos 30° = m\frac{v^2}{150}$

$+\uparrow\Sigma F_b = 0;$

$N_c \cos 30° - 0.2N_c \sin 30° - m(9.81) = 0$

$v = 36.0\text{ m/s}$ *Ans.*

F13–11. Draw FBD.

$\Sigma F_t = ma_t;\quad 10(9.81)\text{ N}\cos 45° = (10\text{ kg})a_t$

$a_t = 6.94\text{ m/s}^2$ *Ans.*

$\Sigma F_n = ma_n;$

$T - 10(9.81)\text{ N}\sin 45° = (10\text{ kg})\,\frac{(3\text{ m/s})^2}{2\text{ m}}$

$T = 114\text{ N}$ *Ans.*

F13–12. Draw FBD.

$\Sigma F_n = ma_n;$

$F_n = (500\text{ kg})\,\frac{(15\text{ m/s})^2}{200\text{ m}} = 562.5\text{ N}$

$\Sigma F_t = ma_t;$

$F_t = (500\text{ kg})(1.5\text{ m/s}^2) = 750\text{ N}$

$F = \sqrt{F_n^2 + F_t^2} = \sqrt{(562.5\text{ N})^2 + (750\text{ N})^2}$

$= 938\text{ N}$ *Ans.*

F13–13. $a_r = \ddot{r} - r\dot{\theta}^2 = 0 - (1.5\text{ m} + (8\text{ m})\sin 45°)\dot{\theta}^2$

$= (-7.157\,\dot{\theta}^2)\text{ m/s}^2$

Draw FBD.

$\Sigma F_z = ma_z;$

$T\cos 45° - m(9.81) = m(0)\quad T = 13.87\,m$

$\Sigma F_r = ma_r;$

$-(13.87m)\sin 45° = m(-7.157\,\dot{\theta}^2)$

$\dot{\theta} = 1.17\text{ rad/s}$ *Ans.*

F13–14. $\theta = \pi t^2\big|_{t=0.5\text{ s}} = (\pi/4)$ rad

$\dot{\theta} = 2\pi t\big|_{t=0.5\text{ s}} = \pi$ rad/s

$\ddot{\theta} = 2\pi$ rad/s^2

$r = 0.6 \sin\theta\big|_{\theta=\pi/4\text{ rad}} = 0.4243$ m

$\dot{r} = 0.6(\cos\theta)\dot{\theta}\big|_{\theta=\pi/4\text{ rad}} = 1.3329$ m/s

$\ddot{r} = 0.6\,[(\cos\theta)\ddot{\theta} - (\sin\theta)\dot{\theta}^2]\big|_{\theta=\pi/4\text{ rad}} = -1.5216$ m/s^2

$a_r = \ddot{r} - r\dot{\theta}^2 = -1.5216\text{ m/s}^2 - (0.4243\text{ m})(\pi\text{ rad/s})^2$

$= -5.7089$ m/s^2

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.4243\text{ m}(2\pi\text{ rad/s}^2)$

$+ 2(1.3329\text{ m/s})(\pi\text{ rad/s})$

$= 11.0404$ m/s^2

Draw FBD.

$\Sigma F_r = ma_r;$

$F\cos 45^\circ - N\cos 45^\circ - 0.2(9.81)\cos 45^\circ$

$= 0.2(-5.7089)$

$\Sigma F_\theta = ma_\theta;$

$F\sin 45^\circ + N\sin 45^\circ - 0.2(9.81)\sin 45^\circ$

$= 0.2(11.0404)$

$N = 2.37$ N $\quad F = 2.72$ N *Ans.*

F13–15. $r = 50e^{2\theta}\big|_{\theta=\pi/6\text{ rad}} = \left[50e^{2(\pi/6)}\right]$ m $= 142.48$ m

$\dot{r} = 50\left(2e^{2\theta}\dot{\theta}\right) = 100e^{2\theta}\dot{\theta}\big|_{\theta=\pi/6\text{ rad}}$

$= \left[100e^{2(\pi/6)}(0.05)\right] = 14.248$ m/s

$\ddot{r} = 100\left((2e^{2\theta}\dot{\theta})\dot{\theta} + e^{2\theta}(\ddot{\theta})\right)\big|_{\theta=\pi/6\text{ rad}}$

$= 100\left[2e^{2(\pi/6)}(0.05^2) + e^{2(\pi/6)}(0.01)\right]$

$= 4.274$ m/s^2

$a_r = \ddot{r} - r\dot{\theta}^2 = 4.274\text{ m/s}^2 - 142.48\text{ m}(0.05\text{ rad/s})^2$

$= 3.918$ m/s^2

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 142.48\text{ m}(0.01\text{ rad/s}^2)$

$+ 2(14.248\text{ m/s})(0.05\text{ rad/s})$

$= 2.850$ m/s^2

Draw FBD.

$\Sigma F_r = ma_r;$

$F_r = (2000\text{ kg})(3.918\text{ m/s}^2) = 7836.55$ N

$\Sigma F_\theta = ma_\theta;$

$F_\theta = (2000\text{ kg})(2.850\text{ m/s}^2) = 5699.31$ N

$F = \sqrt{F_r^2 + F_\theta^2}$

$= \sqrt{(7836.55\text{ N})^2 + (5699.31\text{ N})^2}$

$= 9689.87\text{ N} = 9.69$ kN

F13–16.

$r = (0.6\cos 2\theta)\text{ m}\big|_{\theta=0^\circ} = [0.6\cos 2(0^\circ)]\text{ m} = 0.6$ m

$\dot{r} = (-1.2\sin 2\theta\dot{\theta})\text{ m/s}\big|_{\theta=0^\circ}$

$= \left[-1.2\sin 2(0^\circ)(-3)\right]\text{ m/s} = 0$

$\ddot{r} = -1.2\left(\sin 2\theta\ddot{\theta} + 2\cos 2\theta\dot{\theta}^2\right)\text{ m/s}^2\big|_{\theta=0^\circ}$

$= -21.6$ m/s^2

Thus,

$a_r = \ddot{r} - r\dot{\theta}^2 = -21.6\text{ m/s}^2 - 0.6\text{ m}(-3\text{ rad/s})^2$

$= -27$ m/s^2

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.6\text{ m}(0) + 2(0)(-3\text{ rad/s}) = 0$

Draw FBD.

$\Sigma F_\theta = ma_\theta;\quad F - 0.2(9.81)\text{ N} = 0.2\text{ kg}(0)$

$F = 1.96\text{ N}\uparrow$ *Ans.*

Chapter 14

F14–1. Draw FBD.

$T_1 + \Sigma U_{1-2} = T_2$

$0 + \left(\frac{4}{5}\right)(500\text{ N})(0.5\text{ m}) - \frac{1}{2}(500\text{ N/m})(0.5\text{ m})^2$

$= \frac{1}{2}(10\text{ kg})v^2$

$v = 5.24$ m/s *Ans.*

F14–2. Draw FBD.

$\Sigma F_y = ma_y;\ N_A - 20(9.81)\text{ N}\cos 30^\circ = 0$

$N_A = 169.91$ N

$T_1 + \Sigma U_{1-2} = T_2$

$0 + 300\text{ N}(10\text{ m}) - 0.3(169.91\text{ N})(10\text{ m})$

$- 20(9.81)\text{ N}(10\text{ m})\sin 30^\circ$

$= \frac{1}{2}(20\text{ kg})v^2$

$v = 12.3$ m/s *Ans.*

F14–3. Draw FBD.

$T_1 + \Sigma U_{1-2} = T_2$

$0 + 2\left[\int_0^{15\text{ m}} (600 + 2s^2)\text{ N}\,ds\right] - 100(9.81)\text{ N}(15\text{ m})$

$= \frac{1}{2}(100\text{ kg})v^2$

$v = 12.5$ m/s *Ans.*

F14–4. Draw FBD.

$$T_1 + \Sigma U_{1-2} = T_2$$

$$\tfrac{1}{2}(1800\text{ kg})(125\text{ m/s})^2 - \left[\frac{(50\,000\text{ N} + 20\,000\text{ N})}{2}(400\text{ m})\right] = \tfrac{1}{2}(1800\text{ kg})v^2$$

$$v = 8.33\text{ m/s} \qquad \textit{Ans.}$$

F14–5. Draw FBD.

$$T_1 + \Sigma U_{1-2} = T_2$$

$$\tfrac{1}{2}(10\text{ kg})(5\text{ m/s})^2 + 100\text{ N}s' + [10(9.81)\text{ N}]\,s'\sin 30^\circ - \tfrac{1}{2}(200\text{ N/m})(s')^2 = 0$$

$$s' = 2.09\text{ m}$$

$$s = 0.6\text{ m} + 2.09\text{ m} = 2.69\text{ m} \qquad \textit{Ans.}$$

F14–6. Draw FBD.

$$T_A + \Sigma U_{A-B} = T_B$$

Consider difference in cord length $AC - BC$, which is distance F moves.

$$0 + 50\text{ N}\left(\sqrt{(1.5\text{ m})^2 + (2\text{ m})^2} - 1.5\text{ m}\right) = \tfrac{1}{2}\left(2.5\text{ kg}\right)v_B^2$$

$$v_B = 6.32\text{ m/s} \qquad \textit{Ans.}$$

F14–7. Draw FBD.

$$\xrightarrow{+}\Sigma F_x = ma_x;$$

$$30\left(\tfrac{4}{5}\right) = 20a \quad a = 1.2\text{ m/s}^2 \rightarrow$$

$$v = v_0 + a_c t$$

$$v = 0 + 1.2(4) = 4.8\text{ m/s}$$

$$P = \mathbf{F}\cdot\mathbf{v} = F(\cos\theta)v$$

$$= 30\left(\tfrac{4}{5}\right)(4.8)$$

$$= 115\text{ W} \qquad \textit{Ans.}$$

F14–8. Draw FBD.

$$\xrightarrow{+}\Sigma F_x = ma_x;$$

$$10s = 20a \quad a = 0.5s\text{ m/s}^2 \rightarrow$$

$$v\,dv = a\,ds$$

$$\int_1^v v\,dv = \int_0^{5\text{ m}} 0.5\,s\,ds$$

$$v = 3.674\text{ m/s}$$

$$P = \mathbf{F}\cdot\mathbf{v} = [10(5)](3.674) = 184\text{ W} \qquad \textit{Ans.}$$

F14–9. Draw FBD.

$$+\uparrow\Sigma F_y = 0;$$

$$T_A - 450\text{ N} = 0 \qquad T_A = 450\text{ N}$$

$$+\uparrow\Sigma F_y = 0;$$

$$450\text{ N} + 450\text{ N} - T_B = 0 \quad T_B = 900\text{ N}$$

$$P_{out} = \mathbf{T}_B\cdot\mathbf{v}_B = (900\text{ N})(1\text{ m/s}) = 900\text{ W}$$

$$P_{in} = \frac{P_{out}}{\varepsilon} = \frac{900\text{ W}}{0.8} = 1.12\text{ kW} \qquad \textit{Ans.}$$

F14–10. Draw FBD.

$$\Sigma F_{y'} = ma_{y'};\ N - 20(9.81)\cos 30^\circ = 20(0)$$

$$N = 169.91\text{ N}$$

$$\Sigma F_{x'} = ma_{x'};$$

$$F - 20(9.81)\sin 30^\circ - 0.2(169.91) = 0$$

$$F = 132.08\text{ N}$$

$$P = \mathbf{F}\cdot\mathbf{v} = 132.08(5) = 660\text{ W} \qquad \textit{Ans.}$$

F14–11. Draw FBD.

$$+\uparrow\Sigma F_y = ma_y;$$

$$T - 50(9.81) = 50(0) \quad T = 490.5\text{ N}$$

$$P_{out} = \mathbf{T}\cdot\mathbf{v} = 490.5(1.5) = 735.75\text{ W}$$

Also, for a point on the other cable

$$P_{out} = \left(\frac{490.5}{2}\right)(1.5)(2) = 735.75\text{ W}$$

$$P_{in} = \frac{P_{out}}{\varepsilon} = \frac{735.75}{0.8} = 920\text{ W} \qquad \textit{Ans.}$$

F14–12. $2s_A + s_P = l$

$$2a_A + a_P = 0$$

$$2a_A + 6 = 0$$

$$a_A = -3\text{ m/s}^2 = 3\text{ m/s}^2\uparrow$$

Draw FBD.

$$\Sigma F_y = ma_y; \quad T_A - 490.5\text{ N} = (50\text{ kg})(3\text{ m/s}^2)$$

$$T_A = 640.5\text{ N}$$

$$P_{out} = \mathbf{T}\cdot\mathbf{v} = (640.5\text{ N/2})(12) = 3843\text{ W}$$

$$P_{in} = \frac{P_{out}}{\varepsilon} = \frac{3843}{0.8} = 4803.75\text{ W} = 4.80\text{ kW} \qquad \textit{Ans.}$$

F14–13. Datum at B.

$$T_A + V_A = T_B + V_B$$

$$0 + 2(9.81)(1.5) = \tfrac{1}{2}(2)(v_B)^2 + 0$$

$$v_B = 5.42\text{ m/s} \qquad \textit{Ans.}$$

$$+\uparrow\Sigma F_n = ma_n;\ T - 2(9.81) = 2\left(\frac{(5.42)^2}{1.5}\right)$$

$$T = 58.9\text{ N} \qquad \textit{Ans.}$$

F14–14. Datum at B.

$$T_A + V_A = T_B + V_B$$
$$\tfrac{1}{2}m_A v_A^2 + mgh_A = \tfrac{1}{2}m_B v_B^2 + mgh_B$$
$$\left[\tfrac{1}{2}(2\text{ kg})(1\text{ m/s})^2\right] + [2(9.81)\text{ N}(4\text{ m})] = \left[\tfrac{1}{2}(2\text{ kg})v_B^2\right] + [0]$$
$$v_B = 8.915\text{ m/s} = 8.92\text{ m/s} \qquad \textit{Ans.}$$
$$+\uparrow\Sigma F_n = ma_n;\quad N_B - 2(9.81)\text{ N} = (2\text{ kg})\left(\frac{(8.915\text{ m/s})^2}{2\text{ m}}\right)$$
$$N_B = 99.1\text{ N} \qquad \textit{Ans.}$$

F14–15. Datum at A.

$$T_1 + V_1 = T_2 + V_2$$
$$\tfrac{1}{2}(2)(4)^2 + 0 + \tfrac{1}{2}(30)(2-1)^2 = \tfrac{1}{2}(2)(v)^2 - 2(9.81)(1) + \tfrac{1}{2}(30)(\sqrt{5}-1)^2$$
$$v = 5.26\text{ m/s} \qquad \textit{Ans.}$$

F14–16. Datum at B.

$$T_A + V_A = T_B + V_B$$
$$0 + \tfrac{1}{2}(60)(0.75-0.15)^2 + 2.5(9.81)(0.75) = \tfrac{1}{2}(2.5)v_B^2 + \tfrac{1}{2}(60)(0.3-0.15)^2 + 0$$
$$v_B = 4.78\text{ m/s} \qquad \textit{Ans.}$$

F14–17. Datum at initial position.

$$T_1 + V_1 = T_2 + V_2$$
$$\tfrac{1}{2}mv_1^2 + mgy_1 + \tfrac{1}{2}ks_1^2 = \tfrac{1}{2}mv_2^2 + mgy_2 + \tfrac{1}{2}ks_2^2$$
$$[0] + [0] + [0] = [0] + [-35\text{ kg}(9.81\text{ m/s}^2)(1.5\text{ m} + s)] + \left[2\left(\tfrac{1}{2}(16\,000\text{ N/m})s^2\right) + \tfrac{1}{2}(24\,000\text{ N/m})(s - 0.075\text{ m})^2\right]$$
$$s = s_A = s_C = 0.1704\text{ m} = 0.170\text{ m} \qquad \textit{Ans.}$$

Also,

$$s_B = 0.1704\text{ m} - 0.075\text{ m} = 0.0954\text{ m} \qquad \textit{Ans.}$$

F14–18. Datum at A.

$$T_A + V_A = T_B + V_B$$
$$\tfrac{1}{2}mv_A^2 + \left(\tfrac{1}{2}ks_A^2 + mgy_A\right) = \tfrac{1}{2}mv_B^2 + \left(\tfrac{1}{2}ks_B^2 + mgy_B\right)$$
$$\tfrac{1}{2}(4\text{ kg})(2\text{ m/s})^2 + \tfrac{1}{2}(400\text{ N/m})(0.1\text{ m} - 0.2\text{ m})^2 + 0 = \tfrac{1}{2}(4\text{ kg})v_B^2 + \tfrac{1}{2}(400\text{ N/m})(\sqrt{(0.4\text{ m})^2 + (0.3\text{ m})^2} - 0.2\text{ m})^2 + [4(9.81)\text{ N}](-(0.1\text{ m} + 0.3\text{ m}))$$
$$v_B = 1.962\text{ m/s} = 1.96\text{ m/s} \qquad \textit{Ans.}$$

Chapter 15

F15–1. Draw FBD.

$$(\overset{+}{\rightarrow})\quad m(v_1)_x + \Sigma\int_{t1}^{t2} F_x\,dt = m(v_2)_x$$
$$(0.5\text{ kg})(25\text{ m/s})\cos 45^\circ - \int F_x\,dt = (0.5\text{ kg})(10\text{ m/s})\cos 30^\circ$$
$$I_x = \int F_x\,dt = 4.509\text{ N}\cdot\text{s}$$
$$(+\uparrow)\quad m(v_1)_y + \Sigma\int_{t1}^{t2} F_y\,dt = m(v_2)_y$$
$$-(0.5\text{ kg})(25\text{ m/s})\sin 45^\circ + \int F_y\,dt = (0.5\text{ kg})(10\text{ m/s})\sin 30^\circ$$
$$I_y = \int F_y\,dt = 11.339\text{ N}\cdot\text{s}$$
$$I = \int F\,dt = \sqrt{(4.509\text{ N}\cdot\text{s})^2 + (11.339\text{ N}\cdot\text{s})^2} = 12.2\text{ N}\cdot\text{s} \qquad \textit{Ans.}$$

F15–2. Draw FBD.

$$(+\uparrow)\quad m(v_1)_y + \Sigma\int_{t1}^{t2} F_y\,dt = m(v_2)_y$$
$$0 + N(4\text{ s}) + (500\text{ N})(4\text{ s})\sin 30^\circ - (75\text{ kg})(9.81\text{ m/s}^2)(4\text{ s}) = 0$$
$$N = 485.75\text{ N}$$
$$(\overset{+}{\rightarrow})\quad m(v_1)_x + \Sigma\int_{t1}^{t2} F_x\,dt = m(v_2)_x$$
$$0 + (500\text{ N})(4\text{ s})\cos 30^\circ - 0.2(485.75\text{ N})(4\text{ s}) = (75\text{ kg})v$$
$$v = 17.9\text{ m/s} \qquad \textit{Ans.}$$

F15–3. Draw FBD.

Time to start motion,

$$+\uparrow\Sigma F_y = 0;\quad N - 25(9.81)\text{ N} = 0 \quad N = 245.25\text{ N}$$
$$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad 20t^2 - 0.3(245.25\text{ N}) = 0 \quad t = 1.918\text{ s}$$
$$(\overset{+}{\rightarrow})\quad m(v_1)_x + \Sigma\int_{t1}^{t2} F_x\,dt = m(v_2)_x$$
$$0 + \int_{1.918\text{ s}}^{4\text{ s}} 20t^2\,dt - (0.25(245.25\text{ N}))(4\text{ s} - 1.918\text{ s}) = (25\text{ kg})v$$
$$v = 10.1\text{ m/s} \qquad \textit{Ans.}$$

F15–4. Draw FBD.

$$(\xrightarrow{+})\quad m(v_1)_x + \Sigma\int_{t1}^{t2} F_x\,dt = m(v_2)_x$$

$$(1500\text{ kg})(0) + \left[\tfrac{1}{2}(6000\text{ N})(2\text{ s}) + (6000\text{ N})(6\text{ s} - 2\text{ s})\right] = (1500\text{ kg})\,v$$

$v = 20\text{ m/s}$ *Ans.*

F15–5. Draw FBD.

SUV and trailer,

$$m(v_1)_x + \Sigma\int_{t1}^{t2} F_x\,dt = m(v_2)_x$$

$0 + (9000\text{ N})(20\text{ s}) = (1500\text{ kg} + 2500\text{ kg})v$

$v = 45.0\text{ m/s}$ *Ans.*

Trailer,

$$m(v_1)_x + \Sigma\int_{t1}^{t2} F_x\,dt = m(v_2)_x$$

$0 + T(20\text{ s}) = (1500\text{ kg})(45.0\text{ m/s})$

$T = 3375\text{ N} = 3.375\text{ kN}$ *Ans.*

F15–6. Draw FBD.

Block B:

$(+\downarrow)\ mv_1 + \int F\,dt = mv_2$

$0 + 3.6(9.81)(5) - T(5) = 3.6(0.3)$

$T = 35.1\text{ N}$ *Ans.*

Block A:

$(\xrightarrow{+})\ mv_1 + \int F\,dt = mv_2$

$0 + 35.1(5) - \mu_k(4.5)(9.81)(5) = 4.5(0.3)$

$\mu_k = 0.789$ *Ans.*

F15–7. $(\xrightarrow{+})\ m_A(v_A)_1 + m_B(v_B)_1 = m_A(v_A)_2 + m_B(v_B)_2$

$(20(10^3)\text{ kg})(3\text{ m/s}) + (15(10^3)\text{ kg})(-1.5\text{ m/s})$

$= (20(10^3)\text{ kg})(v_A)_2 + (15(10^3)\text{ kg})(2\text{ m/s})$

$(v_A)_2 = 0.375\text{ m/s} \rightarrow$ *Ans.*

$$(\xrightarrow{+})\quad m(v_B)_1 + \Sigma\int_{t1}^{t2} F\,dt = m(v_B)_2$$

$(15(10^3)\text{ kg})(-1.5\text{ m/s}) + F_{avg}(0.5\text{ s})$

$= (15(10^3)\text{ kg})(2\text{ m/s})$

$F_{avg} = 105(10^3)\text{ N} = 105\text{ kN}$ *Ans.*

F15–8. $(\xrightarrow{+})\quad m_p[(v_p)_1]_x + m_c[(v)_1]_x = (m_p + m_c)v_2$

$5\left[10\left(\tfrac{4}{5}\right)\right] + 0 = (5 + 20)v_2$

$v_2 = 1.6\text{ m/s}$ *Ans.*

F15–9. Datum at B.

$T_1 + V_1 = T_2 + V_2$

$\tfrac{1}{2}m_A(v_A)_1^2 + (V_g)_1 = \tfrac{1}{2}m_A(v_A)_2^2 + (V_g)_2$

$\tfrac{1}{2}(5)(5)^2 + 5(9.81)(1.5) = \tfrac{1}{2}(5)(v_A)_2^2 + 0$

$(v_A)_2 = 7.378\text{ m/s}$

$(\xleftarrow{+})\quad m_A(v_A)_2 + m_B(v_B)_2 = (m_A + m_B)v$

$5(7.378) + 0 = (5 + 8)v$

$v = 2.84\text{ m/s}$ *Ans.*

F15–10. $(\xrightarrow{+})\ m_A(v_A)_1 + m_B(v_B)_1 = m_A(v_A)_2 + m_B(v_B)_2$

$0 + 0 = 10(v_A)_2 + 15(v_B)_2$ (1)

$T_1 + V_1 = T_2 + V_2$

$\tfrac{1}{2}m_A(v_A)_1^2 + \tfrac{1}{2}m_B(v_B)_1^2 + (V_e)_1$

$= \tfrac{1}{2}m_A(v_A)_2^2 + \tfrac{1}{2}m_B(v_B)_2^2 + (V_e)_2$

$0 + 0 + \tfrac{1}{2}\left[5\left(10^3\right)\right]\left(0.2^2\right)$

$= \tfrac{1}{2}(10)(v_A)_2^2 + \tfrac{1}{2}(15)(v_B)_2^2 + 0$

$5(v_A)_2^2 + 7.5\,(v_B)_2^2 = 100$ (2)

Solving Eqs. (1) and (2),

$(v_B)_2 = 2.31\text{ m/s} \rightarrow$ *Ans.*

$(v_A)_2 = -3.464\text{ m/s} = 3.46\text{ m/s} \leftarrow$ *Ans.*

F15–11. $(\xleftarrow{+})\quad m_A(v_A)_1 + m_B(v_B)_1 = (m_A + m_B)v_2$

$0 + 10(15) = (15 + 10)v_2$

$v_2 = 6\text{ m/s}$

$T_1 + V_1 = T_2 + V_2$

$\tfrac{1}{2}(m_A + m_B)v_2^2 + (V_e)_2 = \tfrac{1}{2}(m_A + m_B)v_3^2 + (V_e)_3$

$\tfrac{1}{2}(15 + 10)\left(6^2\right) + 0 = 0 + \tfrac{1}{2}\left[10\left(10^3\right)\right]s_{max}^2$

$s_{max} = 0.3\text{ m} = 300\text{ mm}$ *Ans.*

F15–12. $(\xrightarrow{+})\quad 0 + 0 = m_p(v_p)_x - m_c v_c$

$0 = (20\text{ kg})(v_p)_x - (250\text{ kg})v_c$

$(v_p)_x = 12.5\,v_c$ (1)

$\mathbf{v}_p = \mathbf{v}_c + \mathbf{v}_{p/c}$

$(v_p)_x\mathbf{i} + (v_p)_y\mathbf{j} = -v_c\mathbf{i} + [(400\text{ m/s})\cos 30°\mathbf{i}$

$+ (400\text{ m/s})\sin 30°\mathbf{j}]$

$(v_p)_x\mathbf{i} + (v_p)_y\mathbf{j} = (346.41 - v_c)\mathbf{i} + 200\mathbf{j}$

$(v_p)_x = 346.41 - v_c$

$(v_p)_y = 200\text{ m/s}$

$(v_p)_x = 320.75\text{ m/s}\quad v_c = 25.66\text{ m/s}$

$v_p = \sqrt{(v_p)_x^2 + (v_p)_y^2}$

$= \sqrt{(320.75\text{ m/s})^2 + (200\text{ m/s})^2}$

$= 378\text{ m/s}$ *Ans.*

F15–13. $(\xrightarrow{+})\quad e = \dfrac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$

$$= \frac{(9\text{ m/s}) - (1\text{ m/s})}{(8\text{ m/s}) - (-2\text{ m/s})} = 0.8$$

F15–14. $(\xrightarrow{+})\ m_A(v_A)_1 + m_B(v_B)_1 = m_A(v_A)_2 + m_B(v_B)_2$

$[15(10^3)\text{ kg}](5\text{ m/s}) + [25(10^3)](-7\text{ m/s})$

$= [15(10^3)\text{ kg}](v_A)_2 + [25(10^3)](v_B)_2$

$15(v_A)_2 + 25(v_B)_2 = -100 \qquad (1)$

Using the coefficient of restitution equation,

$(\xrightarrow{+})\quad e = \dfrac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$

$$0.6 = \frac{(v_B)_2 - (v_A)_2}{5\text{ m/s} - (-7\text{ m/s})}$$

$(v_B)_2 - (v_A)_2 = 7.2 \qquad (2)$

Solving,

$(v_B)_2 = 0.2\text{ m/s} \rightarrow$ *Ans.*

$(v_A)_2 = -7\text{ m/s} = 7\text{ m/s} \leftarrow$ *Ans.*

F15–15. Datum at B.

$T_1 + V_1 = T_2 + V_2$

$\frac{1}{2}m(v_A)_1^2 + mg(h_A)_1 = \frac{1}{2}m(v_A)_2^2 + mg(h_A)_2$

$\frac{1}{2}(15\text{ kg})(1.5\text{ m/s})^2 + 15\text{ kg}(9.81\text{ m/s}^2)(3\text{ m})$

$= \frac{1}{2}(15\text{ kg})(v_A)_2^2 + 0$

$(v_A)_2 = 7.817\text{ m/s} \leftarrow$

$(\xleftarrow{+})\quad m_A(v_A)_2 + m_B(v_B)_2 = m_A(v_A)_3 + m_B(v_B)_3$

$(15\text{ kg})(7.817\text{ m/s}) + 0$

$= (15\text{ kg})(v_A)_3 + (40\text{ kg})(v_B)_3$

$15(v_A)_3 + 40(v_B)_3 = 117.26 \qquad (1)$

$(\xleftarrow{+})\quad e = \dfrac{(v_B)_3 - (v_A)_3}{(v_A)_2 - (v_B)_2}$

$$0.6 = \frac{(v_B)_3 - (v_A)_3}{7.817\text{ m/s} - 0}$$

$(v_B)_3 - (v_A)_3 = 4.690 \qquad (2)$

Solving Eqs. (1) and (2) yields

$(v_B)_3 = 3.411\text{ m/s} \leftarrow$

$(v_A)_3 = -1.279\text{ m/s} = 1.28\text{ m/s} \rightarrow$ *Ans.*

F15–16. $(+\uparrow)\quad m[(v_b)_1]_y = m[(v_b)_2]_y$

$[(v_b)_2]_y = [(v_b)_1]_y = (20\text{ m/s})\sin 30° = 10\text{ m/s} \uparrow$

$(\xrightarrow{+})\quad e = \dfrac{(v_w)_2 - [(v_b)_2]_x}{[(v_b)_1]_x - (v_w)_1}$

$$0.75 = \frac{0 - [(v_b)_2]_x}{(20\text{ m/s})\cos 30° - 0}$$

$[(v_b)_2]_x = -12.99\text{ m/s} = 12.99\text{ m/s} \leftarrow$

$(v_b)_2 = \sqrt{[(v_b)_2]_x^2 + [(v_b)_2]_y^2}$

$= \sqrt{(12.99\text{ m/s})^2 + (10\text{ m/s})^2}$

$= 16.4\text{ m/s}$ *Ans.*

$\theta = \tan^{-1}\left(\dfrac{[(v_b)_2]_y}{[(v_b)_2]_x}\right) = \tan^{-1}\left(\dfrac{10\text{ m/s}}{12.99\text{ m/s}}\right)$

$= 37.6°$ *Ans.*

F15–17. $\Sigma m(v_x)_1 = \Sigma m(v_x)_2$

$0 + 0 = 2(1) + 11(v_{Bx})_2$

$(v_{Bx})_2 = -0.1818\text{ m/s}$

$\Sigma m(v_y)_1 = \Sigma m(v_y)_2$

$2(3) + 0 = 0 + 11(v_{By})_2$

$(v_{By})_2 = 0.545\text{ m/s}$

$(v_B)_2 = \sqrt{(-0.1818)^2 + (0.545)^2}$

$= 0.575\text{ m/s}$ *Ans.*

F15–18. $+\nearrow\quad 1(3)\left(\frac{3}{5}\right) - 1(4)\left(\frac{4}{5}\right)$

$= 1(v_B)_{2x'} + 1(v_A)_{2x'}$

$+\nearrow\quad 0.5 = [(v_A)_{2x'} - (v_B)_{2x'}]/[(3)\left(\frac{3}{5}\right) - (-4)\left(\frac{4}{5}\right)]$

Solving,

$(v_A)_{2x'} = 0.550\text{ m/s}, (v_B)_{2x'} = -1.95\text{ m/s}$

Disc A,

$+\nwarrow\quad -1(4)\left(\frac{3}{5}\right) = 1(v_A)_{2y'}$

$(v_A)_{2y'} = -2.40\text{ m/s}$

Disc B,

$+\nwarrow\quad -1(3)\left(\frac{4}{5}\right) = 1(v_B)_{2y'}$

$(v_B)_{2y'} = -2.40\text{ m/s}$

$(v_A)_2 = \sqrt{(0.550)^2 + (-2.40)^2} = 2.46\text{ m/s}$ *Ans.*

$(v_B)_2 = \sqrt{(-1.95)^2 + (-2.40)^2} = 3.09\text{ m/s}$ *Ans.*

F15–19. $H_O = \Sigma mvd;$

$H_O = \left[2(10)\left(\frac{4}{5}\right)\right](4) - \left[2(10)\left(\frac{3}{5}\right)\right](3)$

$= 28 \text{ kg}\cdot\text{m}^2/\text{s}$

F15–20. $H_P = \Sigma mvd;$

$H_P = [2(15)\sin 30°](2) - [2(15)\cos 30°](5)$

$= -99.9 \text{ kg}\cdot\text{m}^2/\text{s} = 99.9 \text{ kg}\cdot\text{m}^2/\text{s}$ ↻

F15–21. Draw an FBD of the rod and block.

$(H_z)_1 + \Sigma\int M_z\,dt = (H_z)_2$

$5(2)(1.5) + 5(1.5)(3) = 5v(1.5)$

$v = 5 \text{ m/s}$ *Ans.*

F15–22. Draw an FBD of the rod and block.

$(H_z)_1 + \Sigma\int M_z\,dt = (H_z)_2$

$0 + \int_0^{4\text{ s}} (10t)\left(\frac{4}{5}\right)(1.5)dt = 5v(1.5)$

$v = 12.8 \text{ m/s}$ *Ans.*

F15–23. Draw an FBD of the rod and sphere.

$(H_z)_1 + \Sigma\int M_z\,dt = (H_z)_2$

$0 + \int_0^{5\text{ s}} 0.9t^2\,dt = 2v(0.6)$

$v = 31.2 \text{ m/s}$ *Ans.*

F15–24. Draw an FBD of the rod and spheres.

$(H_z)_1 + \Sigma\int M_z\,dt = (H_z)_2$

$0 + \int_0^{4\text{ s}} 8t dt + 2(10)(0.5)(4) = 2[10v(0.5)]$

$v = 10.4 \text{ m/s}$ *Ans.*

Chapter 16

F16–1. $\theta = (20 \text{ rev})\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right) = 40\pi \text{ rad}$

$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$

$(30 \text{ rad/s})^2 = 0^2 + 2\alpha_c[(40\pi \text{ rad}) - 0]$

$\alpha_c = 3.581 \text{ rad/s}^2 = 3.58 \text{ rad/s}^2$ *Ans.*

$\omega = \omega_0 + \alpha_c t$

$30 \text{ rad/s} = 0 + (3.581 \text{ rad/s}^2)t$

$t = 8.38 \text{ s}$ *Ans.*

F16–2. $\frac{d\omega}{d\theta} = 2(0.005\theta) = (0.01\theta)$

$\alpha = \omega\frac{d\omega}{d\theta} = \left(0.005\,\theta^2\right)(0.01\theta) = 50\left(10^{-6}\right)\theta^3 \text{ rad/s}^2$

When $\theta = 20 \text{ rev}(2\pi \text{ rad}/1 \text{ rev}) = 40\pi \text{ rad}$,

$\alpha = \left[50\left(10^{-6}\right)(40\pi)^3\right] \text{ rad/s}^2$

$= 99.22 \text{ rad/s}^2 = 99.2 \text{ rad/s}^2$ *Ans.*

F16–3. $\omega = 4\theta^{1/2}$

$150 \text{ rad/s} = 4\,\theta^{1/2}$

$\theta = 1406.25 \text{ rad}$

$dt = \frac{d\theta}{\omega}$

$\int_0^t dt = \int_{1\text{ rad}}^{\theta} \frac{d\theta}{4\theta^{1/2}}$

$t\Big|_0^t = \frac{1}{2}\theta^{1/2}\Big|_{1\text{ rad}}^{\theta}$

$t = \frac{1}{2}\theta^{1/2} - \frac{1}{2}$

$t = \frac{1}{2}(1406.25)^{1/2} - \frac{1}{2} = 18.25 \text{ s}$ *Ans.*

F16–4. $\omega = \frac{d\theta}{dt} = (1.5t^2 + 15) \text{ rad/s}$

$\alpha = \frac{d\omega}{dt} = (3t) \text{ rad/s}^2$

$\omega = [1.5(3^2) + 15] \text{ rad/s} = 28.5 \text{ rad/s}$

$\alpha = 3(3) \text{ rad/s}^2 = 9 \text{ rad/s}^2$

$v = \omega r = (28.5 \text{ rad/s})(0.225 \text{ m}) = 6.41 \text{ m/s}$ *Ans.*

$a = \alpha r = (9 \text{ rad/s}^2)(0.225 \text{ m}) = 2.02 \text{ m/s}^2$ *Ans.*

F16–5. $\omega\,d\omega = \alpha\,d\theta$

$\int_{2\text{ rad/s}}^{\omega} \omega\,d\omega = \int_0^{\theta} 0.5\theta\,d\theta$

$\frac{\omega^2}{2}\Big|_{2\text{ rad/s}}^{\omega} = 0.25\theta^2\Big|_0^{\theta}$

$\omega = (0.5\theta^2 + 4)^{1/2} \text{ rad/s}$

When $\theta = 2 \text{ rev} = 4\pi \text{ rad}$,

$\omega = [0.5(4\pi)^2 + 4]^{1/2} \text{ rad/s} = 9.108 \text{ rad/s}$

$v_P = \omega r = (9.108 \text{ rad/s})(0.2 \text{ m}) = 1.82 \text{ m/s}$ *Ans.*

$(a_P)_t = \alpha r = (0.5\theta \text{ rad/s}^2)(0.2 \text{ m})\Big|_{\theta = 4\pi \text{ rad}}$

$= 1.257 \text{ m/s}^2$

$(a_P)_n = \omega^2 r = (9.108 \text{ rad/s})^2(0.2 \text{ m}) = 16.59 \text{ m/s}^2$

$a_P = \sqrt{(a_P)_t^2 + (a_P)_n^2}$

$= \sqrt{(1.257 \text{ m/s}^2)^2 + (16.59 \text{ m/s}^2)^2}$

$= 16.6 \text{ m/s}^2$ *Ans.*

F16–6. The tangential acceleration of P is the same for A and B.

$$\alpha_B = \alpha_A\left(\frac{r_A}{r_B}\right)$$

$$= (4.5\ \text{rad/s}^2)\left(\tfrac{0.075\ \text{m}}{0.225\ \text{m}}\right) = 1.5\ \text{rad/s}^2$$

$$\omega_B = (\omega_B)_0 + \alpha_B t$$

$$\omega_B = 0 + (1.5\ \text{rad/s}^2)(3\ \text{s}) = 4.5\ \text{rad/s}$$

$$\theta_B = (\theta_B)_0 + (\omega_B)_0 t + \tfrac{1}{2}\alpha_B t^2$$

$$\theta_B = 0 + 0 + \tfrac{1}{2}(1.5\ \text{rad/s}^2)(3\ \text{s})^2$$

$$\theta_B = 6.75\ \text{rad}$$

$$v_C = \omega_B r_D = (4.5\ \text{rad/s})(0.125\ \text{m})$$

$$= 0.5625\ \text{m/s} \qquad \textit{Ans.}$$

$$s_C = \theta_B r_D = (6.75\ \text{rad})(0.125\ \text{m}) = 0.84375\ \text{m}$$

$$= 844\ \text{mm} \qquad \textit{Ans.}$$

F16–7. Vector Analysis

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$

$$-v_B\,\mathbf{j} = (3\mathbf{i})\ \text{m/s}$$

$$+ (\omega\mathbf{k}) \times (-1.5\cos 30°\mathbf{i} + 1.5\sin 30°\mathbf{j})$$

$$-v_B\,\mathbf{j} = [3 - \omega_{AB}(1.5\sin 30°)]\mathbf{i} - \omega(1.5\cos 30°)\mathbf{j}$$

$$0 = 3 - \omega(1.5\sin 30°) \qquad (1)$$

$$-v_B = 0 - \omega(1.5\cos 30°) \qquad (2)$$

$$\omega = 4\ \text{rad/s} \qquad v_B = 5.20\ \text{m/s} \qquad \textit{Ans.}$$

Scalar Analysis

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

$$\left[\downarrow v_B\right] = \left[\underset{\rightarrow}{3}\right] + \left[\omega(1.5)\ \measuredangle\, 30°\right]$$

This yields Eqs. (1) and (2).

F16–8. Vector Analysis

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$

$$(v_B)_x\mathbf{i} + (v_B)_y\mathbf{j} = \mathbf{0} + (-10\mathbf{k}) \times (-0.6\mathbf{i} + 0.6\mathbf{j})$$

$$(v_B)_x\mathbf{i} + (v_B)_y\mathbf{j} = 6\mathbf{i} + 6\mathbf{j}$$

$$(v_B)_x = 6\ \text{m/s and } (v_B)_y = 6\ \text{m/s}$$

$$v_B = \sqrt{(v_B)_x^2 + (v_B)_y^2}$$

$$= \sqrt{(6\ \text{m/s})^2 + (6\ \text{m/s})^2}$$

$$= 8.49\ \text{m/s} \qquad \textit{Ans.}$$

Scalar Analysis

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

$$\left[\underset{\rightarrow}{(v_B)_x}\right] + \left[(v_B)_y \uparrow\right] = \left[0\right] + \left[\measuredangle 45°\ \ 10\left(\frac{0.6}{\cos 45°}\right)\right]$$

$$\overset{+}{\rightarrow}\ (v_B)_x = 0 + 10(0.6/\cos 45°)\cos 45° = 6\ \text{m/s} \rightarrow$$

$$+\uparrow (v_B)_y = 0 + 10(0.6/\cos 45°)\sin 45° = 6\ \text{m/s}\uparrow$$

F16–9. Vector Analysis

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$

$$(1.2\ \text{m/s})\mathbf{i} = (-0.6\ \text{m/s})\mathbf{i} + (-\omega\mathbf{k}) \times (0.9\ \text{m})\mathbf{j}$$

$$1.2\mathbf{i} = (-0.6 + 0.9\omega)\mathbf{i}$$

$$\omega = 2\ \text{rad/s} \qquad \textit{Ans.}$$

Scalar Analysis

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

$$\left[\underset{\rightarrow}{1.2}\right] = \left[\underset{\leftarrow}{0.6}\right] + \left[\underset{\rightarrow}{\omega(0.9)}\right]$$

$$\overset{+}{\rightarrow} \quad 1.2 = -0.6 + \omega(0.9); \qquad \omega = 2\ \text{rad/s}$$

F16–10. Vector Analysis

$$\mathbf{v}_A = \boldsymbol{\omega}_{OA} \times \mathbf{r}_A$$

$$= (12\ \text{rad/s})\mathbf{k} \times (0.3\ \text{m})\mathbf{j}$$

$$= [-3.6\mathbf{i}]\ \text{m/s}$$

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}$$

$$v_B\,\mathbf{j} = (-3.6\ \text{m/s})\mathbf{i}$$

$$+ (\omega_{AB}\,\mathbf{k}) \times (0.6\cos 30°\mathbf{i} - 0.6\sin 30°\mathbf{j})\ \text{m}$$

$$v_B\,\mathbf{j} = [\omega_{AB}(0.6\sin 30°) - 3.6]\mathbf{i} + \omega_{AB}(0.6\cos 30°)\mathbf{j}$$

$$0 = \omega_{AB}(0.6\sin 30°) - 3.6 \qquad (1)$$

$$v_B = \omega_{AB}(0.6\cos 30°) \qquad (2)$$

$$\omega_{AB} = 12\ \text{rad/s} \quad v_B = 6.24\ \text{m/s}\uparrow \qquad \textit{Ans.}$$

Scalar Analysis

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

$$\left[v_B \uparrow\right] = \left[\overset{\leftarrow}{12(0.3)}\right] + \left[\measuredangle\, 30°\,\omega(0.6)\right]$$

This yields Eqs. (1) and (2).

F16–11. Vector Analysis

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$$

$$v_C\,\mathbf{j} = (-18\mathbf{i})\ \text{m/s}$$

$$+ (-\omega_{BC}\mathbf{k}) \times (-0.75\cos 30°\mathbf{i} + 0.75\sin 30°\mathbf{j})\ \text{m}$$

$$v_C\mathbf{j} = (-18)\mathbf{i} + 0.6495\omega_{BC}\mathbf{j} + 0.375\omega_{BC}\,\mathbf{i}$$

$$0 = -18 + 0.375\omega_{BC} \qquad (1)$$

$$v_C = 0.6495\,\omega_{BC} \qquad (2)$$

$$\omega_{BC} = 48\ \text{rad/s} \qquad \textit{Ans.}$$

$$v_C = 31.2\ \text{m/s} \qquad \textit{Ans.}$$

Scalar Analysis

$\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{C/B}$

$\left[v_C \uparrow\right] = \left[\underset{\leftarrow}{v_B}\right] + \left[\measuredangle 30^\circ\ \omega\,(0.75)\right]$

This yields Eqs. (1) and (2).

F16–12. Vector Analysis

$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$

$-v_B \cos 30^\circ\,\mathbf{i} + v_B \sin 30^\circ\,\mathbf{j} = (-3\text{ m/s})\mathbf{j} + (-\omega\mathbf{k}) \times (-2 \sin 45^\circ\mathbf{i} - 2\cos 45^\circ\mathbf{j})\text{ m}$

$-0.8660 v_B\,\mathbf{i} + 0.5 v_B\,\mathbf{j} = -1.4142\omega\mathbf{i} + (1.4142\omega - 3)\mathbf{j}$

$-0.8660 v_B = -1.4142\omega \qquad (1)$

$0.5 v_B = 1.4142\omega - 3 \qquad (2)$

$\omega = 5.02\text{ rad/s} \quad v_B = 8.20\text{ m/s}$ *Ans.*

Scalar Analysis

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$\left[\measuredangle 30^\circ\ v_B\right] = \left[\downarrow 3\right] + \left[\measuredangle 45^\circ\ \omega(2)\right]$

This yields Eqs. (1) and (2).

F16–13. $\omega_{AB} = \dfrac{v_A}{r_{A/IC}} = \dfrac{6}{3} = 2\text{ rad/s}$ *Ans.*

$r_{C/IC} = \sqrt{(3)^2 + (2.5)^2 - 2(3)(2.5)\cos 53.13^\circ} = 2.5\text{ m}$

$v_C = \omega_{AB}\, r_{C/IC} = 2(2.5) = 5\text{ m/s}$ *Ans.*

$\phi = \tan^{-1}\left(\frac{2}{1.5}\right) = 53.13^\circ$

$\theta = 90^\circ - \phi = 90^\circ - 53.13^\circ = 36.9^\circ$ ⦪ *Ans.*

F16–14. $v_B = \omega_{AB}\, r_{B/A} = 12(0.6) = 7.2\text{ m/s} \downarrow$

$v_C = 0$ *Ans.*

$\omega_{BC} = \dfrac{v_B}{r_{B/IC}} = \dfrac{7.2}{1.2} = 6\text{ rad/s}$ *Ans.*

F16–15. $\omega = \dfrac{v_O}{r_{O/IC}} = \dfrac{6}{0.3} = 20\text{ rad/s}$ *Ans.*

$r_{A/IC} = \sqrt{0.3^2 + 0.6^2} = 0.6708\text{ m}$

$\phi = \tan^{-1}\left(\frac{0.3}{0.6}\right) = 26.57^\circ$

$v_A = \omega r_{A/IC} = 20(0.6708) = 13.4\text{ m/s}$ *Ans.*

$\theta = 90^\circ - \phi = 90^\circ - 26.57^\circ = 63.4^\circ$ ⦣ *Ans.*

F16–16. The location of IC can be determined using similar triangles.

$\dfrac{0.5 - r_{C/IC}}{3} = \dfrac{r_{C/IC}}{1.5} \qquad r_{C/IC} = 0.1667\text{ m}$

$\omega = \dfrac{v_C}{r_{C/IC}} = \dfrac{1.5}{0.1667} = 9\text{ rad/s}$ *Ans.*

Also, $r_{O/IC} = 0.3 - r_{C/IC} = 0.3 - 0.1667 = 0.1333\text{ m}$

$v_O = \omega r_{O/IC} = 9(0.1333) = 1.20\text{ m/s}$ *Ans.*

F16–17. $v_B = \omega r_{B/A} = 6(0.2) = 1.2\text{ m/s}$

$r_{B/IC} = 0.8 \tan 60^\circ = 1.3856\text{ m}$

$r_{C/IC} = \frac{0.8}{\cos 60^\circ} = 1.6\text{ m}$

$\omega_{BC} = \dfrac{v_B}{r_{B/IC}} = \dfrac{1.2}{1.3856} = 0.8660\text{ rad/s} = 0.866\text{ rad/s}$ *Ans.*

Then,

$v_C = \omega_{BC}\, r_{C/IC} = 0.8660(1.6) = 1.39\text{ m/s}$ *Ans.*

F16–18. $v_B = \omega_{AB}\, r_{B/A} = 10(0.2) = 2\text{ m/s}$

$v_C = \omega_{CD}\, r_{C/D} = \omega_{CD}\,(0.2) \rightarrow$

$r_{B/IC} = \frac{0.4}{\cos 30^\circ} = 0.4619\text{ m}$

$r_{C/IC} = 0.4 \tan 30^\circ = 0.2309\text{ m}$

$\omega_{BC} = \dfrac{v_B}{r_{B/IC}} = \dfrac{2}{0.4619} = 4.330\text{ rad/s} = 4.33\text{ rad/s}$ *Ans.*

$v_C = \omega_{BC}\, r_{C/IC}$

$\omega_{CD}\,(0.2) = 4.330(0.2309)$

$\omega_{CD} = 5\text{ rad/s}$ *Ans.*

F16–19. $\omega = \dfrac{v_A}{r_{A/IC}} = \dfrac{6}{3} = 2\text{ rad/s}$

Vector Analysis

$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2\,\mathbf{r}_{B/A}$

$a_B\mathbf{i} = -5\mathbf{j} + (\alpha\mathbf{k}) \times (3\mathbf{i} - 4\mathbf{j}) - 2^2(3\mathbf{i} - 4\mathbf{j})$

$a_B\mathbf{i} = (4\alpha - 12)\mathbf{i} + (3\alpha + 11)\mathbf{j}$

$a_B = 4\alpha - 12 \qquad (1)$

$0 = 3\alpha + 11 \qquad (2)$

$\alpha = -3.67\text{ rad/s}^2$ *Ans.*

$a_B = -26.7\text{ m/s}^2$ *Ans.*

The negative signs indicate decelerations.

Scalar Analysis

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

$$\begin{bmatrix} a_B \\ \rightarrow \end{bmatrix} = \begin{bmatrix} \downarrow 5 \end{bmatrix} + \begin{bmatrix} \alpha(5)\ \overset{5}{\nearrow}{}^{3}_{4} \end{bmatrix} + \begin{bmatrix} {}^{4}\overset{5}{\nwarrow}{}_{3}\ (2)^2(5) \end{bmatrix}$$

This yields Eqs. (1) and (2).

F16–20. Vector Analysis

$$\mathbf{a}_A = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} - \omega^2 \mathbf{r}_{A/O}$$
$$= 1.8\mathbf{i} + (-6\mathbf{k}) \times (0.3\mathbf{j}) - 12^2(0.3\mathbf{j})$$
$$= \{3.6\mathbf{i} - 43.2\mathbf{j}\}\ \text{m/s}^2 \qquad \textit{Ans.}$$

Scalar Analysis

$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{A/O}$$

$$\begin{bmatrix} (a_A)_x \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (a_A)_y \uparrow \end{bmatrix} = \begin{bmatrix} (6)(0.3) \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (6)(0.3) \\ \rightarrow \end{bmatrix} + \begin{bmatrix} \downarrow (12)^2(0.3) \end{bmatrix}$$

$\overset{+}{\rightarrow}$ $\quad (a_A)_x = 1.8 + 1.8 = 3.6\ \text{m/s}^2 \rightarrow$

$+\uparrow$ $\quad (a_A)_y = -43.2\ \text{m/s}^2$

F16–21. Using

$$v_O = \omega r; \quad 6 = \omega(0.3)$$
$$\omega = 20\ \text{rad/s}$$
$$a_O = \alpha r; \quad 3 = \alpha(0.3)$$
$$\alpha = 10\ \text{rad/s}^2 \qquad \textit{Ans.}$$

Vector Analysis

$$\mathbf{a}_A = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} - \omega^2 \mathbf{r}_{A/O}$$
$$= 3\mathbf{i} + (-10\mathbf{k}) \times (-0.6\mathbf{i}) - 20^2(-0.6\mathbf{i})$$
$$= \{243\mathbf{i} + 6\mathbf{j}\}\ \text{m/s}^2 \qquad \textit{Ans.}$$

Scalar Analysis

$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{A/O}$$

$$\begin{bmatrix} (a_A)_x \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (a_A)_y \\ \uparrow \end{bmatrix} = \begin{bmatrix} 3 \\ \rightarrow \end{bmatrix} + \begin{bmatrix} 10(0.6) \\ \uparrow \end{bmatrix} + \begin{bmatrix} (20)^2(0.6) \\ \rightarrow \end{bmatrix}$$

$\overset{+}{\rightarrow}$ $\quad (a_A)_x = 3 + 240 = 243\ \text{m/s}^2$

$+\uparrow$ $\quad (a_A)_y = 10(0.6) = 6\ \text{m/s}^2 \uparrow$

F16–22. $\dfrac{r_{A/IC}}{3} = \dfrac{0.5 - r_{A/IC}}{1.5}; \qquad r_{A/IC} = 0.3333\ \text{m}$

$$\omega = \frac{v_A}{r_{A/IC}} = \frac{3}{0.3333} = 9\ \text{rad/s}$$

Vector Analysis

$$\mathbf{a}_A = \mathbf{a}_C + \boldsymbol{\alpha} \times \mathbf{r}_{A/C} - \omega^2 \mathbf{r}_{A/C}$$
$$1.5\mathbf{i} - (a_A)_n\,\mathbf{j} = -0.75\mathbf{i} + (a_C)_n\,\mathbf{j}$$
$$+ (-\alpha\mathbf{k}) \times 0.5\mathbf{j} - 9^2(0.5\mathbf{j})$$
$$1.5\mathbf{i} - (a_A)_n\,\mathbf{j} = (0.5\alpha - 0.75)\mathbf{i} + \left[(a_C)_n - 40.5\right]\mathbf{j}$$
$$1.5 = 0.5\alpha - 0.75$$
$$\alpha = 4.5\ \text{rad/s}^2 \qquad \textit{Ans.}$$

Scalar Analysis

$$\mathbf{a}_A = \mathbf{a}_C + \mathbf{a}_{A/C}$$

$$\begin{bmatrix} 1.5 \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (a_A)_n \\ \downarrow \end{bmatrix} = \begin{bmatrix} 0.75 \\ \leftarrow \end{bmatrix} + \begin{bmatrix} (a_C)_n \\ \uparrow \end{bmatrix} + \begin{bmatrix} \alpha(0.5) \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (9)^2(0.5) \\ \downarrow \end{bmatrix}$$

$\overset{+}{\rightarrow}$ $\quad 1.5 = -0.75 + \alpha(0.5)$

$$\alpha = 4.5\ \text{rad/s}^2$$

F16–23. $v_B = \omega r_{B/A} = 12(0.3) = 3.6\ \text{m/s}$

The IC is at C.

$$\omega_{BC} = \frac{v_B}{r_{B/IC}} = \frac{3.6}{1.2} = 3\ \text{rad/s}, \ \omega_{CD} = 0$$

Vector Analysis

$$\mathbf{a}_B = \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$$
$$= (-6\mathbf{k}) \times (0.3\mathbf{i}) - 12^2(0.3\mathbf{i})$$
$$= \{-43.2\mathbf{i} - 1.8\mathbf{j}\}\ \text{m/s}^2$$
$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2 \mathbf{r}_{C/B}$$
$$a_C\,\mathbf{i} = (-43.2\mathbf{i} - 1.8\mathbf{j})$$
$$+ (\alpha_{BC}\,\mathbf{k}) \times (1.2\mathbf{i}) - 3^2(1.2\mathbf{i})$$
$$a_C\,\mathbf{i} = -54\mathbf{i} + (1.2\alpha_{BC} - 1.8)\mathbf{j}$$
$$a_C = -54\ \text{m/s}^2 = 54\ \text{m/s}^2 \leftarrow \qquad \textit{Ans.}$$
$$0 = 1.2\alpha_{BC} - 1.8 \quad \alpha_{BC} = 1.5\ \text{rad/s}^2 \qquad \textit{Ans.}$$

Scalar Analysis

$$\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_{C/B}$$

$$\begin{bmatrix} a_C \\ \leftarrow \end{bmatrix} = \begin{bmatrix} 6(0.3) \\ \downarrow \end{bmatrix} + \begin{bmatrix} (12)^2(0.3) \\ \leftarrow \end{bmatrix} + \begin{bmatrix} \alpha_{BC}(1.2) \\ \uparrow \end{bmatrix} + \begin{bmatrix} (3)^2(1.2) \\ \leftarrow \end{bmatrix}$$

$\overset{+}{\leftarrow}$ $\quad a_C = 43.2 + 10.8 = 54\ \text{m/s}^2 \leftarrow$

$+\uparrow$ $\quad 0 = -6(0.3) + 1.2\alpha_{BC}$

$$\alpha_{BC} = 1.5\ \text{rad/s}^2$$

F16–24. $v_B = \omega\, r_{B/A} = 6(0.2) = 1.2\text{ m/s} \rightarrow$

$r_{B/IC} = 0.8 \tan 60^\circ = 1.3856\text{ m}$

$$\omega_{BC} = \frac{v_B}{r_{B/IC}} = \frac{1.2}{1.3856} = 0.8660\text{ rad/s}$$

Vector Analysis

$\mathbf{a}_B = \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$

$= (-3\mathbf{k}) \times (0.2\mathbf{j}) - 6^2(0.2\mathbf{j})$

$= (0.6\mathbf{i} - 7.2\mathbf{j})\text{ m/s}$

$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega^2 \mathbf{r}_{C/B}$

$a_C \cos 30^\circ \mathbf{i} + a_C \sin 30^\circ \mathbf{j}$

$= (0.6\mathbf{i} - 7.2\mathbf{j}) + (\alpha_{BC}\mathbf{k} \times 0.8\mathbf{i}) - 0.8660^2(0.8\mathbf{i})$

$0.8660 a_C\,\mathbf{i} + 0.5 a_C\,\mathbf{j} = (0.8\alpha_{BC} - 7.2)\mathbf{j}$

$0.8660 a_C = 0$ (1)

$0.5 a_C = 0.8\alpha_{BC} - 7.2$ (2)

$a_C = 0 \qquad \alpha_{BC} = 9\text{ rad/s}^2$ *Ans.*

Scalar Analysis

$\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_{C/B}$

$$\begin{bmatrix} a_C \\ \measuredangle 30^\circ \end{bmatrix} = \begin{bmatrix} 3(0.2) \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (6)^2(0.2) \\ \downarrow \end{bmatrix} + \begin{bmatrix} \alpha_{BC}(0.8) \\ \uparrow \end{bmatrix} + \begin{bmatrix} (0.8660)^2(0.8) \\ \leftarrow \end{bmatrix}$$

This yields Eqs. (1) and (2).

Chapter 17

F17–1. $\xrightarrow{+} \Sigma F_x = m(a_G)_x;\ 100\left(\tfrac{4}{5}\right) = 100a$

$a = 0.8\text{ m/s}^2 \rightarrow$ *Ans.*

$+\uparrow \Sigma F_y = m(a_G)_y;$

$N_A + N_B - 100\left(\tfrac{3}{5}\right) - 100(9.81) = 0$ (1)

$\circlearrowleft + \Sigma M_G = 0;$

$N_A(0.6) + 100\left(\tfrac{3}{5}\right)(0.7)$

$- N_B(0.4) - 100\left(\tfrac{4}{5}\right)(0.7) = 0$ (2)

$N_A = 430.4\text{ N} = 430\text{ N}$ *Ans.*

$N_B = 610.6\text{ N} = 611\text{ N}$ *Ans.*

F17–2. $+\swarrow \Sigma F_{x'} = m(a_G)_{x'};\quad 80(9.81)\sin 15^\circ = 80a$

$a = 2.54\text{ m/s}^2$ *Ans.*

$+\nwarrow \Sigma F_{y'} = m(a_G)_{y'};$

$N_A + N_B - 80(9.81)\cos 15^\circ = 0$ (1)

$\circlearrowleft +\Sigma M_G = 0;$

$N_A(0.5) - N_B(0.5) = 0$ (2)

$N_A = N_B = 379\text{ N}$ *Ans.*

F17–3. $\circlearrowleft +\Sigma M_A = \Sigma(\mathcal{M}_k)_A;\quad 50\left(\tfrac{3}{5}\right)(2.1) = 10a(1.05)$

$a = 6\text{ m/s}^2$ *Ans.*

$\xrightarrow{+} \Sigma F_x = m(a_G)_x;\quad A_x + 50\left(\tfrac{3}{5}\right) = 10(6)$

$A_x = 30\text{ N}$ *Ans.*

$+\uparrow \Sigma F_y = m(a_G)_y;\quad A_y - 10(9.81) + 50\left(\tfrac{4}{5}\right) = 0$

$A_y = 58.1\text{ N}$ *Ans.*

F17–4. $F_A = \mu_s N_A = 0.2N_A \quad F_B = \mu_s N_B = 0.2N_B$

$\xrightarrow{+} \Sigma F_x = m(a_G)_x;$

$0.2N_A + 0.2N_B = 100a$ (1)

$+\uparrow \Sigma F_y = m(a_G)_y;$

$N_A + N_B - 100(9.81) = 0$ (2)

$\circlearrowleft +\Sigma M_G = 0;$

$0.2N_A(0.75) + N_A(0.9) + 0.2N_B(0.75)$

$- N_B(0.6) = 0$ (3)

Solving Eqs. (1), (2), and (3),

$N_A = 294.3\text{ N} = 294\text{ N}$ *Ans.*

$N_B = 686.7\text{ N} = 687\text{ N}$ *Ans.*

$a = 1.96\text{ m/s}^2$ *Ans.*

Since N_A is positive, the table will indeed slide before it tips.

F17–5. $(a_G)_t = \alpha r = \alpha(1.5\text{ m})$

$(a_G)_n = \omega^2 r = (5\text{ rad/s})^2(1.5\text{ m}) = 37.5\text{ m/s}^2$

$\Sigma F_t = m(a_G)_t;\qquad 100\text{ N} = 50\text{ kg}[\alpha(1.5\text{ m})]$

$\alpha = 1.33\text{ rad/s}^2$ *Ans.*

$\Sigma F_n = m(a_G)_n;\qquad T_{AB} + T_{CD} - 50(9.81)\text{ N}$

$= 50\text{ kg}(37.5\text{ m/s}^2)$

$T_{AB} + T_{CD} = 2365.5$

$\circlearrowleft +\Sigma M_G = 0;\quad T_{CD}(1\text{ m}) - T_{AB}(1\text{ m}) = 0$

$T_{AB} = T_{CD} = 1182.75\text{ N} = 1.18\text{ kN}$ *Ans.*

F17–6. Member CD,

$\circlearrowleft +\Sigma M_C = 0;$

$D_y(0.6) - 450 = 0 \quad D_y = 750\text{ N}$ *Ans.*

$\mathbf{a}_G = \mathbf{a}_D = \mathbf{a}_B$

$(a_G)_n = \omega^2 r = 6^2(0.6) = 21.6\ \text{m/s}^2$

$(a_G)_t = \alpha r = \alpha(0.6)$

Member DB

$+\uparrow \Sigma F_t = m(a_G)_t;$

$750 - 50(9.81) = 50[\alpha(0.6)]$

$\alpha = 8.65\ \text{rad/s}^2$ *Ans.*

$\xrightarrow{+} \Sigma F_n = m(a_G)_n;$

$F_{AB} + D_x = 50(21.6)$ (1)

$\circlearrowleft + \Sigma M_G = 0;$

$D_x(0.4) + 750(0.1) - F_{AB}(0.4) = 0$ (2)

$D_x = 446.25\ \text{N} = 446\ \text{N}$ *Ans.*

$F_{AB} = 633.75\ \text{N} = 634\ \text{N}$ *Ans.*

F17–7. $I_O = mk_O^2 = 100(0.5^2) = 25\ \text{kg}\cdot\text{m}^2$

$\circlearrowleft + \Sigma M_O = I_O\alpha; \quad -100(0.6) = -25\alpha$

$\alpha = 2.4\ \text{rad/s}^2$

$\omega = \omega_0 + \alpha_c t$

$\omega = 0 + 2.4(3) = 7.2\ \text{rad/s}$ *Ans.*

F17–8. $I_O = \frac{1}{2} mr^2 = \frac{1}{2}(50)(0.3^2) = 2.25\ \text{kg}\cdot\text{m}^2$

$\circlearrowleft + \Sigma M_O = I_O\alpha;$

$-9t = -2.25\alpha \qquad \alpha = (4t)\ \text{rad/s}^2$

$d\omega = \alpha\, dt$

$$\int_0^\omega d\omega = \int_0^t 4t\, dt$$

$\omega = (2t^2)\ \text{rad/s}$

$\omega = 2(4^2) = 32\ \text{rad/s}$ *Ans.*

F17–9. $(a_G)_t = \alpha r_G = \alpha(0.15)$

$(a_G)_n = \omega^2 r_G = 6^2(0.15) = 5.4\ \text{m/s}^2$

$I_O = I_G + md^2 = \frac{1}{12}(30)(0.9^2) + 30(0.15^2)$

$= 2.7\ \text{kg}\cdot\text{m}^2$

$\circlearrowleft + \Sigma M_O = I_O\alpha; \quad 60 - 30(9.81)(0.15) = 2.7\alpha$

$\alpha = 5.872\ \text{rad/s}^2 = 5.87\ \text{rad/s}^2$ *Ans.*

$\xleftarrow{+} \Sigma F_n = m(a_G)_n; \quad O_n = 30(5.4) = 162\ \text{N}$ *Ans.*

$+\uparrow \Sigma F_t = m(a_G)_t;$

$O_t - 30(9.81) = 30[5.872(0.15)]$

$O_t = 320.725\ \text{N} = 321\ \text{N}$ *Ans.*

F17–10. $(a_G)_t = \alpha r_G = \alpha(0.3)$

$(a_G)_n = \omega^2 r_G = 10^2(0.3) = 30\ \text{m/s}^2$

$I_O = I_G + md^2 = \frac{1}{2}(30)(0.3^2) + 30(0.3^2)$

$= 4.05\ \text{kg}\cdot\text{m}^2$

$\circlearrowleft + \Sigma M_O = I_O\alpha;$

$50\left(\frac{3}{5}\right)(0.3) + 50\left(\frac{4}{5}\right)(0.3) = 4.05\alpha$

$\alpha = 5.185\ \text{rad/s}^2 = 5.19\ \text{rad/s}^2$ *Ans.*

$+\uparrow \Sigma F_n = m(a_G)_n;$

$O_n + 50\left(\frac{3}{5}\right) - 30(9.81) = 30(30)$

$O_n = 1164.3\ \text{N} = 1.16\ \text{kN}$ *Ans.*

$\xrightarrow{+} \Sigma F_t = m(a_G)_t;$

$O_t + 50\left(\frac{4}{5}\right) = 30[5.185(0.3)]$

$O_t = 6.67\ \text{N}$ *Ans.*

F17–11. $I_G = \frac{1}{12}ml^2 = \frac{1}{12}(15\ \text{kg})(0.9\ \text{m})^2 = 1.0125\ \text{kg}\cdot\text{m}^2$

$(a_G)_n = \omega^2 r_G = 0$

$(a_G)_t = \alpha(0.15\ \text{m})$

$I_O = I_G + md_{OG}^2$

$= 1.0125\ \text{kg}\cdot\text{m}^2 + 15\ \text{kg}(0.15\ \text{m})^2$

$= 1.35\ \text{kg}\cdot\text{m}^2$

$\circlearrowleft + \Sigma M_O = I_O\alpha;$

$[15(9.81)\ \text{N}](0.15\ \text{m}) = (1.35\ \text{kg}\cdot\text{m}^2)\alpha$

$\alpha = 16.35\ \text{rad/s}^2$ *Ans.*

$+\downarrow \Sigma F_t = m(a_G)_t; \quad -O_t + 15(9.81)\ \text{N}$

$= (15\ \text{kg})[16.35\ \text{rad/s}^2(0.15\ \text{m})]$

$O_t = 110.36\ \text{N} = 110\ \text{N}$ *Ans.*

$\xrightarrow{+} \Sigma F_n = m(a_G)_n; \quad O_n = 0$ *Ans.*

F17–12. $(a_G)_t = \alpha r_G = \alpha(0.45)$

$(a_G)_n = \omega^2 r_G = 6^2(0.45) = 16.2\ \text{m/s}^2$

$I_O = \frac{1}{3}ml^2 = \frac{1}{3}(30)(0.9^2) = 8.1\ \text{kg}\cdot\text{m}^2$

$\circlearrowleft + \Sigma M_O = I_O\alpha;$

$300\left(\frac{4}{5}\right)(0.6) - 30(9.81)(0.45) = 8.1\alpha$

$\alpha = 1.428\ \text{rad/s}^2 = 1.43\ \text{rad/s}^2$ *Ans.*

$\xleftarrow{+} \Sigma F_n = m(a_G)_n; \quad O_n + 300\left(\frac{3}{5}\right) = 30(16.2)$

$O_n = 306\ \text{N}$ *Ans.*

$+\uparrow \Sigma F_t = m(a_G)_t; \quad O_t + 300\left(\frac{4}{5}\right) - 30(9.81)$

$= 30[1.428(0.45)]$

$O_t = 73.58\ \text{N} = 73.6\ \text{N}$ *Ans.*

F17–13. $I_G = \frac{1}{12}ml^2 = \frac{1}{12}(60)(3^2) = 45 \text{ kg}\cdot\text{m}^2$

$+\uparrow \Sigma F_y = m(a_G)_y; \quad 80 - 20 = 60a_G$

$a_G = 1 \text{ m/s}^2\uparrow$

$\circlearrowleft +\Sigma M_G = I_G\alpha; \quad 80(1) + 20(0.75) = 45\alpha$

$\alpha = 2.11 \text{ rad/s}^2$ *Ans.*

F17–14. $\circlearrowleft +\Sigma M_{IC} = (\mathcal{M}_k)_{IC};$

$-200(0.3) = -100a_G(0.3) - \frac{1}{2}(100)(0.3)^2\alpha$

$30a_G + 4.5\alpha = 60 \quad (1)$

$a_G = \alpha r = \alpha(0.3) \quad (2)$

$\alpha = 4.44 \text{ rad/s}^2 \quad a_G = 1.33 \text{ m/s}^2 \rightarrow$ *Ans.*

$I_G = 20(0.3)^2 = 1.8 \text{ kg}\cdot\text{m}^2$

F17–15. $I_G = 20(0.3)^2 = 1.8 \text{ kg}\cdot\text{m}^2$

$+\uparrow \Sigma F_y = m(a_G)_y;$

$N - 20(9.81) = 0 \quad N = 196.2 \text{ N}$

$\overset{+}{\rightarrow} \Sigma F_x = m(a_G)_x; \quad 0.5(196.2) = 20a_O$

$a_O = 4.905 \text{ m/s}^2 \rightarrow$ *Ans.*

$\circlearrowleft +\Sigma M_O = I_O\alpha;$

$0.5(196.2)(0.4) - 100 = -1.8\alpha$

$\alpha = 33.8 \text{ rad/s}^2$ *Ans.*

F17–16. Sphere $I_G = \frac{2}{5}(20)(0.15)^2 = 0.18 \text{ kg}\cdot\text{m}^2$

$\circlearrowleft +\Sigma M_{IC} = (\mathcal{M}_k)_{IC};$

$20(9.81)\sin 30°(0.15) = 0.18\alpha + (20a_G)(0.15)$

$0.18\alpha + 3a_G = 14.715$

$a_G = \alpha r = \alpha(0.15)$

$\alpha = 23.36 \text{ rad/s}^2 = 23.4 \text{ rad/s}^2$ *Ans.*

$a_G = 3.504 \text{ m/s}^2 = 3.50 \text{ m/s}^2$ *Ans.*

F17–17. $I_G = 200(0.3)^2 = 18 \text{ kg}\cdot\text{m}^2$

$+\uparrow \Sigma F_y = m(a_G)_y;$

$N - 200(9.81) = 0 \quad N = 1962 \text{ N}$

$\overset{+}{\rightarrow} \Sigma F_x = m(a_G)_x;$

$T - 0.2(1962) = 200a_G \quad (1)$

The IC is at A.

$\circlearrowleft +\Sigma M_A = (\mathcal{M}_k)_A; \quad 450 - 0.2(1962)(1)$

$= 18\alpha + 200a_G(0.4) \quad (2)$

$(a_A)_t = 0 \quad a_A = (a_A)_n$

$\mathbf{a}_G = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{G/A} - \omega^2\mathbf{r}_{G/A}$

$a_G\mathbf{i} = -a_A\mathbf{j} + \alpha\mathbf{k} \times (-0.4\mathbf{j}) - \omega^2(-0.4\mathbf{j})$

$a_G\mathbf{i} = 0.4\alpha\mathbf{i} + (0.4\omega^2 - a_A)\mathbf{j}$

$a_G = 0.4\alpha \quad (3)$

Solving Eqs. (1), (2), and (3),

$\alpha = 1.15 \text{ rad/s}^2 \quad a_G = 0.461 \text{ m/s}^2$

$T = 485 \text{ N}$ *Ans.*

F17–18. $\overset{+}{\rightarrow} \Sigma F_x = m(a_G)_x; \quad 0 = 12(a_G)_x \quad (a_G)_x = 0$

$\circlearrowleft +\Sigma M_A = (\mathcal{M}_k)_A;$

$-12(9.81)(0.3) = 12(a_G)_y(0.3) - \frac{1}{12}(12)(0.6)^2\alpha$

$0.36\alpha - 3.6(a_G)_y = 35.316 \quad (1)$

$\omega = 0$

$\mathbf{a}_G = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{G/A} - \omega^2\mathbf{r}_{G/A}$

$(a_G)_y\,\mathbf{j} = a_A\mathbf{i} + (-\alpha\mathbf{k}) \times (0.3\mathbf{i}) - \mathbf{0}$

$(a_G)_y\,\mathbf{j} = (a_A)\mathbf{i} - 0.3\alpha\mathbf{j}$

$a_A = 0$ *Ans.*

$(a_G)_y = -0.3\alpha \quad (2)$

Solving Eqs. (1) and (2)

$\alpha = 24.5 \text{ rad/s}^2$

$(a_G)_y = -7.36 \text{ m/s}^2 = 7.36 \text{ m/s}^2\downarrow$ *Ans.*

Chapter 18

F18–1. $I_O = mk_O^2 = 80\left(0.4^2\right) = 12.8 \text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2}I_O\omega^2 = \frac{1}{2}(12.8)\omega^2 = 6.4\omega^2$

$s = \theta r = 20(2\pi)(0.6) = 24\pi \text{ m}$

$T_1 + \Sigma U_{1-2} = T_2$

$0 + 50(24\pi) = 6.4\omega^2$

$\omega = 24.3 \text{ rad/s}$ *Ans.*

F18–2. $T_1 = 0$

$T_2 = \frac{1}{2}m(v_G)_2^2 + \frac{1}{2}I_G\omega_2^2$

$= \frac{1}{2}(25 \text{ kg})(0.75\omega_2)^2$

$+ \frac{1}{2}\left[\frac{1}{12}(25 \text{ kg})(1.5 \text{ m})^2\right]\omega_2^2$

$T_2 = 9.375\omega_2^2$

Or,

$I_O = \frac{1}{3}ml^2 = \frac{1}{3}(25 \text{ kg})(1.5 \text{ m})^2$

$= 18.75 \text{ kg}\cdot\text{m}^2$

so that

$T_2 = \frac{1}{2} I_O \omega_2^2 = \frac{1}{2}(18.75 \text{ kg} \cdot \text{m}^2)\omega_2^2$

$= 9.375\omega_2^2$

$T_1 + \Sigma U_{1-2} = T_2$

$T_1 + \left[-W y_G + M\theta\right] = T_2$

$0 + \left[-(25(9.81) \text{ N}(0.75 \text{ m})) + (150 \text{ N} \cdot \text{m})\left(\frac{\pi}{2}\right)\right]$

$= 9.375\omega_2^2$

$\omega_2 = 2.35 \text{ rad/s}$ *Ans.*

F18–3. $(v_G)_2 = \omega_2 r_{G/IC} = \omega_2(2.5)$

$I_G = \frac{1}{12} m l^2 = \frac{1}{12}(50)\left(5^2\right) = 104.17 \text{ kg} \cdot \text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2} m(v_G)_2^2 + \frac{1}{2} I_G \omega_2^2$

$= \frac{1}{2}(50)\left[\omega_2(2.5)\right]^2 + \frac{1}{2}(104.17)\omega_2^2 = 208.33\omega_2^2$

$U_P = P s_P = 600(3) = 1800 \text{ J}$

$U_W = -Wh = -50(9.81)(2.5 - 2) = -245.25 \text{ J}$

$T_1 + \Sigma U_{1-2} = T_2$

$0 + 1800 + (-245.25) = 208.33\omega_2^2$

$\omega_2 = 2.732 \text{ rad/s} = 2.73 \text{ rad/s}$ *Ans.*

F18–4. $T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$

$= \frac{1}{2}(50 \text{ kg})(0.4\omega)^2 + \frac{1}{2}\left[50 \text{ kg}(0.3 \text{ m})^2\right]\omega^2$

$= 6.25\omega^2 \text{ J}$

Or,

$T = \frac{1}{2} I_{IC} \omega^2$

$= \frac{1}{2}\left[50 \text{ kg}(0.3 \text{ m})^2 + 50 \text{ kg}(0.4 \text{ m})^2\right]\omega^2$

$= 6.25\omega^2 \text{ J}$

$s_G = \theta r = 10(2\pi \text{ rad})(0.4 \text{ m}) = 8\pi \text{ m}$

$T_1 + \Sigma U_{1-2} = T_2$

$T_1 + P \cos 30° \, s_G = T_2$

$0 + (50 \text{ N})\cos 30°(8\pi \text{ m}) = 6.25\omega^2 \text{ J}$

$\omega = 13.2 \text{ rad/s}$ *Ans.*

F18–5. $I_G = \frac{1}{12} m l^2 = \frac{1}{12}(30)\left(3^2\right) = 22.5 \text{ kg} \cdot \text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$

$= \frac{1}{2}(30)[\omega(0.5)]^2 + \frac{1}{2}(22.5)\omega^2 = 15\omega^2$

Or,

$I_O = I_G + m d^2 = \frac{1}{12}(30)\left(3^2\right) + 30\left(0.5^2\right)$

$= 30 \text{ kg} \cdot \text{m}^2$

$T_2 = \frac{1}{2} I_O \omega^2 = \frac{1}{2}(30)\omega^2 = 15\omega^2$

$s_1 = \theta r_1 = 8\pi(0.5) = 4\pi \text{ m}$

$s_2 = \theta r_2 = 8\pi(1.5) = 12\pi \text{ m}$

$U_{P_1} = P_1 s_1 = 30(4\pi) = 120\pi \text{ J}$

$U_{P_2} = P_2 s_2 = 20(12\pi) = 240\pi \text{ J}$

$U_M = M\theta = 20[4(2\pi)] = 160\pi \text{ J}$

$U_W = 0$ since bar returns to the same position

$T_1 + \Sigma U_{1-2} = T_2$

$0 + 120\pi + 240\pi + 160\pi = 15\omega^2$

$\omega = 10.44 \text{ rad/s} = 10.4 \text{ rad/s}$ *Ans.*

F18–6. $v_G = \omega r = \omega(0.4)$

$I_G = m k_G^2 = 20\left(0.3^2\right) = 1.8 \text{ kg} \cdot \text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$

$= \frac{1}{2}(20)\left[\omega(0.4)\right]^2 + \frac{1}{2}(1.8)\omega^2$

$= 2.5\omega^2$

$U_M = M\theta = M\left(\frac{s_O}{r}\right) = 50\left(\frac{20}{0.4}\right) = 2500 \text{ J}$

$T_1 + \Sigma U_{1-2} = T_2$

$0 + 2500 = 2.5\omega^2$

$\omega = 31.62 \text{ rad/s} = 31.6 \text{ rad/s}$ *Ans.*

F18–7. $v_G = \omega r = \omega(0.3)$

$I_G = \frac{1}{2} m r^2 = \frac{1}{2}(30)\left(0.3^2\right) = 1.35 \text{ kg} \cdot \text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2} m(v_G)_2^2 + \frac{1}{2} I_G \omega_2^2$

$= \frac{1}{2}(30)[\omega_2(0.3)]^2 + \frac{1}{2}(1.35)\omega_2^2 = 2.025\omega_2^2$

Datum at O.

$\left(V_g\right)_1 = W y_1 = 0$

$\left(V_g\right)_2 = -W y_2 = -30(9.81)(0.3) = -88.29 \text{ J}$

$T_1 + V_1 = T_2 + V_2$

$0 + 0 = 2.025\omega_2^2 + (-88.29)$

$\omega_2 = 6.603 \text{ rad/s} = 6.60 \text{ rad/s}$ *Ans.*

F18–8. $v_O = \omega r_{O/IC} = \omega(0.2)$

$I_O = m k_O^2 = 50\left(0.3^2\right) = 4.5 \text{ kg} \cdot \text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2} m(v_O)_2^2 + \frac{1}{2} I_O \omega_2^2$

$= \frac{1}{2}(50)\left[\omega_2(0.2)\right]^2 + \frac{1}{2}(4.5)\omega_2^2$

$= 3.25\omega_2^2$

Datum at initial position.

$\left(V_g\right)_1 = W y_1 = 0$

$\left(V_g\right)_2 = -W y_2 = -50(9.81)(6 \sin 30°)$

$= -1471.5 \text{ J}$

$$T_1 + V_1 = T_2 + V_2$$
$$0 + 0 = 3.25\omega_2^2 + (-1471.5)$$
$\omega_2 = 21.28 \text{ rad/s} = 21.3 \text{ rad/s}$ *Ans.*

F18–9. $v_G = \omega r_G = \omega(1.5)$

$I_G = \frac{1}{12}(60)(3^2) = 45 \text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2}m(v_G)_2^2 + \frac{1}{2}I_G\omega_2^2$
$= \frac{1}{2}(60)[\omega_2(1.5)]^2 + \frac{1}{2}(45)\omega_2^2$
$= 90\omega_2^2$

Or,

$T_2 = \frac{1}{2}I_O\omega_2^2 = \frac{1}{2}\left[45 + 60(1.5^2)\right]\omega_2^2 = 90\omega_2^2$

$(V_g)_1 = Wy_1 = 0$

$(V_g)_2 = -Wy_2 = -60(9.81)(1.5\sin 45°)$
$= -624.30 \text{ J}$

$(V_e)_1 = \frac{1}{2}ks_1^2 = 0$

$(V_e)_2 = \frac{1}{2}ks_2^2 = \frac{1}{2}(150)(3\sin 45°)^2 = 337.5 \text{ J}$

$T_1 + V_1 = T_2 + V_2$

$0 + 0 = 90\omega_2^2 + [-624.30 + 337.5]$

$\omega_2 = 1.785 \text{ rad/s} = 1.79 \text{ rad/s}$ *Ans.*

F18–10. $v_G = \omega r_G = \omega(0.75)$

$I_G = \frac{1}{12}(30)(1.5^2) = 5.625 \text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2}m(v_G)_2^2 + \frac{1}{2}I_G\omega_2^2$
$= \frac{1}{2}(30)[\omega(0.75)]^2 + \frac{1}{2}(5.625)\omega_2^2 = 11.25\omega_2^2$

Or,

$T_2 = \frac{1}{2}I_O\omega_2^2 = \frac{1}{2}\left[5.625 + 30(0.75^2)\right]\omega_2^2$
$= 11.25\omega_2^2$

Datum at O.

$(V_g)_1 = Wy_1 = 0$

$(V_g)_2 = -Wy_2 = -30(9.81)(0.75)$
$= -220.725 \text{ J}$

$(V_e)_1 = \frac{1}{2}ks_1^2 = 0$

$(V_e)_2 = \frac{1}{2}ks_2^2 = \frac{1}{2}(80)\left(\sqrt{2^2 + 1.5^2} - 0.5\right)^2 = 160 \text{ J}$

$T_1 + V_1 = T_2 + V_2$

$0 + 0 = 11.25\omega_2^2 + (-220.725 + 160)$

$\omega_2 = 2.323 \text{ rad/s} = 2.32 \text{ rad/s}$ *Ans.*

F18–11. $(v_G)_2 = \omega_2 r_{G/IC} = \omega_2(0.75)$

$I_G = \frac{1}{12}(30)(1.5^2) = 5.625 \text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2}m(v_G)_2^2 + \frac{1}{2}I_G\omega_2^2$
$= \frac{1}{2}(30)[\omega_2(0.75)]^2 + \frac{1}{2}(5.625)\omega_2^2 = 11.25\omega_2^2$

Datum at B.

$(V_g)_1 = Wy_1 = 30(9.81)(0.75\sin 45°) = 156.08 \text{ J}$

$(V_g)_2 = -Wy_2 = 0$

$(V_e)_1 = \frac{1}{2}ks_1^2 = 0$

$(V_e)_2 = \frac{1}{2}ks_2^2 = \frac{1}{2}(300)(1.5 - 1.5\cos 45°)^2$
$= 28.95 \text{ J}$

$T_1 + V_1 = T_2 + V_2$

$0 + (156.08 + 0) = 11.25\omega_2^2 + (0 + 28.95)$

$\omega_2 = 3.362 \text{ rad/s} = 3.36 \text{ rad/s}$ *Ans.*

F18–12. $T_1 = 0$

$T_2 = \frac{1}{2}I_A\omega^2 = \frac{1}{2}\left[\frac{1}{3}(20 \text{ kg})(2 \text{ m})^2\right]\omega^2$
$= 13.3333\omega^2$

Datum at A.

$(V_g)_1 = -Wy_1 = -[20(9.81) \text{ N}](1 \text{ m}) = -196.2 \text{ J}$

$(V_g)_2 = 0$

$(V_e)_1 = \frac{1}{2}ks_1^2$
$= \frac{1}{2}(100 \text{ N/m})\left(\sqrt{(3 \text{ m})^2 + (2 \text{ m})^2} - 0.5 \text{ m}\right)^2$
$= 482.22 \text{ J}$

$(V_e)_2 = \frac{1}{2}ks_2^2 = \frac{1}{2}(100 \text{ N/m})(1 \text{ m} - 0.5 \text{ m})^2$
$= 12.5 \text{ J}$

$T_1 + V_1 = T_2 + V_2$

$0 + [-196.2 \text{ J} + 482.22 \text{ J}]$
$= 13.3333\omega_2^2 + [0 + 12.5 \text{ J}]$

$\omega_2 = 4.53 \text{ rad/s}$ *Ans.*

Chapter 19

F19–1. $\circlearrowleft + I_O\omega_1 + \Sigma\int_{t_1}^{t_2} M_O\,dt = I_O\omega_2$

$$0 + \int_0^{4\text{ s}} 3t^2\,dt = \left[60(0.3)^2\right]\omega_2$$

$\omega_2 = 11.85 \text{ rad/s} = 11.9 \text{ rad/s}$ *Ans.*

F19–2. $\circlearrowleft + (H_A)_1 + \Sigma\int_{t_1}^{t_2} M_A dt = (H_A)_2$

$0 + 300(6) = 300(0.4^2)\omega_2 + 300[\omega(0.6)](0.6)$

$\omega_2 = 11.54 \text{ rad/s} = 11.5 \text{ rad/s}$ *Ans.*

$\overset{+}{\rightarrow} \quad m(v_1)_x + \Sigma\int_{t_1}^{t_2} F_x dt = m(v_2)_x$

$0 + F_f(6) = 300[11.54(0.6)]$

$F_f = 346\text{ N}$ *Ans.*

F19–3. $v_A = \omega_A r_{A/IC} = \omega_A(0.15)$

$\zeta + \Sigma M_O = 0; \quad 9 - A_t(0.45) = 0 \quad A_t = 20\text{ N}$

$$\zeta + (H_C)_1 + \Sigma \int_{t_1}^{t_2} M_C\,dt = (H_C)_2$$

$0 + [20(5)](0.15)$

$= 10[\omega_A(0.15)](0.15)$

$+ \left[10\left(0.1^2\right)\right]\omega_A$

$\omega_A = 46.2\text{ rad/s}$ *Ans.*

F19–4. $I_A = mk_A^2 = 10\left(0.08^2\right) = 0.064\text{ kg}\cdot\text{m}^2$

$I_B = mk_B^2 = 50\left(0.15^2\right) = 1.125\text{ kg}\cdot\text{m}^2$

$\omega_A r_A = \omega_B r_B$

$$\omega_A = \left(\frac{r_B}{r_A}\right)\omega_B = \left(\frac{0.2}{0.1}\right)\omega_B = 2\omega_B$$

F is the force between the gears.

$$\zeta + I_A(\omega_A)_1 + \Sigma \int_{t_1}^{t_2} M_A\,dt = I_A(\omega_A)_2$$

$$0 + 10(5) - \int_0^{5\text{s}} F(0.1)dt = 0.064[2(\omega_B)_2]$$

$$\int_0^{5\text{ s}} Fdt = 500 - 1.28(\omega_B)_2 \quad (1)$$

$$\zeta + I_B(\omega_B)_1 + \Sigma \int_{t_1}^{t_2} M_B\,dt = I_B(\omega_B)_2$$

$$0 + \int_0^{5\text{s}} F(0.2)dt = 1.125(\omega_B)_2$$

$$\int_0^{5\text{ s}} Fdt = 5.625(\omega_B)_2 \quad (2)$$

Equating Eqs. (1) and (2),

$500 - 1.28(\omega_B)_2 = 5.625(\omega_B)_2$

$(\omega_B)_2 = 72.41\text{ rad/s} = 72.4\text{ rad/s}$ *Ans.*

F19–5. $(\overset{+}{\rightarrow}) \quad m\left[(v_O)_x\right]_1 + \Sigma \int F_x\,dt = m\left[(v_O)_x\right]_2$

$0 + (150\text{ N})(3\text{ s}) + F_A(3\text{ s})$

$= (50\text{ kg})(0.3\omega_2)$

$$\zeta + I_G\omega_1 + \Sigma \int M_G\,dt = I_G\omega_2$$

$0 + (150\text{ N})(0.2\text{ m})(3\text{ s}) - F_A(0.3\text{ m})(3\text{ s})$

$= [(50\text{ kg})(0.175\text{ m})^2]\,\omega_2$

$\omega_2 = 37.3\text{ rad/s}$ *Ans.*

$F_A = 36.53\text{ N}$

Also,

$$I_{IC}\omega_1 + \Sigma \int M_{IC}\,dt = I_{IC}\omega_2$$

$0 + [(150\text{ N})(0.2 + 0.3)\text{ m}](3\text{ s})$

$= [(50\text{ kg})(0.175\text{ m})^2 + (50\text{ kg})(0.3\text{ m})^2]\omega_2$

$\omega_2 = 37.3\text{ rad/s}$ *Ans.*

F19–6. $(+\uparrow) \quad m\left[(v_G)_1\right]_y + \Sigma \int F_y\,dt = m\left[(v_G)_2\right]_y$

$0 + N_A(3\text{ s}) - [70(9.81\text{ N})](3\text{ s}) = 0$

$N_A = 686.7\text{ N}$

$$\zeta + (H_{IC})_1 + \Sigma \int M_{IC}\,dt = (H_{IC})_2$$

$0 + (35\text{ N}\cdot\text{m})(3\text{ s}) - \left[0.15(686.7\text{ N})(3\text{ s})\right](0.15\text{ m})$

$= \left[70\text{ kg}(0.375\text{ m})^2\right]\omega_2 + (70\text{ kg})\left[\omega_2(0.3\text{ m})\right](0.3\text{ m})$

$\omega_2 = 3.63\text{ rad/s}$ *Ans.*

Review Problem Answers

Chapter 12

R12–1. $0 \le t \le 5 \quad a = 4\ \text{m/s}^2$ *Ans.*
$5 \le t \le 20 \quad a = 0\ \text{m/s}^2$ *Ans.*
$20 \le t \le 30 \quad a = -2\ \text{m/s}^2$ *Ans.*
$s_1 = 50\ \text{m}$ *Ans.*
$s_2 = 350\ \text{m}$ *Ans.*
$s_3 = 450\ \text{m}$ *Ans.*

R12–2. $a_{\max} = 42.0\ \text{m/s}^2$ *Ans.*
$v_{\max} = 135\ \text{m/s}$ *Ans.*

R12–3. $s = 20.0\ \text{m}$ *Ans.*

R12–4. $s = 1.84\ \text{m}$ *Ans.*

R12–5. $v_0 = 20.6\ \text{m/s}$ *Ans.*
$\theta = 58.9°$ *Ans.*

R12–6. $\mathbf{a}_{AB} = \{0.404\mathbf{i} + 7.07\mathbf{j}\}\ \text{m/s}^2$ *Ans.*
$\mathbf{a}_{AC} = \{2.50\mathbf{i}\}\ \text{m/s}^2$ *Ans.*

R12–7. $v = 3.19\ \text{m/s}$ *Ans.*
$a = 4.22\ \text{m/s}^2$ *Ans.*

R12–8. $v = 4.58\ \text{m/s}$ *Ans.*
$a = 0.653\ \text{m/s}^2$ *Ans.*

R12–9. $v_{B/A} = 875\ \text{km/h}$ *Ans.*
$\theta = 41.5°$ ⦪ *Ans.*

R12–10. $a = 29.0\ \text{m/s}^2$ *Ans.*

R12–11. $t = 160\ \text{s}$ *Ans.*

Chapter 13

R13–1. $T = 158\ \text{N}$ *Ans.*

R13–2. $T = 46.7\ \text{N}$ *Ans.*

R13–3. $F = 85.7\ \text{N}$ *Ans.*

R13–4. $a_A = 22.1\ \text{m/s}^2$ *Ans.*
$a_B = 1.18\ \text{m/s}^2$ *Ans.*

R13–5. $v = 2.10\ \text{m/s}$ *Ans.*

R13–6. $F_r = 68\ \text{N}$ *Ans.*
$N = 153\ \text{N}$ *Ans.*

R13–7. $a_t = 3.36\ \text{m/s}^2 \ \swarrow$ *Ans.*
$T = 361\ \text{N}$ *Ans.*

R13–8. $N = 29.9\ \text{N}$ *Ans.*
$a_t = 6.97\ \text{m/s}^2$ *Ans.*

Chapter 14

R14–1. $v_B = 18.8\ \text{m/s}$ *Ans.*

R14–2. $F = 38.5\ \text{N}$ *Ans.*

R14–3. $v_2 = 6.51\ \text{m/s}$ *Ans.*

R14–4. $v_B = 9.90\ \text{m/s}$ *Ans.*

R14–5. $P_i = 1.43\ \text{kW}$ *Ans.*

R14–6. $v_A = v_B = 4.18\ \text{m/s}$ *Ans.*

R14–7. $P_i = 2.72\ \text{kW}$ *Ans.*

R14–8. $v_B = 10.4\ \text{m/s}$ *Ans.*

Chapter 15

R15–1. $v_2 = 1.92\ \text{m/s}$ *Ans.*

R15–2. $v_{ABC} = 2.18\ \text{m/s} \leftarrow$ *Ans.*

R15–3. $t = 1.02\ \text{s}$ *Ans.*

R15–4. $v_2 = 31.7\ \text{m/s}$ *Ans.*

R15–5. $(v_A)_2 = 0.400\ \text{m/s} \rightarrow$ *Ans.*
$(v_B)_2 = 2.40\ \text{m/s} \rightarrow$ *Ans.*
$d = 0.951\ \text{m}$ *Ans.*

R15–6. $v = 13.4\ \text{m/s}$ *Ans.*

R15–7. $(v_B)_2 = 8\ \text{m/s} \rightarrow$
$t = 4.077\ \text{s} = 4.08\ \text{s}$ *Ans.*

R15–8. $(v_B)_2 = 1.34\ \text{m/s} \quad \leftarrow$ *Ans.*
$(v_A)_2 = 1.30\ \text{m/s}$ *Ans.*
$(\theta_A)_2 = 8.47°$ ∡ *Ans.*

Chapter 16

R16–1. $v_B = 0.394 \text{ m/s}$ *Ans.*

R16–2. $v_C = 1.34 \text{ m/s}$ *Ans.*

R16–3. $v_W = 4.28 \text{ m/s}$ *Ans.*
$a_W = 75.0 \text{ mm/s}^2$ *Ans.*

R16–4. $v_B = 0.860 \text{ m/s}$ *Ans.*

R16–5. $\alpha_{CB} = 7.52 \text{ rad/s}^2 \searrow 60.8^\circ$ *Ans.*
$a_C = 4.22 \text{ m/s}^2 \searrow 60.8^\circ$ *Ans.*

R16–6. $v_C = 0.234 \text{ m/s} \swarrow$ (slope 3, 4, 5) *Ans.*
$a_C = 1.37 \text{ m/s}^2 \swarrow$ (slope 3, 4, 5) *Ans.*

R16–7. $a_C = 50.6 \text{ m/s}^2 \nearrow 9.09^\circ$ *Ans.*
The cylinder moves up with an acceleration
$a_B = 8.00 \text{ m/s}^2 \uparrow$ *Ans.*

R16–8. $a_B = 2.25 \text{ m/s}^2$ *Ans.*
$\theta = 32.6^\circ \searrow$ *Ans.*

Chapter 17

R17–1. $E_x = 79.4 \text{ N}$ *Ans.*
$E_y = 58.0 \text{ N}$ *Ans.*
$M_E = 43.5 \text{ M} \cdot \text{m}$ *Ans.*

R17–2. $T = 218 \text{ N}$ *Ans.*
$\alpha = 21.0 \text{ rad/s}^2$ *Ans.*

R17–3. At each wheel
$N'_A = 383 \text{ N}$ *Ans.*
$N'_B = 620 \text{ N}$ *Ans.*

R17–4. $a_G = 1.38 \text{ m/s}^2$ *Ans.*

R17–5. $F_o = 30.1 \text{ N} \uparrow$ *Ans.*

R17–6. $\omega = 100 \text{ rad/s}$ *Ans.*

R17–7. $\alpha = 7.50 \text{ rad/s}^2$ *Ans.*
$F = 150 \text{ N}$ *Ans.*

R17–8. $\alpha = 8.89 \text{ rad/s}^2$ *Ans.*

Chapter 18

R18–1. $\omega = 33.0 \text{ rad/s}$ *Ans.*

R18–2. $s_G = 0.859 \text{ m}$ *Ans.*

R18–3. $\omega_2 = 2.83 \text{ rad/s}$ *Ans.*

R18–4. $v_B = 2.58 \text{ m/s}$ *Ans.*
$P = 141 \text{ N}$ *Ans.*

R18–5. $\omega = 2.30 \text{ rad/s}$ *Ans.*

R18–6. $d = 3.38 \text{ m}$ *Ans.*

R18–7. $v_2 = 3.46 \text{ m/s}$ *Ans.*

R18–8. $v_C = 4.01 \text{ m/s}$ *Ans.*

Chapter 19

R19–1. $\omega_2 = 0.386 \text{ rad/s}$ *Ans.*

R19–2. $(v_O)_2 = 6 \text{ m/s}$ *Ans.*
$\omega_2 = 26.7 \text{ rad/s}$ *Ans.*

R19–3. $\omega_2 = 12.8 \text{ rad/s}$ *Ans.*

R19–4. $(v_G)_2 = 9.42 \text{ m/s}$ *Ans.*

R19–5. $\omega_A = 1.70 \text{ rad/s}$ *Ans.*
$\omega_B = 5.10 \text{ rad/s}$ *Ans.*

R19–6. $\omega = 3.56 \text{ rad/s}$ *Ans.*

R19–7. $\omega = 217 \text{ rad/s}$ *Ans.*

R19–8. $\omega_2 = \frac{1}{3}\omega_1$ *Ans.*

Answers to Selected Problems

Chapter 12

12-1 a) $\Delta s = 240$ mm, b) $v_{avg} = 60.0$ mm/s, c) $a = 20.0$ mm/s^2

12-2 $v = 0, s = 80.7$ m

12-3 $v_{avg} = 0, (v_{sp})_{avg} = 3.00$ m/s, $a\Big|_{t=6\text{ s}} = 2.00$ m/s^2

12-5 $v = 32.0$ m/s, $s = 67.0$ m, $d = 66.0$ m

12-6 $a = -12.0$ m/s^2, $s = 0$, $s_T = 8.00$ m, $(v_{sp})_{avg} = 2.67$ m/s

12-7 $s = 11.9$ m, $v = 0.250$ m/s

12-9 $s_{BA} = \left|\dfrac{2v_Av_B - v_A^2}{2a_A}\right|$

12-10 $v = 2.63$ m/s

12-11 $v = 1.29$ m/s

12-13 a) $a = -25.0$ mm/s^2, b) $s = 100$ mm, c) $t \rightarrow \infty$

12-14 $v = 322$ m/s, $t = 19.3$ s

12-15 $t|_{s=60\text{ m}} = 2.45$ s, $t_{stop} = 6.67$ s

12-17 $s = 6.53$ m, $t = 3.27$ s

12-18 $s_A = 960$ m

12-19 $s = 28.4$ km

12-21 $v = 3.93$ m/s, $s = 9.98$ m

12-22 $t = 0.549\left(\dfrac{v_f}{g}\right)$

12-23 $v = 22.4$ m/s

12-25 $h_{max} = \dfrac{1}{2k}\ln\left(1 + \dfrac{k}{g}v_0^2\right)$

12-26 (a) $v = 45.5$ m/s, (b) $v_{max} = 100$ m/s

12-27 $h = 4.54$ m

12-29 $v = -R\sqrt{\dfrac{2g_0(y_0 - y)}{(R + y)(R + y_0)}}$, $v_{imp} = 3.02$ km/s

12-30 $a_c = -4.40$ m/s^2

12-31 $s_{AB} = 35.7$ m

12-33 $s = \dfrac{v_0}{k}(1 - e^{-kt}), a = -kv_0e^{-kt}$

12-34 $h = 20.4$ m, $t = 2.00$ s

12-35 $v = \dfrac{3\pi}{4}\cos\left(\dfrac{\pi}{4}t\right)$

$a = -\dfrac{3\pi^2}{16}\sin\left(\dfrac{\pi}{4}t\right)$

12-37 For $0 \le t < 5$ s, $s = \{2t^2\}$ m and $a = 4.00$ m/s^2. For 5 s $< t < 20$ s, $s = \{20t - 50\}$ m and $a = 0$. For 20 s $< t \le 30$ s, $s = \{2t^2 - 60t + 750\}$ m and $a = 4.00$ m/s^2.

12-38 For $0 \le t < 12$ s, $a_A = 20$ m/s^2, $v_A = (20t)$ m/s, $s_A = (10t^2)$ m
For 12 s $< t \le 20$ s, $a_A = 0$, $v_A = 240$ m/s, $s_A = 240(t - 6)$ m
For $0 \le t < 10$ s, $a_B = 15$ m/s^2, $v_B = (15t)$ m/s, $s_B = (7.5t^2)$ m
For 10 s $< t \le 20$ s, $a_B = 0$, $v_B = 150$ m/s, $s_B = 150(t - 5)$ m
$\Delta s = 1.11$ km

12-39 $a|_{t=0} = -4.00$ m/s^2, $a|_{t=2\text{ s}} = 0$, $a|_{t=4\text{ s}} = 4.00$ m/s^2, $v|_{t=0} = 3.00$ m/s, $v|_{t=2\text{ s}} = -1.00$ m/s, $v|_{t=4\text{ s}} = 3.00$ m/s

12-41 For $0 \le t < 10$ s, $v = (0.4t^2)$ m/s
For 10 s $< t \le 11.2$ s, $v = (8t - 40)$ m/s
$t = 11.2$ s

12-42 $s = 600$ m. For $0 \le t < 40$ s, $a = 0$.
For 40 s $< t \le 80$ s, $a = -0.250$ m/s^2.

12-43 $v_{max} = 0.894$ m/s, $t' = 0.447$ s
For $0 \le t < 0.224$ s, $s = (2t^2)$ m;
For 0.224 s $< t \le 0.447$ s, $s = (-2t^2 + 1.79t - 0.2)$ m

12-45 For $0 \le t < 30$ s, $v = \left\{\dfrac{1}{5}t^2\right\}$ m/s, $s = \left\{\dfrac{1}{15}t^3\right\}$ m
For $30 \le t \le 60$ s, $v = \{24t - 540\}$ m/s, $s = \{12t^2 - 540t + 7200\}$ m

12-46 For $0 \le t < 9$ s,
$v = (4t^{3/2})$ m/s
$s = \left(\dfrac{8}{5}t^{5/2}\right)$ m
For 9 s $< t \le 14$ s,
$v = (2t^2 - 18t + 108)$ m/s
$s = \left(\dfrac{2}{3}t^3 - 9t^2 + 108t - 340\right)$ m

12-47 For $0 \le t < 4$ s, $a = 3.50$ m/s^2;
For 4 s $< t < 5$ s, $a = 0$; For 5 s $< t \le 8$ s, $a = 4.00$ m/s^2; $a_{max} = 4.00$ m/s^2

12-49 For $0 \le t < 30$ s, $v = (0.800t)$ m/s, $a = 0.800$ m/s^2;
For 30 s $< t \le 40$ s, $v = 24.0$ m/s, $a = 0$

12-50 For $0 \le t < 60$ s, $s = \left\{\dfrac{1}{20}t^2\right\}$ m, $a = 0.100$ m/s^2.
For 60 s $< t < 120$ s, $s = \{6t - 180\}$ m, $a = 0$.
For 120 s $< t \le 180$ s, $s = \left\{\dfrac{1}{30}t^2 - 2t + 300\right\}$ m, $a = 0.0667$ m/s^2.

12-51 For $0 \le t < 10$ s,
$a = 8.00$ m/s^2
$s = (4t^2)$ m
For 10 s $< t \le 40$ s,
$a = 0$
$s = (80t - 400)$ m

12-53 $s = 52.5$ m; For $0 \le t < 4$ s, $a = 1.25$ m/s^2,
$s = (0.625t^2)$ m;
For 4 s $< t <$ 10 s, $a = 0$, $s = (5t - 10)$ m;
For 10 s $< t \le$ 15 s, $a = -1.00$ m/s^2,
$s = (15t - 0.5t^2 - 60)$ m

12-54 At $t = 8$ s, $a = 0$ and $s = 30.0$ m. At $t = 12$ s,
$a = -1.00$ m/s^2 and $s = 48.0$ m.

12-55 For $0 \le t < 15$ s, $v = \left\{\frac{1}{2}t^2\right\}$ m/s, $s = \left\{\frac{1}{6}t^3\right\}$ m.
For 15 s $< t \le$ 40 s,
$v = \{20t - 187.5\}$ m/s,
$s = \{10t^2 - 187.5t + 1125\}$ m

12-57 $s = 144$ m; For $0 \le t < 30$ s, $s = (t^2/10)$ m;
For 30 s $< t \le$ 48 s, $s = (-t^2/6 + 16t - 240)$ m

12-58 For $0 \le t < 10$ s, $a_A = 4$ m/s^2, $v_A = (4t)$ m/s,
$s_A = (2t^2)$ m
For 10 s $< t \le$ 15 s, $a_A = 0$, $v_A = 40.0$ m/s,
$s_A = 40(t - 5)$ m
For $0 \le t < 5$ s, $a_B = 5$ m/s^2, $v_B = (5t)$ m/s,
$s_B = (2.5t^2)$ m
For 5 s $< t \le$ 15 s, $a_B = 0$, $v_B = 25.0$ m/s,
$s_B = 25(t - 25)$ m
$\Delta s = 87.5$ m

12-59 For $0 \le t < 6$ s
$v = \left(\frac{1}{18}t^3\right)$ m/s
For 6 s $< t \le$ 10 s
$v = (6t - 24)$ m/s
$s = 114$ m

12-61 $v = (3t^2 - 6t + 2)$ m/s, $a = (6t - 6)$ m/s^2

12-62 $v_{max} = 80.0$ m/s, $t = 49.3$ s

12-63 $v = \{5 - 6t\}$ m/s, $a = -6.00$ m/s^2

12-65 For $0 \le s < 100$ m
$v = \{0.141s\}$ m/s,
For 100 m $< s \le$ 200 m
$v = \left\{\sqrt{4s - 200}\right\}$ m/s

12-66 For $0 \le s < 300$ m, $v = \left(\frac{1}{10}s\right)$ m/s;
For 300 m $< s \le$ 600 m,
$v = \sqrt{12s - \frac{1}{100}s^2 - 1800}$ m/s

12-67 For $0 \le t < 10$ s, $s = (t^2)$ m
For 10 s $< t \le$ 16.9 s, $s = (13.5e^{t/5})$ m
For $0 \le s < 100$ m, $a = 2$ m/s^2
For 100 m $< s \le$ 400 m, $a = (0.04s)$ m/s^2
$t = 16.9$ s

12-69 $v = 201$ m/s, $a = 405$ m/s^2

12-70 $a = 5.31$ m/s^2, $\alpha = 53.0°$ $\beta = 37.0°$, $\gamma = 90.0°$

12-71 $\Delta\mathbf{r} = \{6\mathbf{i} + 4\mathbf{j}\}$ m

12-73 $a_x = 4r\cos 2t$, $a_y = -4r\sin 2t$

12-74 $(v_{sp})_{avg} = 5.11$ m/s

12-75 $a = 80.2$ m/s^2, $(x, y, z) = (42.7, 16.0, 14.0)$ m

12-77 $v_x = v_0\left[1 + \left(\frac{\pi}{L}c\right)^2\cos^2\left(\frac{\pi}{L}x\right)\right]^{-\frac{1}{2}}$,
$v_y = \frac{v_0\pi c}{L}\cos\left(\frac{\pi}{L}x\right)\left[1 + \left(\frac{\pi}{L}c\right)^2\cos^2\left(\frac{\pi}{L}x\right)\right]^{-\frac{1}{2}}$

12-78 $\theta = 45°$

12-79 $v = 1.00$ km/s, $a = 103$ m/s^2

12-81 $v = \sqrt{c^2k^2 + b^2}$, $a = ck^2$

12-82 $(\mathbf{a}_{AB})_{avg} = \{1.20\mathbf{i} + 1.60\mathbf{j}\}$ m/s^2,
$(\mathbf{a}_{AC})_{avg} = \{-1.12\mathbf{i}\}$ m/s^2

12-83 $d = 204$ m, $v = 41.8$ m/s, $a = 4.66$ m/s^2

12-85 $v_A = 23.2$ m/s, $\theta_A = 30.5°$

12-86 $v_A = 11.7$ m/s, $v_B = 10.8$ m/s, $\theta_B = 20.7°$ ⦩

12-87 $\theta = 58.3°$, $(v_0)_{min} = 9.76$ m/s

12-89 $d = \frac{v_0^2}{g\cos\theta}\left(\sin 2\phi - 2\tan\theta\cos^2\phi\right)$

12-90 $\phi = \frac{1}{2}\tan^{-1}(-\text{ctn}\,\theta)$

12-91 $R_{max} = 10.2$ m, $\theta = 45.0°$

12-93 $\theta_A = 7.19°$ and $80.5°$

12-94 $v_A = 19.4$ m/s, $t_{AB} = 4.54$ s

12-95 $v_A = 19.4$ m/s, $v_B = 40.4$ m/s

12-97 $\theta_A = 51.4°$, $d = 7.18$ m

12-98 $v_A = 18.2$ m/s, $d = 12.7$ m

12-99 $v_B = 160$ m/s, $h_B = 427$ m,
$h_C = 1.08$ km, $R = 2.98$ km

12-101 $\theta_A = 11.6°$, $t = 0.408$ s, $\theta_B = 11.6°$ ⦩,
$v_B = 10.0$ m/s

12-102 $\theta_A = 78.4°$, $t = 2.00$ s, $\theta_B = 78.4°$ ⦩,
$v_B = 10.0$ m/s

12-103 $v_{min} = 0.838$ m/s, $v_{max} = 1.76$ m/s

12-105 $\theta_C = 75.3°$, $\theta_D = 14.7°$, $\Delta t = 1.45$ s

12-106 $t = 3.55$ s, $v = 32.0$ m/s

12-107 $R = 19.0$ m, $t = 2.48$ s

12-109 $R_{min} = 0.189$ m, $R_{max} = 1.19$ m

12-110 $a = 4.22$ m/s^2

12-111 $v = 38.7$ m/s

12-113 $v_n = 0$, $v_t = 7.21$ m/s,
$a_n = 0.555$ m/s^2, $a_t = 2.77$ m/s^2

12-114 $v = 4.40$ m/s, $a_t = 5.04$ m/s^2, $a_n = 1.39$ m/s^2

12-115 $v = 5.66$ m/s, $a = 1.02$ m/s^2

12-117 $a = 1.30$ m/s^2

12-118 $a = 0.730$ m/s^2

12-119 $v = 19.9$ m/s, $a = 24.2$ m/s^2

12-121 $a = 0.525$ m/s^2

12-122 $a = 2.75$ m/s^2

12-123 $a = 1.68$ m/s^2

12-125 $a_t = 3.62$ m/s^2, $\rho = 29.6$ m

12-126 $a_A = 4.44$ m/s^2

12-127 $a_B = 0.556$ m/s^2

12-129 $t = 1.08$ s

12-130 $t = 7.00$ s, $s = 98.0$ m
12-131 $a = 32.2\ \text{m/s}^2$
12-133 $v = 1.96$ m/s, $a = 0.930\ \text{m/s}^2$
12-134 $a = 5.02\ \text{m/s}^2$
12-135 $a = 0.763\ \text{m/s}^2$
12-137 $a_t = 17.1\ \text{m/s}^2$, $\rho = 6.44$ km
12-138 $y = -0.0766x^2$, $v = 8.37$ m/s, $\theta_v = 17.0°$ ↙
$a_n = 9.38\ \text{m/s}^2$, $a_t = 2.88\ \text{m/s}^2$
12-139 $v_B = 19.1$ m/s, $a = 8.22\ \text{m/s}^2$, $\phi = 17.3°$
up from negative t axis
12-141 $a = 2.01\ \text{m/s}^2$
12-142 $a_{\text{max}} = v^2a/b^2$
12-143 $a_{\text{min}} = v^2b/a^2$
12-145 $\theta = 10.6°$, $a = 51.3\ \text{m/s}^2$
12-146 $a = 26.9\ \text{m/s}^2$
12-147 $d = 11.0$ m, $a_A = 19.0\ \text{m/s}^2$, $a_B = 12.8\ \text{m/s}^2$
12-149 $a = 322\ \text{mm/s}^2$, $\theta = 26.6°$ ↗
12-150 $a = 0.511\ \text{m/s}^2$
12-151 $a = 0.309\ \text{m/s}^2$
12-153 $\alpha = 52.5°$, $\beta = 142°$, $\gamma = 85.1°$ or $\alpha = 128°$,
$\beta = 37.9°$, $\gamma = 94.9°$
12-154 $v = 43.0$ m/s, $a = 6.52\ \text{m/s}^2$
12-155 $\mathbf{v} = \{-14.4\mathbf{u}_r - 24.0\mathbf{u}_z\}$ m/s
$\mathbf{a} = \{-3.61\mathbf{u}_r - 6.00\mathbf{u}_z\}\ \text{m/s}^2$
12-157 $v_r = 691$ m/s, $v_\theta = 402$ m/s
12-158 $v_r = a \sin\theta\dot{\theta}$, $v_\theta = (b - a\cos\theta)\dot{\theta}$,
$a_r = (2a\cos\theta - b)\dot{\theta}^2 + a\sin\theta\ddot{\theta}$,
$a_\theta = (b - a\cos\theta)\ddot{\theta} + 2a\dot{\theta}^2\sin\theta$
12-159 $a = 1.33\ \text{m/s}^2$
12-161 $\dot{\theta} = 0.378$ rad/s
12-162 $v = 20.0$ m/s, $a = 5.39\ \text{m/s}^2$
12-163 $v_r = ae^{at}$, $v_\theta = e^{at}$,
$a_r = e^{at}(a^2-1)$, $a_\theta = 2ae^{at}$
12-165 $v_r = -16.9$ m/s, $v_\theta = 4.87$ m/s, $a_r = -89.4\ \text{m/s}^2$,
$a_\theta = -53.7\ \text{m/s}^2$
12-166 $\mathbf{a} = (\dddot{r} - 3\dot{r}\dot{\theta}^2 - 3r\dot{\theta}\ddot{\theta})\mathbf{u}_r$
$+ (3\dot{r}\ddot{\theta} + r\dddot{\theta} + 3\ddot{r}\dot{\theta} - r\dot{\theta}^3)\mathbf{u}_\theta + (\dddot{z})\mathbf{u}_z$
12-167 $v = 146$ mm/s, $a = 90.0\ \text{mm/s}^2$
12-169 $v = 0.242$ m/s, $a = 0.169\ \text{m/s}^2$,
$(v_A)_x = 0.162$ m/s ←, $(v_A)_y = 0.180$ m/s ↑
12-170 $v_r = 1.20$ m/s, $v_\theta = 1.50$ m/s,
$a_r = -4.50\ \text{m/s}^2$, $a_\theta = 7.20\ \text{m/s}^2$
12-171 $a = 19.3\ \text{m/s}^2$
12-173 $v = 3.49$ m/s
12-174 $a = 27.8\ \text{m/s}^2$
12-175 $v = lc\sec^2 ct$, $a = 2lc^2\sec^2 ct\tan ct$
12-177 $a = 7.26\ \text{m/s}^2$
12-178 $v = 4.16$ m/s, $a = 33.1\ \text{m/s}^2$
12-179 $a = 2.32\ \text{m/s}^2$
12-181 $v_{AB} = 25.9$ mm/s, $a_{AB} = 42.5\ \text{mm/s}^2$
12-182 $v = 8.49$ m/s, $a = 88.2\ \text{m/s}^2$
12-183 $v = 5.16$ m/s, $a = 39.1\ \text{m/s}^2$
12-185 $a = 8.66\ \text{m/s}^2$
12-186 $v_r = 306$ m/s, $v_\theta = 177$ m/s, $a_r = -128\ \text{m/s}^2$,
$a_\theta = 67.7\ \text{m/s}^2$
12-187 $v_r = -14.1$ m/s, $v_\theta = 14.1$ m/s, $a_r = -1.41\ \text{m/s}^2$,
$a_\theta = -1.41\ \text{m/s}^2$
12-189 $v_{AB} = -250$ mm/s, $a_{AB} = -2.17\ \text{m/s}^2$
12-190 $v_{AB} = -250$ mm/s, $a_{AB} = -2.47\ \text{m/s}^2$
12-191 $v = 2a\dot{\theta}$, $a = 4a\dot{\theta}^2$
12-193 $\dot{\theta} = 0.333$ rad/s, $a = 6.67\ \text{m/s}^2$
12-194 $a_r = -6.67\ \text{m/s}^2$, $a_\theta = 3\ \text{m/s}^2$
12-195 a) $t = 1.07$ s, b) $v_{A/B} = 5.93$ m/s →
12-197 a) $t = 1.22$ s, b) $v_{A/B} = 2.90$ m/s ↑
12-198 $v_B = 1.67$ m/s ↑
12-199 $v_A = 32.0$ m/s ↓
12-201 $\Delta s_B = 0.400$ m →
12-202 $t = 5.00$ s
12-203 $v_B = 5.33$ m/s ↑
12-205 $v_B = 1.50$ m/s ←
12-206 $v_B = 1.00$ m/s ↑
12-207 $v_A = 1.33$ m/s ↑
12-209 $v_B = 2.40$ m/s ↑, $a_B = 3.25\ \text{m/s}^2$ ↑
12-210 $v_C = 0.667$ m/s ↑
12-211 $v_A = 14.7$ m/s
12-213 $v_C = 1.20$ m/s ↑, $a_C = 0.512\ \text{m/s}^2$ ↑
12-214 $v_A = 10.0$ m/s ←, $a_A = 46.0\ \text{m/s}^2$ ←
12-215 $v_C = (1.8\sec\theta)$ m/s →
12-217 $v_A = -v_B\left[1 + \left(\dfrac{h}{s_A}\right)^2\right]^{1/2}$,

$$a_A = -a_B\left[1 + \left(\frac{h}{s_B}\right)^2\right]^{1/2} + \frac{v_A v_B h^2}{s_A^3}\left[1 + \left(\frac{h}{s_A}\right)^2\right]^{-1/2}$$

12-218 $\theta = 15.1°$
12-219 $v_{B/A} = 1.04$ Mm/h, $\theta = 54.5°$ ↗
12-221 $v_{A/C} = 21.5$ m/s, $\theta_v = 34.9°$ ↖,
$a_{A/C} = 4.20\ \text{m/s}^2$, $\theta_a = 75.4°$ ↙
12-222 $v_{B/C} = 18.6$ m/s, $\theta_v = 66.2°$ ↗, $a_{B/C} = 0.959\ \text{m/s}^2$,
$\theta_a = 8.57°$ ↘
12-223 $v_{A/B} = 15.7$ m/s, $\theta = 7.11°$ ↙, $t = 38.1$ s
12-225 $v_{A/B} = 120$ km/h ↓; $a_{A/B} = 4.00\ \text{Mm/h}^2$,
$\theta_a = 0.716°$ ↙
12-226 $v_{B/A} = 20.5$ m/s, $\theta_v = 43.1°$ ↙, $a_{B/A} = 4.92\ \text{m/s}^2$,
$\theta_a = 6.04°$ ↙
12-227 $v_{B/A} = 20.5$ m/s, $\theta_v = 43.1°$ ↙; $a_{B/A} = 9.17\ \text{m/s}^2$,
$\theta_a = 74.8°$ ↙
12-229 $v_r = 34.6$ km/h ↓
12-230 $v_b = 6.21$ m/s, $t = 11.4$ s
12-231 $v_{w/s} = 19.9$ m/s, $\theta = 74.0°$ ↙

Chapter 13

13-1 $v = 3.36\text{ m/s}, s = 5.04\text{ m}$
13-2 $P = 224\text{ N}$
13-3 $a = 1.66\text{ m/s}^2$
13-5 $a = 1.75\text{ m/s}^2$
13-6 $F = \dfrac{m(a_B + g)\sqrt{4y^2 + d^2}}{4y}$
13-7 $v = 3.29\text{ m/s}$
13-9 $v = 32.2\text{ m/s}$
13-10 $t = 2.04\text{ s}$
13-11 $s = 8.49\text{ m}$
13-13 $A_x = 685\text{ N}, A_y = 1.19\text{ kN}, M_A = 4.74\text{ kN}\cdot\text{m}$
13-14 $a = \dfrac{1}{2}(1 - \mu_k)\,g$
13-15 $T = 11.25\text{ kN}, F = 33.75\text{ kN}$
13-17 $R = 2.45\text{ m}, t_{AB} = 1.72\text{ s}$
13-18 Team A, $a_B = 0.434\text{ m/s}^2$
13-19 $P = 574\text{ N}$
13-21 $v = 14.1\text{ m/s}$
13-22 $s = 5.43\text{ m}$
13-23 $v_B = 4.52\text{ m/s}$
13-25 $a_A = \dfrac{5}{11}g\uparrow, a_B = \dfrac{3}{11}g\downarrow, a_C = \dfrac{7}{11}g\downarrow$
13-26 $t = 0.815\text{ s}$
13-27 $R = \{150t\}\text{ N}$
13-29 $v = 0.301\text{ m/s}$
13-30 $T = 1.63\text{ kN}$
13-31 $T = 1.80\text{ kN}$
13-33 $P = 31.9\text{ N}, t = 3.07\text{ s}$
13-34 $P = 2mg\left(\dfrac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right)$,
$a = \left(\dfrac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right)g$
13-35 $v_A = 20.7\text{ m/s}\downarrow$
13-37 $a = 6.92\text{ m/s}^2$ (The initial velocity of block B will not affect its acceleration)
13-38 $A_x = 0, A_y = 2.11\text{ kN}, B_y = 1.92\text{ kN}$
13-39 $s = 16.7\text{ m}, T = 1.20\text{ kN}$
13-41 $a_C = 7.49\text{ m/s}^2\ \theta = 22.8°\measuredangle$
13-42 a) $a_C = 6.94\text{ m/s}^2\ \measuredangle 45°$
b) $a_C = 6.94\text{ m/s}^2\ \measuredangle 45°$
c) $a_C = 7.08\text{ m/s}^2, \theta = 56.5°\measuredangle$
13-43 $T_A = 90.6\text{ N}, T_B = 22.6\text{ N}, a_A = 1.51\text{ m/s}^2\uparrow$,
$a_B = 6.04\text{ m/s}^2\downarrow$
13-45 $P = 2mg\tan\theta$
13-46 $P = 2mg\left(\dfrac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right)$
13-47 $x = \dfrac{m\,v_0}{k}\cos\theta_0(1 - e^{-(k/m)t}), y = -\dfrac{mgt}{k} +$
$\dfrac{m}{k}\left(v_0\sin\theta_0 + \dfrac{mg}{k}\right)(1 - e^{-(k/m)t})$,
$x_{\max} = \dfrac{m\,v_0\cos\theta_0}{k}$
13-49 $v_{B/AC} = a_0\sin\theta\,t, s = a_0\sin\theta\,t^2/2$
13-50 $d = \dfrac{(m_A+m_B)g}{k}$
13-51 $v = \dfrac{mg}{c}[1 - e^{-(c/m)t}]$,
$s = \dfrac{mg}{c}\left\{t + \dfrac{m}{c}[e^{-(c/m)t} - 1]\right\}$
13-53 $v = 10.5\text{ m/s}$
13-54 When $v = 40\text{ m/s}, F_L = 42.4\text{ kN}, \theta = 22.2°$;
When $v = 60\text{ m/s}, \theta = 42.5°$
13-55 $\rho = 9.32\text{ m}$
13-57 $v = 0.969\text{ m/s}$
13-58 $v = 1.48\text{ m/s}$
13-59 $v = 9.37\text{ m/s}, N_z = 589\text{ N}, N_n = 340\text{ N}, N_t = 0$
13-61 $L = 51.5\text{ kN}, \theta = 17.8°$
13-62 $L = 50.8\text{ kN}, r = 3.60\text{ km}$
13-63 $F_{y'} = 15.9\text{ kN}, \rho = 282\text{ m}$
13-65 $v = 6.30\text{ m/s}, F_n = 283\text{ N}, F_t = 0, F_b = 490\text{ N}$
13-66 $\theta = 26.7°$
13-67 $v = 22.1\text{ m/s}$
13-69 $F_s = 117\text{ N}$
13-70 $d = 0.500\text{ m}$
13-71 $N = 6.73\text{ kN}, F_f = 3.51\text{ kN}$
13-73 $v = 49.5\text{ m/s}$
13-74 $N = 466\text{ N}, a_t = 0.986\text{ m/s}^2$
13-77 $a = 7.32\text{ m/s}^2, N = 59.2\text{ N}$
13-78 $v_G = 2.83\text{ m/s}$
13-79 At $\theta = 0°, T = 64.0\text{ N}$; At $\theta = 30°, T = 34.6\text{ N}$
13-81 $N = 4.16\text{ kN}, a_t = 1.92\text{ m/s}^2$
13-82 $\theta = 37.7°$
13-83 $T = 9.09\text{ kN}, \theta = 43.3°$
13-85 $F = 7.72\text{ N}$
13-86 $(F_z)_{\min} = 15.6\text{ N}, (F_z)_{\max} = 23.6\text{ N}$
13-87 $F = 210\text{ N}$
13-89 $v_r = 2.50\text{ m/s}, v_\theta = 2.00\text{ m/s}$
13-90 $N = 4.90\text{ N}, F = 4.17\text{ N}$
13-91 $F_r = 102\text{ N}, N_z = 375\text{ N}, N_\theta = 79.7\text{ N}$
13-93 $T = 5.66\text{ N}$
13-94 $\psi = 84.3°, a_t = 12\text{ m/s}^2$
13-95 $N = 12.1\text{ N}, F = 7.67\text{ N}$
13-97 $\dot{\theta}_{\min} = 4.67\text{ rad/s}$
13-98 $\dot{\theta}_{\max} = 7.00\text{ rad/s}$
13-99 $F_r = -1.54\text{ kN}, F_\theta = 0, F_z = 2.45\text{ kN}$
13-101 $\dot{\theta} = 5.70\text{ rad/s}$

13-102 $F_r = -131$ N, $F_\theta = -38.4$ N, $F_z = 215$ N
13-103 $a_t = 12$ m/s^2; $\theta = 5.71°$ ↖
13-105 $F = 24.8$ N, $N = 24.8$ N
13-106 $N = 2.95$ kN
13-107 $N = 0.328$ N, $F = 10.5$ N
13-109 $N_C = 20.7$ N, $F_{OA} = 0$
13-110 $P = 17.8$ N, $N = 59.3$ N
13-111 $N = 2.86$ kN
13-113 $T^2 = \left(\frac{4\pi^2}{GM_s}\right)a^3$
13-114 $v_0 = 30.4$ km/s, $\frac{1}{r} = 0.113\left(10^{-12}\right)\cos\theta + 6.74\left(10^{-12}\right)$
13-115 $h = 35.9$ Mm, $v_s = 3.07$ km/s
13-117 $v_{r/s} = 1.68$ km/s
13-118 $v_0 = 7.82$ km/s, $T = 1.97$ h
13-119 $v_p = 7.76$ km/s, $v_A = 4.52$ km/s, $T = 3.35$ hr
13-121 $v_A = 6.33$ km/s, $t = 2.9$ hr, $\Delta v = 578$ m/s
13-122 $v = 7.31$ km/s
13-123 $v_A = 7.31$ km/s
13-125 $(v_A)_C = 5.27(10^3)$ m/s, $\Delta v = 684$ m/s
13-126 $t = 3.87$ h
13-127 a) $v_{A'} = 2.43$ km/s, b) $(v_A)_C = 6.38$ km/s
c) $T_c = 2.19$ h, $T_e = 6.78$ h
13-129 $v_A = 7.01(10^3)$ m/s
13-130 $\Delta v = -1.05$ km/s
13-131 $v_A = 2.01(10^3)$ m/s

Chapter 14

14-1 $x_{max} = 0.825$ m
14-2 $s = 59.7$ m
14-3 $s = 5.99$ m
14-5 $d = 120$ m
14-6 $h = 39.3$ m, $\rho = 26.2$ m
14-7 $v = 5.63$ m/s
14-9 Observer A: $v_2 = 6.08$ m/s,
Observer B: $v_2 = 4.08$ m/s
14-10 $v = 2.84$ m/s
14-11 $s = 20.5$ m
14-13 $v = 3.58$ m/s
14-14 $v = 4.08$ m/s
14-15 $k = 15.0$ MN/m^2
14-17 $\theta = 41.4°$
14-18 $v_B = 3.34$ m/s
14-19 $v_C = 2.36$ m/s
14-21 $k_B = 11.1$ kN/m
14-22 $s = 0.730$ m
14-23 $x = 0.688$ m
14-25 $v_A = 28.3$ m/s
14-26 $F = 43.9$ N
14-27 $v = 10.7$ m/s
14-29 $R = 2.83$ m, $v_C = 7.67$ m/s
14-30 $s = 1.35$ m
14-31 $v_2 = 15.4$ m/s
14-33 $v_A = 1.98$ m/s ↓
14-34 $v_A = 3.82$ m/s ↑
14-35 $v_D = 17.7$ m/s, $R = 33.0$ m
14-37 $h_A = 22.5$ m, $h_C = 12.5$ m
14-38 $v = 1.07$ m/s
14-39 $v_B = 14.9$ m/s, $N = 1.25$ kN
14-41 $F = 367$ N
14-42 $P_{in} = 102$ kW
14-43 $\varepsilon = 0.460$
14-45 $v = 4.86$ m/s
14-46 $t = 4.44$ s
14-47 $t = 51.4$ min
14-49 $P_{max} = 113$ kW, $P_{avg} = 56.5$ kW
14-50 $P = 12.6$ kW
14-51 $P = 8.31t$ MW
14-53 $P_i = 622$ kW
14-54 $v = 0.285$ m/s
14-55 $\varepsilon = 0.654$
14-57 $P_i = 22.2$ kW
14-58 $P_{in} = 19.5$ kW
14-59 $(P_{in})_{avg} = 8.76$ kW
14-61 $P = \left\{400(10^3)t\right\}$ W
14-62 For $0 \le t < 0.2$ s, $P = \{53.3t\}$ kW,
For 0.2 s $< t \le 0.3$ s, $P = \left\{160\,t - 533t^2\right\}$ kW,
$U = 1.69$ kJ
14-63 $P_{max} = 10.7$ kW
14-65 $P = 58.1$ kW
14-66 $h = 1.15$ mm
14-67 $F = 227$ N
14-69 $v_B = 5.33$ m/s, $N = 694$ N
14-70 $v_B = 3.56$ m/s, $N = 622$ N
14-71 $h = 18.3$ m, $N_B = 0$, $N_C = 17.2$ kN
14-73 $k_B = 287$ N/m
14-74 $v_A = 1.54$ m/s, $v_B = 4.62$ m/s
14-75 $s_B = 5.70$ m
14-77 $v = 16.4$ m/s
14-78 $d = 8.53$ m, $v_D = 10.0$ m/s
14-79 $v_B = 15.5$ m/s
14-81 $v = 5.94$ m/s, $N = 8.53$ N
14-82 $U_{1-2} = GM_e m\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$
14-85 $s_{max} = 1.96$ m
14-86 $s_B = 0.638$ m, $s_A = 1.02$ m
14-87 $v_2 = 2.15$ m/s
14-89 $\theta = 22.3°$, $s = 0.587$ m
14-90 $v_B = 10.6$ m/s, $T_B = 142$ N, $v_C = 9.47$ m/s,
$T_C = 48.7$ N
14-91 $N = 78.6$ N
14-93 $v = 1.68$ m/s

14-94 $v_2 = \sqrt{\frac{2}{\pi}(\pi - 2)gr}$
14-95 $d = 1.34\ \text{m}$
14-97 $v = 6.97\ \text{m/s}$

Chapter 15

15-1 $v = 1.75\ \text{N}\cdot\text{s}$
15-2 $t = 0.432\ \text{s}$
15-3 $v = 5.07\ \text{m/s}$
15-5 $v = 104\ \text{m/s}$
15-6 $I = 5.68\ \text{N}\cdot\text{s}$
15-7 $F = 19.4\ \text{kN}, T = 12.5\ \text{kN}$
15-9 $v_2 = \frac{2Ct'}{\pi m}, s = \frac{Ct'^2}{\pi m}$
15-10 $v = 6.62\ \text{m/s}$
15-11 $\mu_k = 0.340$
15-13 $v_2 = 15.6\ \text{m/s}$
15-14 $v = 4.05\ \text{m/s}$
15-15 $I = 15.0\ \text{kN}\cdot\text{s}$ in both cases.
15-17 $I_u = 6.05\ \text{mN}\cdot\text{s}, I_c = 6.55\ \text{mN}\cdot\text{s}$
15-18 $v = 4.00\ \text{m/s} \rightarrow$
15-19 $v|_{t=3\text{ s}} = 5.68\ \text{m/s}\downarrow, v|_{t=6\text{ s}} = 21.1\ \text{m/s}\uparrow$
15-21 $v_{max} = 90.0\ \text{m/s}$
15-22 $v = 14.3\ \text{m/s}$
$F_D = 15.7\ \text{kN}$
15-23 $v = 0.849\ \text{m/s}$
15-25 Observer A: $v = 7.40\ \text{m/s}$,
Observer B: $v = 5.40\ \text{m/s}$
15-26 $v = 7.21\ \text{m/s}\uparrow$
15-27 $I = 162\ \text{N}\cdot\text{s}$
15-29 $v_2 = 16.6\ \text{m/s}$
15-30 $T = 520.5\ \text{N}$
15-31 $v = 2.34\ \text{m/s}$
15-33 $v = 7.65\ \text{m/s}$
15-34 $F = 12.7\ \text{kN}$
15-35 $v = 5.21\ \text{m/s} \leftarrow, \Delta T = 32.6\ \text{kJ}$
15-37 $s = 2.11\ \text{m}$
15-38 $v_M = 0.178\ \text{m/s}, N = 771\ \text{N}$
15-39 $v = 0.5\ \text{m/s}, \Delta T = -16.9\ \text{kJ}$
15-41 $s = 4.00\ \text{m}$
15-42 $d = 3.18\ \text{m}$
15-43 $d = 1.40\ \text{m}$
15-45 $v_t = 8.62\ \text{m/s}$
15-46 $v_A = 3.29\ \text{m/s}, v_B = 2.19\ \text{m/s}$
15-47 $v_A = 1.86\ \text{m/s} \leftarrow, v_C = 0.781\ \text{m/s} \rightarrow$
15-49 $s_B = 6.67\ \text{m} \rightarrow$
15-50 $v = \frac{Mv_0}{m + M}, t = \frac{v_0}{\mu g\left(1 + \frac{m}{M}\right)}$
15-51 $x_{c/b} = \left|\frac{v_0^2}{2\mu g\left(1 + \frac{m}{M}\right)}\right|$
15-53 $s_B = 71.4\ \text{mm} \rightarrow$
15-54 $v_B = 9.37\ \text{m/s}$
15-55 $d = 6.87\ \text{mm}$
15-57 $v_c = 5.04\ \text{m/s} \leftarrow$
15-58 $(v_A)_2 = 1.53\ \text{m/s} \leftarrow, (v_B)_2 = 1.27\ \text{m/s} \rightarrow$
15-59 $(v_A)_2 = 0.353\ \text{m/s}, (v_B)_2 = 2.35\ \text{m/s}$
15-61 $v_A = 0.160\ \text{m/s} \leftarrow, v_B = 8.24\ \text{m/s} \rightarrow$
15-62 $x_{max} = 0.839\ \text{m}$
15-65 $(v_P)_2 = 0.940\ \text{m/s}$
15-66 $d = 34.8\ \text{mm}$
15-67 $(v_B)_2 = \frac{1}{3}\sqrt{2gh}(1 + e)$
15-69 a) $(v_A)_2 = 0.600\ \text{m/s} \leftarrow, (v_B)_2 = 3.40\ \text{m/s} \leftarrow$,
b) $s = 304\ \text{mm}$
15-70 $h = 21.8\ \text{mm}$
15-71 $F = 2.62\ \text{kN}$
15-73 $h = 1.57\ \text{m}$
15-74 $v_A = 1.35\ \text{m/s} \rightarrow, v_B = 5.89\ \text{m/s}, \theta = 32.9°$
15-75 $e = 0.0113$
15-77 a) $(v_B)_1 = 8.81\ \text{m/s}, \theta_1 = 10.5°$,
b) $(v_B)_2 = 4.62\ \text{m/s}, \theta_2 = 20.3°$, c) $s = 3.96\ \text{m}$
15-78 $(v_A)_2 = 3.80\ \text{m/s} \leftarrow, (v_B)_2 = 6.51\ \text{m/s}$,
$(\theta_B)_2 = 68.6°$
15-79 $(v_B)_2 = \frac{e(1 + e)}{2}v_0$
15-81 $v_b = v_0\frac{m - eM}{m + M}, v_c = v_0\frac{(1 + e)m}{m + M}$,
$t = \frac{d}{v_0}\left(1 + \frac{1}{e}\right)^2$
15-82 $(v_A)_2 = 6.90\ \text{m/s}, (v_B)_2 = 75.7\ \text{m/s}$
15-83 $e = \sqrt{\frac{h_1}{h_0}}$
15-85 $e = \frac{\sin\phi}{\sin\theta}\left(\frac{\cos\theta - \mu\sin\theta}{\mu\sin\phi + \cos\phi}\right)$
15-86 $\mu = 0.250$
15-87 $(v_A)_2 = 12.1\ \text{m/s}, (v_B)_2 = 12.4\ \text{m/s}$
15-89 $v_1 = 4.43\ \text{m/s}$
15-90 $h = 0.390\ \text{m}$
15-91 $(v_B)_3 = 1.50\ \text{m/s}$
15-93 $e = 2/3$
15-94 $(\mathbf{H}_A)_O = \{-57.6\mathbf{k}\}\ \text{kg}\cdot\text{m}^2/\text{s}$,
$(\mathbf{H}_B)_O = \{-95.1\mathbf{k}\}\ \text{kg}\cdot\text{m}^2/\text{s}$
15-95 $(\mathbf{H}_A)_P = \{-52.8\mathbf{k}\}\ \text{kg}\cdot\text{m}^2/\text{s}$,
$(\mathbf{H}_B)_P = \{-118\mathbf{k}\}\ \text{kg}\cdot\text{m}^2/\text{s}$

15-97 $(H_A)_P = 66.0 \text{ kg}\cdot\text{m}^2/\text{s}$ ↻, $(H_B)_P = 73.9 \text{ kg}\cdot\text{m}^2/\text{s}$ ↻
15-98 $v = 3.47 \text{ m/s}$
15-99 $\mathbf{H}_O = \{-21.5\mathbf{i} + 21.5\mathbf{j} + 37.6\mathbf{k}\} \text{ kg}\cdot\text{m}^2/\text{s}$
15-101 $v = 4.67 \text{ m/s}$
15-102 $v = 6.15 \text{ m/s}$
15-103 $t = 1.34 \text{ s}$
15-105 $v_2 = 6.74 \text{ m/s}, (v_2)_r = 2.52 \text{ m/s}$
15-106 $v = 3.33 \text{ m/s}$
15-107 $v = 9.50 \text{ m/s}$
15-109 $H_O = 6.76(10^6) \text{ kg}\cdot\text{m}^2/\text{s}$ ↻
15-110 $h = 196 \text{ mm}, \theta = 75.0°$
15-111 $v_2 = 4.03 \text{ m/s}, \Sigma U_{1-2} = 725 \text{ J}$
15-113 $d' = 0.414 \text{ m}, T_1 = 20.3 \text{ N}, T_2 = 21.6 \text{ N}$
15-114 $C_x = 4.97 \text{ kN}, D_x = 2.23 \text{ kN}, D_y = 7.20 \text{ kN}$
15-115 $F_x = 11.0 \text{ N}$
15-117 $a = 0.528 \text{ m/s}^2$
15-118 $F = 6.24 \text{ N}, P = 3.12 \text{ N}$
15-119 $F = v^2\rho_w A$ for all cases
15-121 $F_x = 795 \text{ N}, F_y = 1.20 \text{ kN}$
15-122 $T = 82.8 \text{ N}, N = 396 \text{ N}$
15-123 $F_B = 402 \text{ N}$
15-125 $T = 9.72 \text{ N}$
15-126 $A_y = 4.18 \text{ kN}\uparrow, B_x = 65.0 \text{ N}\rightarrow, B_y = 3.72 \text{ kN}\uparrow$
15-127 $v = 25.0 \text{ m/s}$
15-129 $F = 858 \text{ N}$
15-130 $a = 0.0476 \text{ m/s}^2$
15-131 $v = \left\{\dfrac{8000}{2000 + 50t}\right\} \text{ m/s}$
15-133 $T = 2.50 \text{ kN}$
15-134 $a = 0.125 \text{ m/s}^2, v = 4.05 \text{ m/s}$
15-135 $F = \{7.85t + 0.320\} \text{ N}$
15-137 $v = \sqrt{\dfrac{2}{3}g\left(\dfrac{y^3 - h^3}{y^2}\right)}$
15-138 $a = 0.104 \text{ m/s}^2$
15-139 $F = m'v^2$
15-141 $a = 24.2 \text{ m/s}^2$
15-142 $v = 3.12 \text{ m/s}$

Chapter 16

16-1 $v_A = 22.0 \text{ m/s}, (a_A)_t = 12.0 \text{ m/s}^2, (a_A)_n = 968 \text{ m/s}^2$
16-2 $v_A = 26.0 \text{ m/s}, (a_A)_t = 10.0 \text{ m/s}^2, (a_A)_n = 1352 \text{ m/s}^2$
16-3 $v_B = 10.2 \text{ m/s}, (a_B)_t = 8.00 \text{ m/s}^2, (a_B)_n = 259 \text{ m/s}^2$
16-5 $\omega_B = 528 \text{ rad/s}, \theta_B = 288 \text{ rad}$
16-6 $\omega_B = 329 \text{ rad/s}$
16-7 $t = 6.98 \text{ s}, \theta_D = 34.9 \text{ rev}$
16-9 $\alpha = 60 \text{ rad/s}^2, \omega = 90.0 \text{ rad/s}, \theta = 90.0 \text{ rad}$
16-10 $\alpha_A = 60.8 \text{ rad/s}^2$
16-11 $\omega_A = 225 \text{ rad/s}$
16-13 $a_t = 2.83 \text{ m/s}^2, a_n = 35.6 \text{ m/s}^2$
16-14 $(a_P)_t = 0.754 \text{ m/s}^2, (a_P)_n = 5.14 \text{ m/s}^2$
16-15 $a_B = 29.0 \text{ m/s}^2$
16-17 $v_A = 8.10 \text{ m/s}, (a_A)_t = 4.95 \text{ m/s}^2, (a_A)_n = 437 \text{ m/s}^2$
16-18 $\omega_B = 21.9 \text{ rad/s}$ ↺
16-19 $\omega_B = 31.7 \text{ rad/s}$ ↺
16-21 $\omega_B = 12.0 \text{ rad/s}$
16-22 $\omega_D = 12.0 \text{ rad/s}, \alpha_D = 0.600 \text{ rad/s}^2$
16-23 $\omega_D = 4.00 \text{ rad/s}, \alpha_D = 0.400 \text{ rad/s}^2$
16-25 $v = 0.627 \text{ m/s}, a = 0.982 \text{ m/s}^2$
16-26 $\theta_{\max} = 18.1°$
16-27 $\omega_B = 300 \text{ rad/s}, \omega_C = 600 \text{ rad/s}$
16-29 $\omega_B = 22.3 \text{ rad/s}$
16-30 $\omega_E = 0.160 \text{ rad/s}$
16-31 $s_W = 2.89 \text{ m}$
16-33 $v_E = 3 \text{ m/s}, (a_E)_t = 2.70 \text{ m/s}^2, (a_E)_n = 600 \text{ m/s}^2$
16-34 $\omega_B = 312 \text{ rad/s}, \alpha_B = 176 \text{ rad/s}^2$
16-35 $\mathbf{v}_D = \{4.80\mathbf{i} + 3.60\mathbf{j} + 1.20\mathbf{k}\} \text{ m/s}, \mathbf{a}_D = \{-36.0\mathbf{i} + 66.6\mathbf{j} - 40.2\mathbf{k}\} \text{ m/s}^2$
16-37 $r_A = 31.8 \text{ mm}, r_B = 31.8 \text{ mm}$, 1.91 canisters per minute
16-38 $v_C = 2.50 \text{ m/s}, a_C = 13.1 \text{ m/s}^2$
16-39 $v_{AB} = l\omega\cos\theta, a_{AB} = l\alpha\cos\theta - l\omega^2\sin\theta$
16-41 $v = 0.416 \text{ m/s}, a = -0.752 \text{ m/s}^2$
16-42 $\omega = 8.70 \text{ rad/s}, \alpha = -50.5 \text{ rad/s}^2$
16-43 $v_{CD} = r\omega\sec\theta\tan\theta\rightarrow, a_{CD} = r\omega^2(\sec^3\theta + \sec\theta\tan^2\theta)\rightarrow$
16-45 $\omega = 14.4 \text{ rad/s}$ ↻
16-46 $v = \omega d\left(\sin\theta + \dfrac{d\sin 2\theta}{2\sqrt{(R+r)^2 - d^2\sin^2\theta}}\right)$
16-47 $v = -r\omega\sin\theta$
16-49 $\omega = 19.2 \text{ rad/s}$ ↻, $\alpha = 183 \text{ rad/s}^2$ ↻
16-50 $\omega = \dfrac{h}{h^2 + x^2}v_B, \alpha = \dfrac{-2xh}{(h^2 + x^2)^2}v_B^2$
16-51 $\omega = -\left(\dfrac{r}{x\sqrt{x^2 - r^2}}\right)v_A, \alpha = \left[\dfrac{r(2x^2 - r^2)}{x^2(x^2 - r^2)^{3/2}}\right]v_A^2$
16-53 $v_B = \left(\dfrac{h}{d}\right)v_A$
16-54 $\omega_{CD} = 4.17 \text{ rad/s}$ ↺
16-55 $\alpha_{CD} = 1.67 \text{ rad/s}^2$ ↻
16-57 $v_A = 2.45 \text{ m/s}\uparrow$
16-58 $v_C = 1.06 \text{ m/s}\leftarrow, \omega_{BC} = 0.707 \text{ rad/s}$ ↻
16-59 $\omega_{AB} = 2.83 \text{ rad/s}$ ↻, $\omega_{BC} = 2.83 \text{ rad/s}$ ↺
16-61 $v_C = 24.6 \text{ m/s}\downarrow$

16-62 $\omega_{BC} = 10.6\ \text{rad/s} \circlearrowleft, v_C = 29.0\ \text{m/s} \rightarrow$
16-63 $v_G = 9.00\ \text{m/s} \leftarrow$
16-65 $v_C = 1.04\ \text{m/s} \rightarrow$
16-66 $v_P = 4.88\ \text{m/s} \leftarrow, \omega_{BDP} = 4.00\ \text{rad/s} \circlearrowright$
16-67 $v_C = 2.45\ \text{m/s}$ ⊿, $\omega_{BC} = 7.81\ \text{rad/s} \circlearrowleft$
16-69 $\omega = 14.0\ \text{rad/s} \circlearrowright$
16-70 $\omega_D = 105\ \text{rad/s} \circlearrowright$
16-71 $v_D = 7.07\ \text{m/s}$
16-73 $v_B = 3.00\ \text{m/s} \rightarrow, v_C = 0.587\ \text{m/s} \rightarrow$
16-74 $v_E = 4.00\ \text{m/s}, \theta = 52.7°$ ⊿
16-75 $\omega_P = 5\ \text{rad/s} \circlearrowright, \omega_A = 1.67\ \text{rad/s} \circlearrowleft$
16-77 $v_C = -1.70\ \text{m/s}$
16-78 $v_O = \left(\dfrac{R}{R-r}\right)v \rightarrow$
16-79 $v_A = \left(\dfrac{2R}{R-r}\right)v \rightarrow$
16-81 $\omega_S = 57.5\ \text{rad/s} \circlearrowleft, \omega_{OA} = 10.6\ \text{rad/s} \circlearrowleft$
16-82 $\omega_S = 15.0\ \text{rad/s}, \omega_R = 3.00\ \text{rad/s}$
16-85 $\omega_R = 4\ \text{rad/s} \circlearrowleft$
16-86 $\omega_R = 4\ \text{rad/s} \circlearrowleft$
16-87 $\omega_R = 11.4\ \text{rad/s} \circlearrowright$
16-89 $v_C = 2.45\ \text{m/s} \swarrow, \omega_{CB} = 7.81\ \text{rad/s} \circlearrowleft$
16-90 $\omega_{AB} = 13.1\ \text{rad/s} \circlearrowleft$
16-91 $\omega_{AB} = 7.17\ \text{rad/s} \circlearrowright$
16-93 $v_D = 1.20\ \text{m/s}, \theta = 60.0°$ ⊿, $\omega_{BPD} = 3.00\ \text{rad/s} \circlearrowright$,
$\omega_{CD} = 6.00\ \text{rad/s} \circlearrowleft$
16-94 $\omega_{BPD} = 3.00\ \text{rad/s} \circlearrowright, v_P = 1.79\ \text{m/s} \leftarrow$
16-95 $\omega_W = 22.8\ \text{rad/s} \circlearrowright$
16-97 $\omega_{CD} = 57.7\ \text{rad/s} \circlearrowleft$
16-98 $v_C = 8.69\ \text{m/s}, \theta = 22.9°$ ⊿
16-99 $v_D = 5.72\ \text{m/s}, \theta = 36.2°$ ⊿
16-101 $v_D = 0.518\ \text{m/s} \searrow$
16-102 $\omega_C = 1.67\ \text{rad/s} \circlearrowright, v_D = 1.10\ \text{m/s} \downarrow$
16-103 $v_C = 3.86\ \text{m/s} \leftarrow, a_C = 17.7\ \text{m/s}^2 \leftarrow$
16-105 $v_B = 0.6\ \text{m/s} \downarrow$,
$a_B = 1.84\ \text{m/s}^2, \theta = 60.6°$ ⊿
16-106 $\omega = 6.67\ \text{rad/s} \circlearrowleft, v_B = 4.00\ \text{m/s} \searrow$
$\alpha = 15.7\ \text{rad/s}^2 \circlearrowright, a_B = 24.8\ \text{m/s}^2 \nwarrow$
16-107 $a_C = 13.0\ \text{m/s}^2 \swarrow, v_C = 2.50\ \text{m/s} \swarrow$
16-109 $\omega_C = 20.0\ \text{rad/s} \circlearrowleft, \alpha_C = 127\ \text{rad/s}^2 \circlearrowright$
16-110 $\alpha_{AB} = 4.62\ \text{rad/s}^2 \circlearrowleft, a_B = 13.3\ \text{m/s}^2, \theta = 37.0°$ ⊿
16-111 $\alpha_{AB} = 36\ \text{rad/s}^2 \circlearrowright$
16-113 $a_B = 6.21\ \text{m/s}^2, \theta = 45.8°$ ⊿
16-114 $\alpha_{CD} = 474\ \text{rad/s}^2 \circlearrowleft$
16-115 $v_B = 4v \rightarrow$,
$v_A = 2\sqrt{2}v, \theta = 45°$ ⊿,
$a_B = \dfrac{2v^2}{r}\downarrow, a_A = \dfrac{2v^2}{r}\rightarrow$
16-117 $a_C = 10.0\ \text{m/s}^2, \theta = 2.02°$ ⊿
16-118 $\alpha_B = 1.43\ \text{rad/s}^2 \circlearrowright$
16-119 $a_E = 0.0714\ \text{m/s}^2 \uparrow$
16-121 $v_C = 1.56\ \text{m/s} \leftarrow, a_C = 29.7\ \text{m/s}^2, \theta = 24.1°$ ⊿
16-122 $a_A = 1.34\ \text{m/s}^2, \theta = 26.6°$ ⊿, $a_B = 1.65\ \text{m/s}^2 \rightarrow$,
$\alpha_{AB} = 1.50\ \text{rad/s}^2 \circlearrowleft$
16-123 $\alpha_B = 7.50\ \text{rad/s}^2 \circlearrowright$
16-125 $\alpha_{ABC} = 41.6\ \text{rad/s}^2 \circlearrowleft, a_C = 38.2\ \text{m/s}^2, \theta = 39.4°$ ⊿
16-126 $\alpha_{AB} = 0.750\ \text{rad/s}^2 \circlearrowleft, \alpha_{BC} = 3.94\ \text{rad/s}^2 \circlearrowleft$
16-127 $\omega_{AC} = 0, \omega_F = 10.7\ \text{rad/s} \circlearrowright$,
$\alpha_{AC} = 28.7\ \text{rad/s}^2 \circlearrowright$
16-129 $\mathbf{v}_B = \{0.6\mathbf{i} + 2.4\mathbf{j}\}\ \text{m/s}$,
$\mathbf{a}_B = \{-14.2\mathbf{i} + 8.40\mathbf{j}\}\ \text{m/s}^2$
16-130 $(\mathbf{v}_{B/A})_{xyz} = \{31.0\mathbf{i}\}\ \text{m/s}$,
$(\mathbf{a}_{B/A})_{xyz} = \{-14.0\mathbf{i} - 206\mathbf{j}\}\ \text{m/s}^2$
16-131 $\mathbf{v}_A = \{-17.2\mathbf{i} + 12.5\mathbf{j}\}\ \text{m/s}$,
$\mathbf{a}_A = \{349\mathbf{i} + 597\mathbf{j}\}\ \text{m/s}^2$
16-133 $\mathbf{a}_B = \{0.850\mathbf{i} - 12.4\mathbf{j}\}\ \text{m/s}^2$
16-134 $v_C = 2.40\ \text{m/s}, \theta = 30°$ ⊿
16-135 $v_C = 2.40\ \text{m/s} \leftarrow, a_C = 14.4\ \text{m/s}^2 \downarrow$
16-137 $v_B = 7.70\ \text{m/s}, a_B = 201\ \text{m/s}^2$
16-138 $\omega_{CD} = 9.00\ \text{rad/s} \circlearrowright, \alpha_{CD} = 249\ \text{rad/s}^2 \circlearrowleft$
16-139 $\omega_{CD} = 6.93\ \text{rad/s} \circlearrowright, \alpha_{CD} = 56.2\ \text{rad/s}^2 \circlearrowleft$
16-141 $\omega_{AB} = 1.18\ \text{rad/s} \circlearrowright, \alpha_{AB} = 15.4\ \text{rad/s}^2 \circlearrowleft$
16-142 $\omega_{AC} = 0, \alpha_{AC} = 14.4\ \text{rad/s}^2 \circlearrowright$
16-143 $\mathbf{v}_B = \{7.12\mathbf{j}\}\ \text{m/s}, \mathbf{a}_B = \{-15.9\mathbf{i}\}\ \text{m/s}^2$
16-145 $\omega_{AB} = 0.667\ \text{rad/s} \circlearrowleft, \alpha_{AB} = 3.08\ \text{rad/s}^2 \circlearrowright$
16-146 $(\mathbf{v}_{A/C})_{xyz} = \{27.0\mathbf{i} + 25.0\mathbf{j}\}\ \text{m/s}$,
$(\mathbf{a}_{A/C})_{xyz} = \{2.40\mathbf{i} + 0.380\mathbf{j}\}\ \text{m/s}^2$
16-147 $(\mathbf{v}_{B/C})_{xyz} = \mathbf{0}, (\mathbf{a}_{B/C})_{xyz} = \{1\mathbf{i}\}\ \text{m/s}^2$

Chapter 17

17-1 $m = \pi h R^2\left(k + \dfrac{aR^2}{2}\right), I_z = \dfrac{\pi h R^4}{2}\left[k + \dfrac{2aR^2}{3}\right]$
17-2 $I_y = \dfrac{1}{3}ml^2$
17-3 $I_x = \dfrac{3}{10}mr^2$
17-5 $k_x = 57.7\ \text{mm}$
17-6 $I_z = \dfrac{3}{10}mr^2$
17-7 $I_z = m\left(R^2 + \dfrac{3}{4}a^2\right)$
17-9 $L = 2.39\ \text{m}, I_O = 5.93\ \text{kg}\cdot\text{m}^2$
17-10 $I_z = 34.2\ \text{kg}\cdot\text{m}^2$
17-11 $I_x = \dfrac{2}{5}mr^2$
17-13 $I_O = 1.36\ \text{kg}\cdot\text{m}^2$

17-14 $I_A = 7.67\ \text{kg}\cdot\text{m}^2$
17-15 $I_O = 0.113\ \text{kg}\cdot\text{m}^2$
17-17 $I_y = \dfrac{m}{6}(a^2 + h^2)$
17-18 $I_G = 0.230\ \text{kg}\cdot\text{m}^2, \bar{y} = 0.203\ \text{m}$
17-19 $I_O = 0.560\ \text{kg}\cdot\text{m}^2$
17-21 $\bar{y} = 1.78\ \text{m}, I_G = 4.45\ \text{kg}\cdot\text{m}^2$
17-22 $I_{x'} = 0.00719\ \text{kg}\cdot\text{m}^2$
17-23 $I_x = 0.00325\ \text{kg}\cdot\text{m}^2$
17-25 $N_A = 113\ \text{N}, N_B = 325\ \text{N}, a = 4.33\ \text{m/s}^2 \leftarrow$
17-26 $P = 579\ \text{N}$
17-27 $N_A = 559\ \text{N}, N_B = 226\ \text{N}, a_G = 2.26\ \text{m/s}^2;$ $N_B = 454\ \text{N}$
17-29 $a = 4\ \text{m/s}^2 \rightarrow, N_B = 1.14\ \text{kN}, N_A = 327\ \text{N}$
17-30 $N = 29.6\ \text{kN}, V = 0, M = 51.2\ \text{kN}\cdot\text{m}$
17-31 $a_G = 0.3\ \text{m/s}^2, N_A = 5.43\ \text{kN}, N_B = 7.09\ \text{kN}$
17-33 $N_f = 597\ \text{N}, N_r = 7.70\ \text{kN}, F = 5.72\ \text{kN}$
17-34 $a_G = 4.33\ \text{m/s}^2, (\mu_s)_{\text{min}} = 0.833$
17-35 $P_{\text{max}} = \dfrac{mgb}{2(d - \frac{h}{2})}$
17-37 $a = 2.74\ \text{m/s}^2, T = 25.1\ \text{kN}$
17-38 $N = 0.433wx, V = 0.250wx, M = 0.125wx^2$
17-39 $a_m = 1.45\ \text{m/s}^2, a_b = 1.94\ \text{m/s}^2$
17-41 $P = 314\ \text{N}$
17-42 $a = 2.01\ \text{m/s}^2$. The crate slips.
17-43 $\theta = \tan^{-1}\left(\dfrac{a}{g}\right)$
17-45 $T = 15.7\ \text{kN}, C_x = 8.92\ \text{kN}, C_y = 16.3\ \text{kN}$
17-46 $a = 9.81\ \text{m/s}^2, C_x = 12.3\ \text{kN}, C_y = 12.3\ \text{kN}$
17-47 $a_G = 6.78\ \text{m/s}^2$
17-49 $T = 1.52\ \text{kN}, \theta = 18.6^\circ$
17-50 $a = 1.33\ \text{m/s}^2, T = 2.38\ \text{kN}$
17-51 $F_{AB} = 112\ \text{N}, C_x = 26.2\ \text{N}, C_y = 49.8\ \text{N}$
17-53 $\alpha = 2.62\ \text{rad/s}^2$
17-54 $N_E = 27.5\ \text{N}, V_E = 43.7\ \text{N}, M_E = 32.8\ \text{N}\cdot\text{m}$
17-55 $F_{AB} = 1.22\ \text{kN}, F_{CD} = 564\ \text{N}$
17-57 $F_A = \dfrac{3}{2}mg$
17-58 $\omega = 56.2\ \text{rad/s}, A_x = 0, A_y = 98.1\ \text{N}$
17-59 $\alpha = 14.7\ \text{rad/s}^2, A_x = 88.3\ \text{N}, A_y = 147\ \text{N}$
17-61 $\alpha = 4.36\ \text{rad/s}^2 \circlearrowleft, A_x = 29.4\ \text{N},$ $A_y = 49.1\ \text{N}$
17-62 $I_G = 60.0\ \text{g}\cdot\text{m}^2$
17-63 $t = 9.90\ \text{s}$
17-66 $\alpha = 0.694\ \text{rad/s}^2$
17-67 $\omega = \sqrt{\dfrac{3g}{L}(1 - \cos\theta)},$
$O_x = mg\sin\theta\,(2.25\cos\theta - 1.5),$
$O_y = mg\,(1 - 1.5\cos\theta + 1.5\cos^2\theta - 0.75\sin^2\theta)$
17-69 $\omega = 0.474\ \text{rad/s}$
17-70 $\alpha = 0, B_x = 0, B_y = 761\ \text{kN}$
17-71 $\theta = 30.1^\circ$
17-73 $t = 1.91\ \text{s}, A_x = 294\ \text{N}, A_y = 294\ \text{N}$
17-74 $\alpha = 0.476\ \text{rad/s}^2 \circlearrowright$
17-75 $\alpha = 14.2\ \text{rad/s}^2$
17-77 $\alpha = 3.38\ \text{rad/s}^2$
17-78 $t = 6.40\ \text{s}$
17-79 $P = 192\ \text{N}$
17-81 $N_A = 177\ \text{kN}, V_A = 5.86\ \text{kN}, M_A = 50.7\ \text{kN}\cdot\text{m}$
17-82 $A_x = 0, A_y = 289\ \text{N}, \alpha = 23.1\ \text{rad/s}^2$
17-83 $\alpha = 3.60\ \text{rad/s}^2, A_x = 163\ \text{N}, A_y = 214\ \text{N}$
17-85 $N = 0, V_t = \dfrac{3M_0}{2L^3}(2Lx - x^2), V_z = \dfrac{mg}{L}x,$
$T = 0, M_t = \dfrac{mg}{2L}x^2, M_z = \dfrac{M_0}{2L^3}(3Lx^2 - x^3)$
17-86 $N = \dfrac{wx}{2g}(2\omega^2 L - \omega^2 x + 2g\cos\theta), V = wx\sin\theta,$
$M = \dfrac{1}{2}wx^2\sin\theta$
17-87 $\omega = 800\ \text{rad/s}$
17-89 $\alpha = 12.5\ \text{rad/s}^2 \circlearrowright,\ a_G = 18.8\ \text{m/s}^2 \downarrow$
17-90 $\alpha = 28.0\ \text{rad/s}^2, T = 5.61\ \text{N}$
17-93 $\alpha = 3.89\ \text{rad/s}^2$
17-94 $\alpha = 9.51\ \text{rad/s}^2$
17-95 $\alpha = 2.45\ \text{rad/s}^2 \circlearrowright, N_B = 2.23\ \text{N}, N_A = 33.3\ \text{N}$
17-97 $\alpha = 8.20\ \text{rad/s}^2$
17-98 $\alpha = 12.3\ \text{rad/s}^2, (\mu_s)_{\text{min}} = 0.0585$
17-99 $\alpha = 0.560\ \text{rad/s}^2 \circlearrowright, a_G = 0.224\ \text{m/s}^2 \rightarrow$
17-101 $\alpha = 5.62\ \text{rad/s}^2, T = 196\ \text{N}$
17-102 $\alpha = 4.65\ \text{rad/s}^2, T = 359\ \text{N}$
17-103 $a_G = 0.150\ \text{m/s}^2 \rightarrow, T_{AB} = 1.96\ \text{kN}\uparrow,$
$\alpha = 0.225\ \text{rad/s}^2 \circlearrowleft, a_A = 0.300\ \text{m/s}^2 \leftarrow$
17-105 $\alpha = 0.500\ \text{rad/s}^2$
17-106 $\alpha = 1.25\ \text{rad/s}^2, T = 2.32\ \text{kN}$
17-107 $T = 3.13\ \text{kN}, \alpha = 1.68\ \text{rad/s}^2, a_C = 1.35\ \text{m/s}^2$
17-109 $\alpha = \dfrac{6(P - \mu_k mg)}{mL}, a_B = \dfrac{2(P - \mu_k mg)}{m}$
17-110 The disk does not slip.
17-111 $\alpha = 3\ \text{rad/s}^2$
17-113 $\alpha_A = 43.6\ \text{rad/s}^2 \circlearrowright, \alpha_B = 43.6\ \text{rad/s}^2 \circlearrowleft, T = 19.6\ \text{N}$
17-114 $a_G = \mu_k g \leftarrow, \alpha = \dfrac{2\mu_k g}{r} \circlearrowright$
17-115 $\omega = \dfrac{1}{3}\omega_0, t = \dfrac{\omega_0 r}{3\mu_k g}$
17-117 $N = m\left(\dfrac{\omega^2 r^2}{R - r} + g\cos\theta\right), \alpha = \dfrac{2g}{3r}\sin\theta$
17-118 $T_A = \dfrac{4}{7}W$

Chapter 18

18-1 $s = 0.661$ m
18-2 $s = 0.859$ m
18-5 $\omega = 1.88$ rad/s
18-6 $\omega = 3.96$ rad/s
18-7 $\omega = 21.5$ rad/s
18-9 $\omega = 14.1$ rad/s
18-10 $\omega = 14.9$ rad/s
18-11 $v = 4.05$ m/s
18-13 $\omega = 8.64$ rad/s
18-14 $\omega = 30.7$ rad/s
18-15 $v_F = 4.28$ m/s
18-17 $v_C = 7.49$ m/s
18-18 $\omega = \sqrt{\omega_0^2 + \frac{g}{r^2} s \sin\theta}$
18-19 $v = 2.10$ m/s
18-21 $\theta_0 = 1.66$ rad
18-22 $\omega_2 = 2.06$ rad/s
18-23 $\omega_2 = 4.97$ rad/s
18-25 $s_G = 1.60$ m
18-26 $\omega = 44.6$ rad/s
18-27 $\omega_A = 29.8$ rad/s
18-29 $\omega = 4.60$ rad/s
18-30 $\omega_2 = \frac{\sqrt{7}}{7r}\sqrt{7v_G^2 - 10\,gR}$
18-31 $v_G = 3\sqrt{\frac{3}{7}gR}$
18-33 $\omega = 10.5$ rad/s
18-34 $\omega = 7.81$ rad/s
18-35 $\theta = 0.891$ rev, regardless of orientation
18-37 $\omega = 39.3$ rad/s
18-38 $v_b = 1.52$ m/s
18-39 $s_C = 78.0$ mm
18-41 $v_A = 1.29$ m/s
18-42 $(\omega_{ABC})_2 = 7.24$ rad/s
18-43 $\omega_{AB} = \sqrt{\frac{3g}{L}\sin\theta}$
18-45 $\omega = 19.8$ rad/s
18-46 $\omega = 3.78$ rad/s
18-47 $\omega = 3.75$ rad/s
18-49 $\omega = 3.28$ rad/s
18-50 $\theta_0 = 8.94$ rev
18-51 $\omega_{BC} = 1.34$ rad/s
18-53 $(\omega_{BC})_2 = 3.00$ rad/s
18-54 $v_A = 4.00$ m/s
18-55 $v_D = 3.67$ m/s
18-57 $\omega_{AB} = \sqrt{\frac{4g}{3(R - r)}}$
18-58 $v = 20.7$ m/s
18-59 $k = 232$ N·m/rad
18-61 $v_C = 0.900$ m/s
18-62 $\omega = 3.44$ rad/s
18-63 $\omega_{AB} = 3.70$ rad/s
18-65 $\omega_r = 6.33$ rad/s

Chapter 19

19-3 $\omega_2 = \frac{Fd}{mk_G^2}t$
19-5 $v_G = 2$ m/s, $\omega = 3.90$ rad/s
19-6 $\omega_s = \left(\frac{5g\sin\theta}{7r}\right)t, \omega_C = \left(\frac{2g\sin\theta}{3r}\right)t$
19-7 $d = \frac{2}{3}L$
19-9 $t = 0.530$ s
19-10 $F_{AB} = 49.0$ N, $t = 1.53$ s
19-11 $\theta = \tan^{-1}\left[\frac{\mu_s(k_G^2 + r^2)}{k_G^2}\right]$
19-13 $\int F\,dt = 15.2$ kN·s
19-14 $v_G = 1.39$ m/s, $\omega = 9.49$ rad/s
19-15 (a) $\omega_{BC} = 68.7$ rad/s, (b) $\omega_{BC} = 66.8$ rad/s, (c) $\omega_{BC} = 68.7$ rad/s
19-17 $\boldsymbol{\omega}_2 = \{-31.8\mathbf{k}\}$ rad/s
19-18 $v_A = 24.1$ m/s
19-19 $\omega = 18.4$ rad/s ↻
19-21 $\omega = 36.5$ rad/s ↻, $v = 5.48$ m/s ←
19-22 $t = 5.08$ s
19-23 $t = 1.32$ s
19-25 $\omega = 21.0$ rad/s ↻
19-26 $\omega = 9$ rad/s
19-27 $h = 0.533$ m
19-29 $\omega_2 = 0.658$ rad/s, $\theta = 18.8°$
19-30 $\omega_2 = 0.578$ rad/s, $\theta = 15.8°$
19-31 $\omega = 1.91$ rad/s
19-33 $\omega_2 = 2.55$ rev/s
19-34 $\omega_T = 1.19(10^{-3})$ rad/s
19-35 $\omega_1 = 2.41$ rad/s, $\omega_2 = 1.86$ rad/s
19-37 $\omega' = \frac{11}{3}\omega_1$
19-38 a) $\omega = \frac{1}{4}\omega_0$, b) $\omega = 1.00$ rad/s
19-39 $\omega_{\text{off}} = \frac{19}{28}\omega$, $v_{\text{off}} = 0.985\omega L$
19-41 $\omega_2 = 57$ rad/s, $U_F = 367$ J
19-42 $(\omega_z)_2 = 5.10$ rad/s
19-43 $\omega = 6.45$ rad/s
19-45 $\theta = 50.2°$
19-46 $\omega_1 = 3.98$ rad/s
19-47 $h = \frac{7}{5}r$

19-49 $\omega_3 = 2.73 \text{ rad/s}$

19-50 $\omega_2 = \dfrac{7}{34}\omega_0$

19-51 $\omega_1 = 7.17 \text{ rad/s}$

19-53 $h = \dfrac{7}{5}r$

19-54 $\omega = 2.25 \text{ rad/s} \circlearrowright, v_G = 0.149 \text{ m/s},$
$\theta = 38.4° \angle$

19-55 $\omega = \sqrt{7.5\dfrac{g}{L}}$

19-57 $v_2 = 0.195 \text{ m/s}$

19-58 $v_G = 1.00 \text{ m/s}, \theta = 60° \angle, \omega = 5.00 \text{ rad/s} \circlearrowright$

Chapter 20

20-1 $\mathbf{v}_A = \{-0.225\mathbf{i}\} \text{ m/s},$
$\mathbf{a}_A = \{-0.112\mathbf{j} - 0.130\mathbf{k}\} \text{ m/s}^2$

20-2 a) $\boldsymbol{\alpha} = \omega_x\omega_t\,\mathbf{j}$, b) $\boldsymbol{\alpha} = -\omega_x\omega_t\,\mathbf{k}$

20-3 $\boldsymbol{\omega} = \{-8.00\mathbf{j}\} \text{ rad/s}, \boldsymbol{\alpha} = \{64.0\mathbf{i}\} \text{ rad/s}^2,$
$\mathbf{v}_A = \{-0.905\mathbf{i}\} \text{ m/s},$
$\mathbf{a}_A = \{-7.24\mathbf{j} - 7.24\mathbf{k}\} \text{ m/s}^2$

20-5 $\mathbf{v}_B = \{-1.56\mathbf{i} - 3.60\mathbf{j} + 6.24\mathbf{k}\} \text{ m/s},$
$\mathbf{a}_B = \{-0.998\mathbf{i} - 6.38\mathbf{j} + 2.00\mathbf{k}\} \text{ m/s}^2$

20-6 $\mathbf{v}_A = \{-2.60\mathbf{i} - 0.750\mathbf{j} + 1.30\mathbf{k}\} \text{ m/s},$
$\mathbf{a}_A = \{2.77\mathbf{i} - 11.7\mathbf{j} + 2.34\mathbf{k}\} \text{ m/s}^2$

20-7 $\mathbf{v}_A = \{0.0828\mathbf{i} - 0.464\mathbf{j} + 0.124\mathbf{k}\} \text{ m/s},$
$\mathbf{a}_A = \{-0.371\mathbf{i} - 0.108\mathbf{j} - 0.278\mathbf{k}\} \text{ m/s}^2$

20-9 $\boldsymbol{\omega} = \{2\mathbf{i} + 42.4\mathbf{j} + 42.4\mathbf{k}\} \text{ rad/s},$
$\boldsymbol{\alpha} = \{-42.4\mathbf{i} - 82.9\mathbf{j} + 84.9\mathbf{k}\} \text{ rad/s}^2$

20-10 $\mathbf{v}_A = \{-4.00\mathbf{j}\} \text{ m/s},$
$\mathbf{a}_A = \{-24.0\mathbf{i} - 2.00\mathbf{j} - 40.0\mathbf{k}\} \text{ m/s}^2$

20-11 $\mathbf{v}_B = \{-0.4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}\} \text{ m/s},$
$\mathbf{a}_B = \{-8.20\mathbf{i} + 40.6\mathbf{j} - \mathbf{k}\} \text{ rad/s}^2$

20-13 $\boldsymbol{\omega} = \{5.66\mathbf{j} + 6.26\mathbf{k}\} \text{ rad/s}, \boldsymbol{\alpha} = \{-3.39\mathbf{i}\} \text{ rad/s}^2$

20-14 $\mathbf{v}_C = [2.60\mathbf{i} + 4.00\mathbf{j}] \text{ m/s},$
$\mathbf{a}_C = [1.73\mathbf{i} + 5.39\mathbf{j} - 2.28\mathbf{k}] \text{ m/s}^2$

20-15 $\boldsymbol{\omega}_B = \{5\mathbf{j} + 5\mathbf{k}\} \text{ rad/s}$

20-17 $\mathbf{v}_B = \{473\mathbf{i} - 4.50\mathbf{j} + 1.80\mathbf{k}\} \text{ m/s}$
$\mathbf{a}_B = \{123\mathbf{i} - 406\mathbf{j} + 0.450\mathbf{k}\} \text{ m/s}^2$

20-18 $\mathbf{v}_P = \{-1.60\mathbf{i}\} \text{ m/s}$
$\mathbf{a}_P = \{-0.640\mathbf{i} - 12.0\mathbf{j} - 8.00\mathbf{k}\} \text{ m/s}^2$

20-19 $\boldsymbol{\omega} = \{30.0\mathbf{j} - 15.0\mathbf{k}\} \text{ rad/s}, \boldsymbol{\alpha} = \{450\mathbf{i}\} \text{ rad/s}^2$

20-21 $\boldsymbol{\omega} = \{4.35\mathbf{i} + 12.7\mathbf{j}\} \text{ rad/s},$
$\boldsymbol{\alpha} = \{-26.1\mathbf{k}\} \text{ rad/s}^2$

20-22 $\omega_A = 47.8 \text{ rad/s}, \omega_B = 7.78 \text{ rad/s}$

20-23 $\omega = 41.2 \text{ rad/s}, v_P = 4.00 \text{ m/s},$
$\alpha = 400 \text{ rad/s}^2, a_P = 100 \text{ m/s}^2$

20-25 $\boldsymbol{\alpha}_{AB} = \{-7.90\mathbf{i} - 3.95\mathbf{j} + 4.75\mathbf{k}\} \text{ rad/s}^2,$
$\mathbf{a}_B = \{-19.8\mathbf{j} - 19.8\mathbf{k}\} \text{ m/s}^2$

20-26 $v_B = 5.00 \text{ m/s},$
$\boldsymbol{\omega}_{AB} = \{-4.00\mathbf{i} - 0.600\mathbf{j} - 1.20\mathbf{k}\} \text{ rad/s}$

20-27 $\mathbf{a}_B = \{31.5\mathbf{i} - 42.0\mathbf{j}\} \text{ m/s}^2$

20-29 $\mathbf{v}_A = \{2.25\mathbf{k}\} \text{ m/s}$

20-30 $\mathbf{a}_A = [-13.9\mathbf{k}] \text{ m/s}^2$

20-31 $\boldsymbol{\omega}_{BC} = \{0.204\mathbf{i} - 0.612\mathbf{j} + 1.36\mathbf{k}\} \text{ rad/s},$
$\mathbf{v}_B = [-0.333\mathbf{j}] \text{ m/s}$

20-33 $\boldsymbol{\omega}_{BD} = \{-1.20\mathbf{j}\} \text{ rad/s}$

20-34 $\boldsymbol{\alpha}_{BD} = \{-8.00\mathbf{j}\} \text{ rad/s}^2$

20-35 $\mathbf{v}_B = \{-1.33\mathbf{i} + 1.00\mathbf{j}\} \text{ m/s}$

20-37 $\mathbf{v}_A = \{-5.70\mathbf{i} + 1.20\mathbf{j} - 1.60\mathbf{k}\} \text{ m/s},$
$\mathbf{a}_A = \{-1.44\mathbf{i} - 3.74\mathbf{j} - 0.240\mathbf{k}\} \text{ m/s}^2$

20-38 $\mathbf{v}_A = \{-5.70\mathbf{i} + 1.20\mathbf{j} - 1.60\mathbf{k}\} \text{ m/s},$
$\mathbf{a}_A = \{-7.14\mathbf{i} - 1.94\mathbf{j} - 2.64\mathbf{k}\} \text{ m/s}^2$

20-39 $\mathbf{v}_C = \{-1.00\mathbf{i} + 5.00\mathbf{j} + 0.800\mathbf{k}\} \text{ m/s},$
$\mathbf{a}_C = \{-28.8\mathbf{i} - 5.45\mathbf{j} + 32.3\mathbf{k}\} \text{ m/s}^2$

20-41 $\mathbf{v}_B = \{-17.8\mathbf{i} - 3.00\mathbf{j} + 5.20\mathbf{k}\} \text{ m/s},$
$\mathbf{a}_B = \{9.00\mathbf{i} - 29.4\mathbf{j} - 1.50\mathbf{k}\} \text{ m/s}^2$

20-42 $\mathbf{v}_B = \{-17.8\mathbf{i} - 3\mathbf{j} + 5.20\mathbf{k}\} \text{ m/s},$
$\mathbf{a}_B = \{3.05\mathbf{i} - 30.9\mathbf{j} + 1.10\mathbf{k}\} \text{ m/s}^2$

20-43 $\mathbf{v}_A = \{13.9\mathbf{i} + 40.0\mathbf{j} - 8.00\mathbf{k}\} \text{ m/s},$
$\mathbf{a}_A = \{-62.4\mathbf{i} + 115\mathbf{j} - 17.5\mathbf{k}\} \text{ m/s}^2$

20-45 $\mathbf{v}_C = \{3.00\mathbf{i} + 6.00\mathbf{j} - 3.00\mathbf{k}\} \text{ m/s},$
$\mathbf{a}_C = \{-13.0\mathbf{i} + 28.5\mathbf{j} - 10.2\mathbf{k}\} \text{ m/s}^2$

20-46 $\mathbf{v}_P = \{-0.849\mathbf{i} + 0.849\mathbf{j} + 0.566\mathbf{k}\} \text{ m/s},$
$\mathbf{a}_P = \{-5.09\mathbf{i} - 7.35\mathbf{j} + 6.79\mathbf{k}\} \text{ m/s}^2$

20-47 $\mathbf{v}_A = \{-8.66\mathbf{i} + 2.26\mathbf{j} + 2.26\mathbf{k}\} \text{ m/s},$
$\mathbf{a}_A = \{-22.6\mathbf{i} - 47.8\mathbf{j} + 45.3\mathbf{k}\} \text{ m/s}^2$

20-49 $\mathbf{v}_B = \{-10.2\mathbf{i} - 30.0\mathbf{j} + 52.0\mathbf{k}\} \text{ m/s},$
$\mathbf{a}_B = \{-31.0\mathbf{i} - 161\mathbf{j} - 90.0\mathbf{k}\} \text{ m/s}^2$

20-50 $\mathbf{v}_B = \{-10.2\mathbf{i} - 28.0\mathbf{j} + 52.0\mathbf{k}\} \text{ m/s},$
$\mathbf{a}_B = \{-33.0\mathbf{i} - 159\mathbf{j} - 90.0\mathbf{k}\} \text{ m/s}^2$

20-51 $\mathbf{v}_T = \{-9.50\mathbf{i} + 3\mathbf{j} + 1.20\mathbf{k}\} \text{ m/s},$
$\mathbf{a}_T = \{-14.4\mathbf{i} + 3.77\mathbf{j} + 4.20\mathbf{k}\} \text{ m/s}^2$

20-53 $\mathbf{v}_B = \{5.00\mathbf{i} - 0.500\mathbf{j} + 6.40\mathbf{k}\} \text{ m/s},$
$\mathbf{a}_B = \{-0.250\mathbf{i} - 7.12\mathbf{j}\} \text{ m/s}^2$

20-54 $\mathbf{v}_B = \{5.00\mathbf{i} - 0.500\mathbf{j} + 6.40\mathbf{k}\} \text{ m/s},$
$\mathbf{a}_B = \{2.95\mathbf{i} - 7.52\mathbf{j} + 7.20\mathbf{k}\} \text{ m/s}^2$

Chapter 21

21-2 $I_{yz} = \dfrac{m}{6}ah$

21-2 $I_{xy} = \dfrac{m}{12}a^2$

21-3 $I_{x'} = \dfrac{13}{24}mr^2, I_{y'} = \dfrac{7}{12}mr^2, I_{z'} = \dfrac{13}{24}mr^2$

21-6 $I_{\bar{y}} = \dfrac{3m}{80}(h^2 + 4a^2), I_{y'} = \dfrac{m}{20}(2h^2 + 3a^2)$

21-7 $I_{aa} = \dfrac{m}{12}(3a^2 + 4h^2)$

21-9 $I_{xx} = 0.626 \text{ kg}\cdot\text{m}^2, I_{yy} = 0.547 \text{ kg}\cdot\text{m}^2,$
$I_{zz} = 1.09 \text{ kg}\cdot\text{m}^2$

21-10 $I_z = 91.5\ \text{g}\cdot\text{m}^2$
21-11 $I_{Oa} = 0.481\ \text{kg}\cdot\text{m}^2$
21-13 $I_{xx} = 38.1\ \text{g}\cdot\text{m}^2$
21-14 $I_{yz} = 0$
21-15 $I_{xy} = 320\ \text{g}\cdot\text{m}^2, I_{yz} = 80.0\ \text{g}\cdot\text{m}^2, I_{xz} = 0$
21-17 $I_x = 4.50\ \text{kg}\cdot\text{m}^2, I_y = 4.38\ \text{kg}\cdot\text{m}^2,$
$I_z = 0.125\ \text{kg}\cdot\text{m}^2$
21-18 $I_z = 0.429\ \text{kg}\cdot\text{m}^2$
21-19 $I_{xy} = 0.32\ \text{kg}\cdot\text{m}^2, I_{yz} = 0.257\ \text{kg}\cdot\text{m}^2,$
$I_{xz} = 0\ \text{kg}\cdot\text{m}^2$
21-21 $I_{OA} = 1.21\ \text{kg}\cdot\text{m}^2$
21-23 $\mathbf{u}_\omega = 0.141\mathbf{j} - 0.990\mathbf{k}, \int \mathbf{F}_O\, dt = \{8.57\mathbf{i}\}\ \text{N}\cdot\text{s}$
21-25 $\mathbf{H}_O = \{21.9\mathbf{i} + 1.69\mathbf{k}\}\ \text{kg}\cdot\text{m}^2/\text{s}, T = 78.5\ \text{J}$
21-26 $\mathbf{H}_O = \{21.9\mathbf{i} + 0.562\mathbf{j} + 1.69\mathbf{k}\}\ \text{kg}\cdot\text{m}^2/\text{s},$
$T = 81.3\ \text{J}$
21-27 $\mathbf{H} = \{-0.447\mathbf{i} + 0.198\mathbf{j} + 169\mathbf{k}\}\ \text{g}\cdot\text{m}^2/\text{s}$
21-29 $\boldsymbol{\omega} = \{8.73\mathbf{i} - 122\mathbf{j}\}\ \text{rad/s}$
21-30 $\boldsymbol{\omega} = \{-0.444\mathbf{j} + 0.444\mathbf{k}\}\ \text{rad/s}$
21-31 $\mathbf{H}_O = \{-1.69\mathbf{j} + 0.422\mathbf{k}\}\ \text{kg}\cdot\text{m}^2/\text{s}, T = 9.49\ \text{J}$
21-33 $\omega_y = 25.2\ \text{rad/s}$
21-34 $\omega_{AB} = 21.4\ \text{rad/s}$, Yes
21-35 $\mathbf{H}_A = \{-2000\mathbf{i} - 55\,000\mathbf{j} + 22\,500\mathbf{k}\}\ \text{kg}\cdot\text{m}^2/\text{s}$
21-37 $T = 1.14\ \text{J}$
21-38 $\mathbf{H}_z = \{0.458\mathbf{k}\}\ \text{kg}\cdot\text{m}^2/\text{s}$
21-39 $\boldsymbol{\omega} = \{-0.750\mathbf{j} + 1.00\mathbf{k}\}\ \text{rad/s}$
21-41 $\Sigma M_x = \dfrac{d}{dt}(I_x\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)$
$-\Omega_z(I_y\omega_y - I_{yz}\omega_z - I_{yx}\omega_x)$
$+\Omega_y(I_z\omega_z - I_{zx}\omega_x - I_{zy}\omega_y)$
21-42 $\Sigma M_x = (I_x\dot{\omega}_x - I_{xy}\dot{\omega}_y - I_{xz}\dot{\omega}_z),$
$-\Omega_z(I_y\omega_y - I_{yz}\omega_z - I_{yx}\omega_x),$
$+\Omega_y(I_z\omega_z - I_{zx}\omega_x - I_{zy}\omega_y)$
21-45 $\dot{\omega}_x = -14.7\ \text{rad/s}^2, B_z = 77.7\ \text{N}, B_y = 3.33\ \text{N},$
$A_x = 0, A_y = 6.67\ \text{N}, A_z = 81.8\ \text{N}$
21-46 $\dot{\omega}_x = 9.28\ \text{rad/s}^2, B_z = 97.7\ \text{N}, B_y = 3.33\ \text{N},$
$A_x = 0, A_y = 6.67\ \text{N}, A_z = 122\ \text{N}$
21-47 $F_A = 277\ \text{N}, F_B = 166\ \text{N}$
21-49 $\dot{\omega}_z = 200\ \text{rad/s}^2, D_y = -12.9\ \text{N}, D_x = -37.5\ \text{N},$
$C_x = -37.5\ \text{N}, C_y = -11.1\ \text{N}, C_z = 36.8\ \text{N}$
21-50 $\dot{\omega}_y = -102\ \text{rad/s}^2, A_x = B_x = 0, A_y = 0,$
$A_z = 297\ \text{N}, B_z = -143\ \text{N}$
21-51 $(M_O)_x = 72.0\ \text{N}\cdot\text{m}, (M_O)_z = 0$
21-53 $T = 23.3\ \text{N}, F_A = 41.3\ \text{N}$
21-54 $A_z = -1.09\ \text{kN}, B_z = 1.38\ \text{kN}$
21-55 $N = 148\ \text{N}, F_f = 0$
21-57 $M_x = -\dfrac{4}{3}ml^2\omega_s\omega_p\cos\theta,$
$M_y = \dfrac{1}{3}ml^2\omega_p^{\,2}\sin 2\theta, M_z = 0$
21-58 $\theta_D = 139°, m_D = 0.661\ \text{kg}, \theta_F = 40.9°,$
$m_F = 1.32\ \text{kg}$
21-59 $\Sigma M_x = 0, \Sigma M_y = (-36.0\sin\theta)\ \text{mN}\cdot\text{m},$
$\Sigma M_z = (3.00\sin 2\theta)\ \text{mN}\cdot\text{m}$
21-61 $\omega_p = 27.9\ \text{rad/s}$
21-62 $M_x = 2\ \text{kN}\cdot\text{m}$
21-65 $\alpha = 90°, \beta = 9.12°, \gamma = 80.9°$
21-66 a) $A_y = 1.49\ \text{kN}, B_y = 2.43\ \text{kN},$
b) $A_y = -1.25\ \text{kN}, B_y = 5.17\ \text{kN},$
c) $A_y = 1.49\ \text{kN}, B_y = 2.43\ \text{kN}$
21-67 $M_x = 328\ \text{N}\cdot\text{m}$
21-69 $\Delta\text{F} = 53.4\ \text{N}$
21-70 $M_x = 27.0\ \text{N}\cdot\text{m}$
21-71 $\dot{\phi} = 81.7\ \text{rad/s}, \dot{\psi} = 212\ \text{rad/s},$
regular precession
21-73 $\dot{\psi} = 2.33\ \text{rev/h}$
21-75 $\dot{\phi} = 25.9\ \text{rad/s}, H_G = 2.10\ \text{Mg}\cdot\text{m}^2/\text{s}$
21-77 $\dot{\phi} = \left(\dfrac{2g\cot\theta}{a + r\cos\theta}\right)^{1/2}$
21-78 $H_G = 12.5\ \text{Mg}\cdot\text{m}^2/\text{s}$

Chapter 22

22-1 $\theta = -0.101\sin(4.95t) + 0.3\cos(4.95t)$
22-2 $\ddot{y} + 56.1\,y = 0, y|_{t=0.22\ \text{s}} = 0.192\ \text{m}$
22-3 $y = -0.05\cos(20t)$
22-5 $f = 4.98\ \text{Hz}, \tau = 0.201\ \text{s}$
22-6 $\omega_n = 49.5\ \text{rad/s}, \tau = 0.127\ \text{s}$
22-7 $y = \{-0.126\sin(3.16t) - 0.09\cos(3.16t)\}\ \text{m}$
$C = 0.155\ \text{m}$
22-9 $\tau = 2\pi\sqrt{\dfrac{Mr^2 + 2mk_O{}^2}{kr^2}}$
22-10 $y = -0.05\cos(8.16t), C = 50\ \text{mm}, f = 1.30\ \text{Hz}$
22-11 $\tau = 2\pi\sqrt{\dfrac{2mL}{3mg + 6kL}}$
22-13 $C = 230\ \text{mm}, f = 12.2\ \text{Hz}$
22-14 $\tau = 2\pi\sqrt{\dfrac{l}{3g}}$
22-15 $\tau = 2\pi\sqrt{\dfrac{m}{3k}}$
22-17 a) $k_{eq} = k_1 + k_2, \tau = 2\pi\sqrt{\dfrac{m}{k_1 + k_2}},$
b) $k_{eq} = \dfrac{k_1k_2}{k_1 + k_2}, \tau = 2\pi\sqrt{\dfrac{m(k_1 + k_2)}{k_1k_2}}$
22-18 $k_1 = 2.07\ \text{kN/m}, k_2 = 302\ \text{N/m}$, or vice versa
22-19 $f = \dfrac{1}{2\pi}\sqrt{\dfrac{\mu g}{d}}$

22-21 $k = 1.36\ \text{N/m}, m_B = 3.58\ \text{kg}$

22-22 $\omega_n = \sqrt{\dfrac{3g(4R^2 - l^2)^{1/2}}{6R^2 - l^2}}$

22-23 $\tau = 1.11\ \text{s}$

22-25 $l = 0.457\ \text{m}$

22-26 $f = \dfrac{1}{\pi}\sqrt{\dfrac{k}{m}}$

22-27 $\tau = 2\pi\left(\dfrac{a}{b}\right)\sqrt{\dfrac{m}{k}}$

22-29 $f = \dfrac{1}{2\pi}\sqrt{\dfrac{2g}{3(R - r)}}$

22-30 $\ddot{x} + 333x = 0$

22-31 $f = \dfrac{1}{\pi}\sqrt{\dfrac{k}{m}}$

22-33 $\tau = 0.487\ \text{s}$

22-34 $\tau = 2\pi\sqrt{\dfrac{Mk_z^2}{k}}$

22-35 $\ddot{\theta} + 26.0\theta = 0$

22-37 $f = 1.17\ \text{Hz}$

22-38 $\tau = 3.85\sqrt{\dfrac{m}{k}}$

22-39 $k_G = \dfrac{r}{2\pi}\sqrt{\dfrac{\tau^2 g - 4\pi^2 R}{R}}$

22-41 $k \le 714\ \text{N/m}$

22-42 $x = A \sin \omega_n t + B \cos \omega_n t + \dfrac{F_0/k}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2} \cos \omega t$

22-43 $y = A \sin \omega_n t + B \cos \omega_n t + \dfrac{F_0}{k}$

22-45 $y = (-3.66 \sin 12.2t + 50 \cos 12.2t + 11.2 \sin 4t)\ \text{mm}$

22-46 $\tau_d = 0.734\ \text{s}$

22-47 $y = (361 \sin 7.75t + 100 \cos 7.75t - 350 \sin 8t)\ \text{mm}$

22-49 $C = \dfrac{3F_O}{\frac{3}{2}(mg + Lk) - mL\omega^2}$

22-50 $X = 29.5\ \text{mm}$

22-51 $\omega = 14.0\ \text{rad/s}$

22-53 $(x_p)_{\max} = 35.5\ \text{mm}$

22-54 $y_P = 0.111 \sin (5t - 0.588)\ \text{m}$

22-55 $\ddot{y} + \dfrac{k}{m} y = \dfrac{k\delta_0}{m} \cos \omega_0 t,$

$y = A \sin \omega_n t + B \cos \omega_n t + \dfrac{k\delta_0}{(k - m\omega_0^2)} \cos \omega_0 t$

22-57 $\omega_0 = 19.8\ \text{rad/s}$

22-58 $X = 21.9\ \text{mm}$

22-59 $\omega_0 = 19.2\ \text{rad/s}$

22-61 $v_R = 1.20\ \text{m/s}$

22-62 $(x_p)_{\max} = 4.53\ \text{mm}$

22-63 $y = [-0.0702\, e^{-3.57t} \sin (8.54t)]\ \text{m}$

22-65 $y_{\max} = 3.57\ \text{mm}$

22-66 $\phi = 9.72°$

22-67 $\text{MF} = 0.997$

22-69 $F = 2c\dot{y}, c_c = 2m\sqrt{\dfrac{k}{m}}, c < \sqrt{mk}$

22-70 $Lq + Rq + \left(\dfrac{1}{C}\right)q = E_0 \cos \omega t$

22-73 $c = \sqrt{8(m + M)k}, x_{\max} = \left[\dfrac{m}{e}\sqrt{\dfrac{1}{2k(m + M)}}\right]v_0$

22-74 $x_{\max} = \dfrac{2mv_0}{\sqrt{8k(m + M) - c^2}}\, e^{-\pi c/(2\sqrt{8k(m+M) - c^2})}$

22-75 $L\ddot{q} + R\dot{q} + \left(\dfrac{2}{C}\right)q = 0$

22-77 $\ddot{y} + 16\dot{y} + 12y = 0$, overdamped motion

22-78 $L\ddot{q} + R\dot{q} + \dfrac{1}{C}q = 0$

22-79 $\ddot{y} + 16\dot{y} + 4y = 0$, overdamped motion

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SI Prefixes

Multiple	*Exponential Form*	*Prefix*	*SI Symbol*
1 000 000 000	10^9	giga	G
1 000 000	10^6	mega	M
1 000	10^3	kilo	k
Submultiple			
0.001	10^{-3}	milli	m
0.000 001	10^{-6}	micro	μ
0.000 000 001	10^{-9}	nano	n

Conversion Factors (SI) to (FPS)

Quantity	*Unit of Measurement (SI)*	*Equals*	*Unit of Measurement (FPS)*
Force	N		0.2248 lb
Mass	kg		0.06852 slug
Length	m		3.281 ft

FPS system: Foot-Pound-Second system

Fundamental Equations of Dynamics

KINEMATICS

Particle Rectilinear Motion

Variable a	*Constant* $a = a_c$
$a = \frac{dv}{dt}$	$v = v_0 + a_c t$
$v = \frac{ds}{dt}$	$s = s_0 + v_0 t + \frac{1}{2}a_c t^2$
$a\,ds = v\,dv$	$v^2 = v_0^2 + 2a_c(s - s_0)$

Particle Curvilinear Motion

x, y, z Coordinates		*r, θ, z Coordinates*	
$v_x = \dot{x}$	$a_x = \ddot{x}$	$v_r = \dot{r}$	$a_r = \ddot{r} - r\dot{\theta}^2$
$v_y = \dot{y}$	$a_y = \ddot{y}$	$v_\theta = r\dot{\theta}$	$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$
$v_z = \dot{z}$	$a_z = \ddot{z}$	$v_z = \dot{z}$	$a_z = \ddot{z}$

n, t, b Coordinates

$v = \dot{s}$	$a_t = \dot{v} = v\frac{dv}{ds}$
	$a_n = \frac{v^2}{\rho} \quad \rho = \frac{[1 + (dy/dx)^2]^{3/2}}{\lvert d^2y/dx^2 \rvert}$

Relative Motion

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$ $\qquad$ $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$

Rigid Body Motion about a Fixed Axis

Variable α	*Constant* $\alpha = \alpha_c$
$\alpha = \frac{d\omega}{dt}$	$\omega = \omega_0 + \alpha_c t$
$\omega = \frac{d\theta}{dt}$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha_c t^2$
$\omega\,d\omega = \alpha\,d\theta$	$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$

For Point P

$s = \theta r \quad v = \omega r \quad a_t = \alpha r \quad a_n = \omega^2 r$

Relative General Plane Motion—Translating Axes

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A(\text{pin})} \qquad \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A(\text{pin})}$

Relative General Plane Motion—Trans. and Rot. Axes

$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz}$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}$$

KINETICS

Mass Moment of Inertia $I = \int r^2\,dm$

Parallel-Axis Theorem $\quad I = I_G + md^2$

Radius of Gyration $\quad k = \sqrt{\frac{I}{m}}$

Equations of Motion

Particle	$\Sigma\mathbf{F} = m\mathbf{a}$
Rigid Body (Plane Motion)	$\Sigma F_x = m(a_G)_x$ $\Sigma F_y = m(a_G)_y$ $\Sigma M_G = I_G\alpha$ or $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$

Principle of Work and Energy

$T_1 + \Sigma U_{1-2} = T_2$

Kinetic Energy

Particle	$T = \frac{1}{2}mv^2$
Rigid Body (Plane Motion)	$T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$

Work

Variable Force	$U_F = \int F\cos\theta\,ds$
Constant Force	$U_F = (F_c\cos\theta)\,\Delta s$
Weight	$U_W = -W\Delta y$
Spring	$U_s = -\left(\frac{1}{2}ks_2^2 - \frac{1}{2}ks_1^2\right)$
Couple Moment	$U_M = M\Delta\theta$

Power and Efficiency

$P = \frac{dU}{dt} = \mathbf{F}\cdot\mathbf{v} \qquad \varepsilon = \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{U_{\text{out}}}{U_{\text{in}}}$

Conservation of Energy Theorem

$T_1 + V_1 = T_2 + V_2$

Potential Energy

$V = V_g + V_e$, where $V_g = \pm Wy$, $V_e = +\frac{1}{2}ks^2$

Principle of Linear Impulse and Momentum

Particle	$m\mathbf{v}_1 + \Sigma\int \mathbf{F}\,dt = m\mathbf{v}_2$
Rigid Body	$m(\mathbf{v}_G)_1 + \Sigma\int \mathbf{F}\,dt = m(\mathbf{v}_G)_2$

Conservation of Linear Momentum

$\Sigma(\text{syst. } m\mathbf{v})_1 = \Sigma(\text{syst. } m\mathbf{v})_2$

Coefficient of Restitution

$e = \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$

Principle of Angular Impulse and Momentum

Particle	$(\mathbf{H}_O)_1 + \Sigma\int \mathbf{M}_O\,dt = (\mathbf{H}_O)_2$ where $H_O = (d)(mv)$
Rigid Body (Plane Motion)	$(H_G)_1 + \Sigma\int M_G\,dt = (H_G)_2$ where $H_G = I_G\omega$ $(H_O)_1 + \Sigma\int M_O\,dt = (H_O)_2$ where $H_O = I_O\omega$

Conservation of Angular Momentum

$\Sigma(\text{syst. } \mathbf{H})_1 = \Sigma(\text{syst. } \mathbf{H})_2$