Fourier series for an arbitrary Period

Ex: p.13

$f(x)$ periodic with period $2T$

$x \to 2T$

$v \to 2\pi \Rightarrow x = \frac{T}{\pi} v \quad , v = \frac{\pi}{T} x \quad , \quad dv = \frac{\pi}{T} dx$

$$f(v) = f\left(\frac{T}{\pi} v\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nv + b_n \sin nv$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(v)\, dv = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{T}{\pi} v\right) dv = \frac{1}{\pi} \int_{-T}^{T} f(x) \frac{\pi}{T}\, dx$$

$$\Rightarrow a_0 = \frac{1}{T} \int_{-T}^{T} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(v) \cos nv\, dv = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{T}{\pi} v\right) \cos nv\, dv$$

$$= \frac{1}{\pi} \int_{-T}^{T} f(x) \cdot \cos\left(\frac{n\pi x}{T}\right) \cdot \frac{\pi}{T}\, dx$$

$$\Rightarrow a_n = \frac{1}{T} \int_{-T}^{T} f(x) \cos\left(\frac{n\pi x}{T}\right) dx$$

Same as :

$$b_n = \frac{1}{T} \int_{-T}^{T} f(x) \sin\left(\frac{n\pi x}{T}\right) dx$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{T} + b_n \sin \frac{n\pi x}{T}$$

Note :

Ex: p.13

Find F.S. for $f(x) = |x| - 1 \quad , -1 < x < 1$

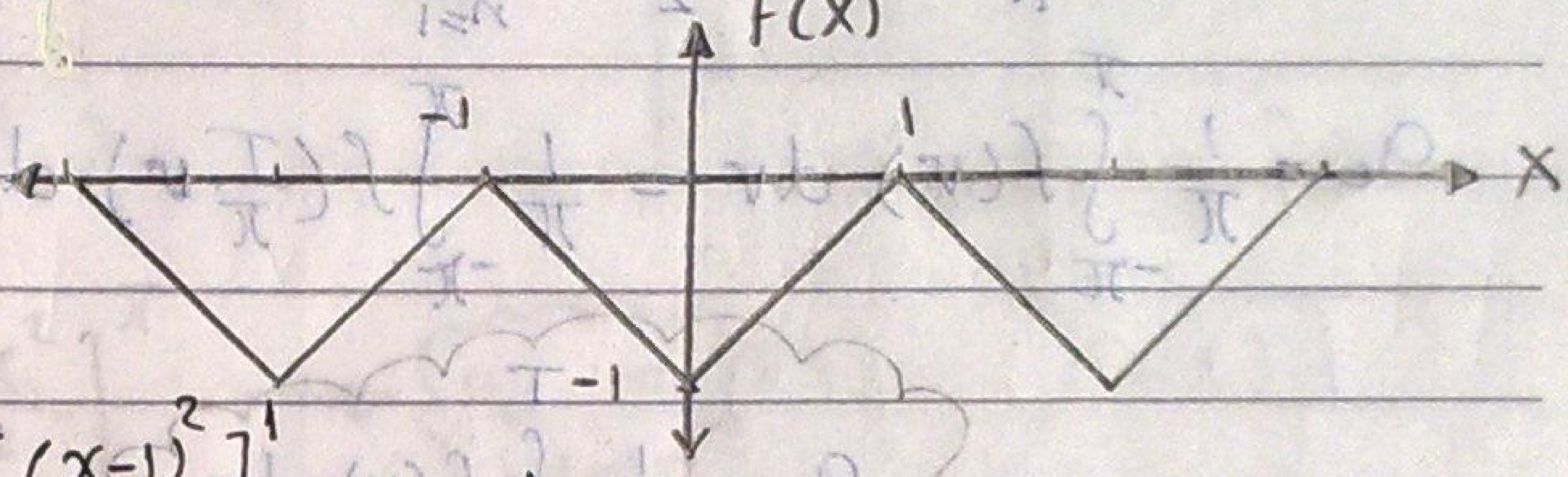
$f(x+2) = f(x)$

$\rightarrow 2T = 2 \longrightarrow \{T = 1\}$ & $|x| = \begin{cases} x & , x > 0 \\ -x & , x < 0 \end{cases}$

$$f(x) = |x| - 1 = \begin{cases} x - 1 & , 0 < x < 1 \\ -x - 1 & , -1 < x < 0 \end{cases}$$

$f(x)$ = even function

$\rightarrow \{b_n = 0\}$



$$a_0 = \frac{2}{1}\int_0^1 (x-1)\,dx = 2\left[\frac{(x-1)^2}{2}\right]_0^1 = -1$$

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\left(\frac{n\pi x}{T}\right)dx = \frac{2}{1}\int_0^1 (x-1)\cos\frac{n\pi x}{1}\,dx$$

$u = x - 1 \qquad dv = \cos(n\pi x)\,dx$

$du = dx \qquad v = \frac{\sin(n\pi x)}{n\pi}$

$$a_n = 2\left[\underbrace{(x-1)\cdot\frac{\sin n\pi x}{n\pi}\Big|_0^1}_{\to 0} + \frac{\cos n\pi x}{(n\pi^2)}\Big|_0^1\right] = \frac{2}{n^2\pi^2}\left[\cos n\pi - 1\right]$$

$$\boxed{a_n = \frac{2}{n^2\pi^2}\left((-1)^n - 1\right)}$$

$$\Rightarrow \boxed{f(x) = \frac{-1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left((-1)^n - 1\right)\cos n\pi x}$$

Note :

لحل التكامل بالتجزئ استخدم

$$\boxed{u\,dv = u\left(\int dv\right) - u'\left(\iint dv\right) + u''\left(\iiint dv\right)}$$

Ex. Problem 21 P. 20

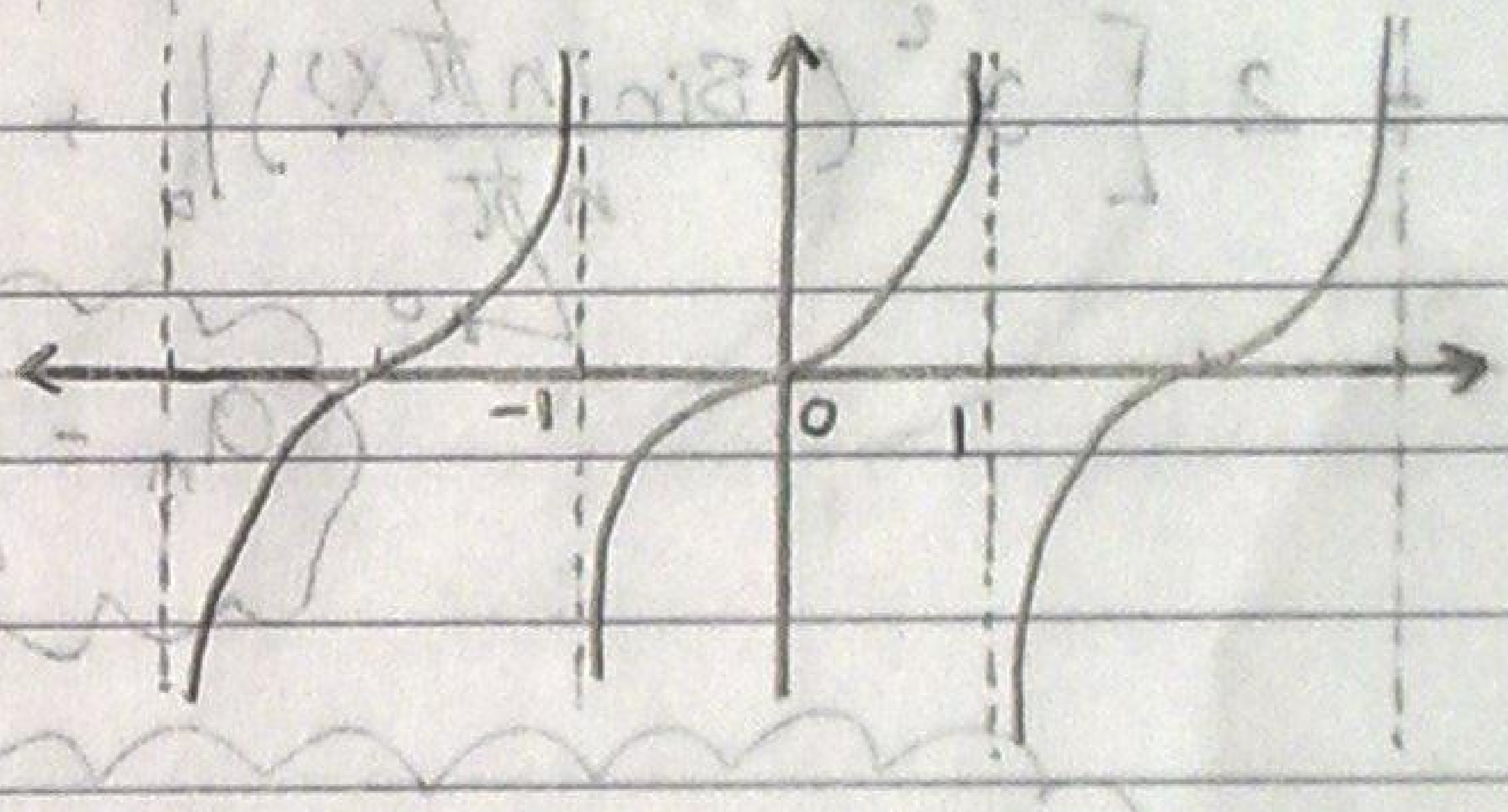
Expand $f(x) = x^2$, $0 < x < 1$, by a Fourier series which contains only:

(i) Sine harmonics (ii) Cosine harmonics

(iii) odd harmonics (iv) Even harmonics

Half Range Expansion

$f(x) = x^2$, $0 < x < 1$



(i) Sine harmonics

odd function

$$\boxed{a_0 = a_n = 0}$$

$$b_n = \frac{2}{1} \int_0^1 x^2 \sin\left(\frac{n\pi x}{1}\right) dx$$

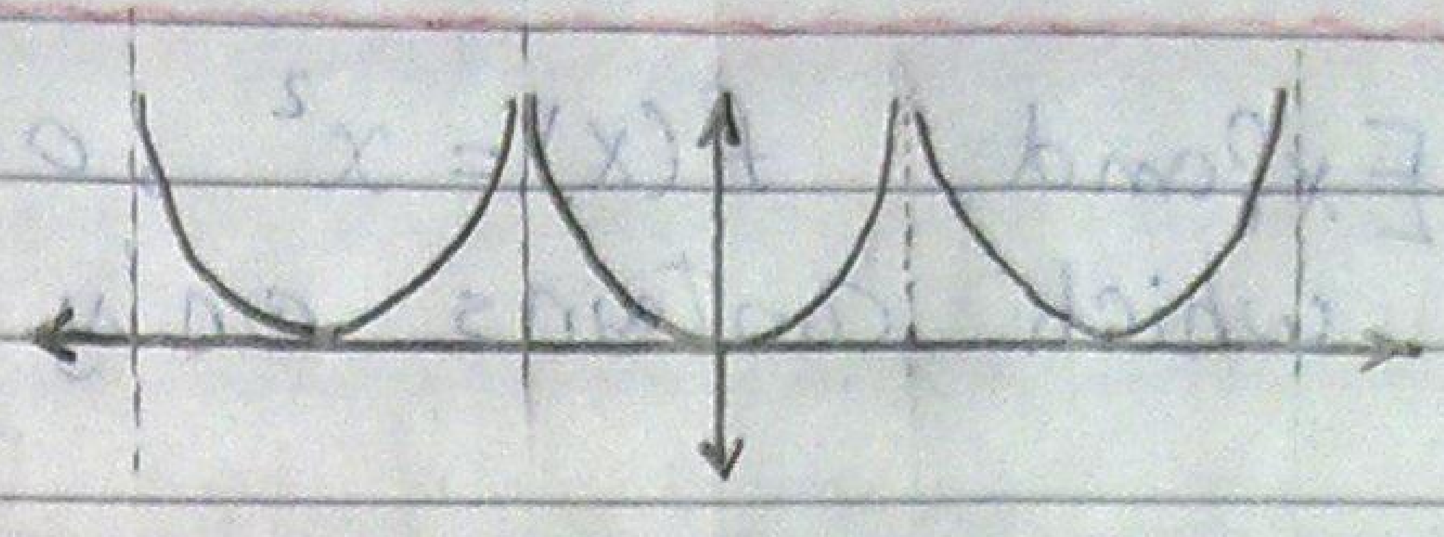
$$= 2\left[x^2 \left(-\frac{\cos n\pi x}{n\pi} \right)\Big|_0^1 + 2x \left(\frac{\sin n\pi x}{n^2\pi^2} \right)\Big|_0^1 + 2\left(\frac{\cos n\pi x}{n^3\pi^3} \right)\Big|_0^1 \right]$$

$$\boxed{b_n = 2\left[\frac{(-1)^{n+1}}{n\pi} + \frac{2}{n^3\pi^3}\left((-1)^n - 1\right) \right]}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} 2\left[\frac{(-1)^{n+1}}{n\pi} + \frac{2}{n^3\pi^3}\left((-1)^n - 1\right) \right] \sin n\pi x}$$

(ii) Cosine harmonic

even function



$$\boxed{b_n = 0}$$

$$a_0 = \frac{2}{1}\int_0^1 x^2\,dx = 2\left[\frac{x^3}{3}\right]_0^1 \Rightarrow \boxed{a_0 = \frac{2}{3}}$$

$$a_n = \frac{2}{1}\int_0^1 x^2 \cos\left(\frac{n\pi x}{1}\right)dx$$

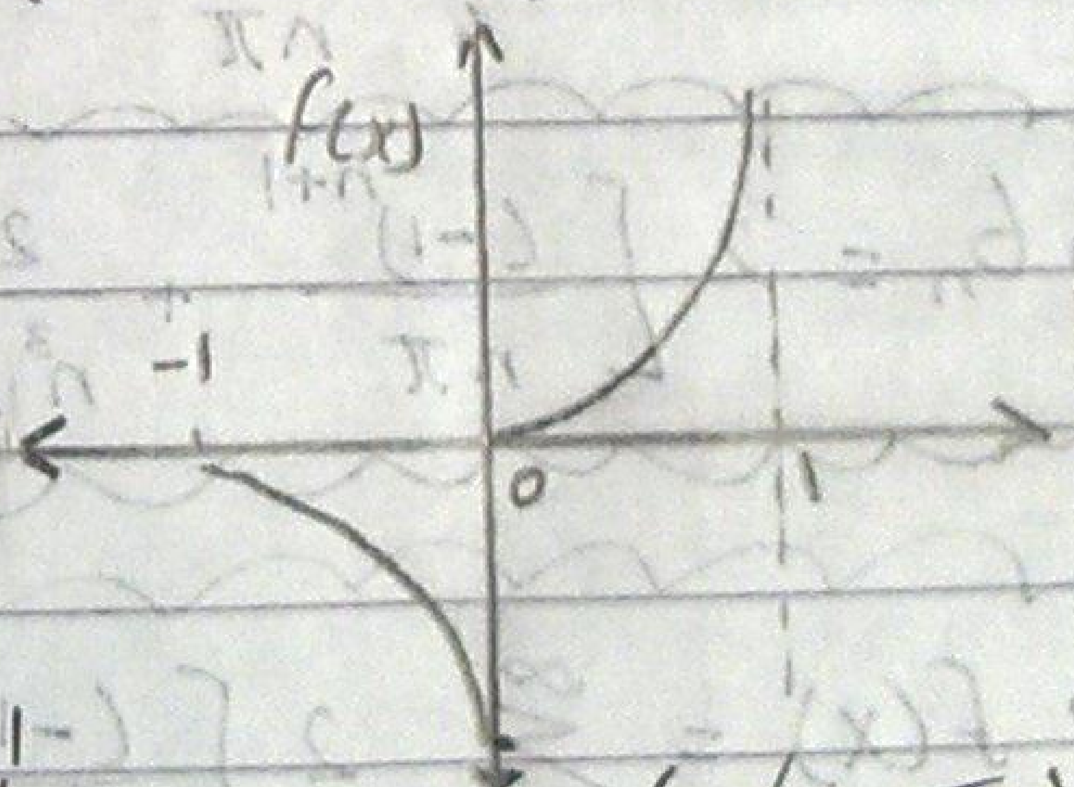
$$= 2\left[x^2\left(\frac{\sin(n\pi x)}{n\pi}\right)\Big|_0^1 + 2x\left(\frac{\cos(n\pi x)}{n^2\pi^2}\right)\Big|_0^1 + 2\left(\frac{-\sin(n\pi x)}{n^3\pi^3}\right)\Big|_0^1\right]$$

$\to 0$ $\to 0$

$$\boxed{a_n = \frac{4}{n^2\pi^2}(-1)^n}$$

$$\therefore \boxed{f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(-1)^n \cos(n\pi x)}$$

(iii) odd harmonics



$$a_0 = a_{2n} = b_{2n} = 0$$

$$a_{2n-1} = \frac{2}{1}\int_0^1 x^2 \cos((2n-1)\pi x)\,dx$$

$$= 2\left[x^2\frac{\sin((2n-1)\pi x)}{(2n-1)\pi}\Big|_0^1 - 2x\frac{-\cos((2n-1)\pi x)}{(2n-1)^2\pi^2}\Big|_0^1 + 2\frac{-\sin((2n-1)\pi x)}{(2n-1)^3\pi^3}\Big|_0^1\right]$$

$\to 0$ $\to 0$

$$a_{2n-1} = \frac{4x}{(2n-1)^2\pi^2}\cdot\cos((2n-1)\pi x) = \boxed{\frac{-4}{(2n-1)^2\pi^2}}$$

$$b_{2n-1} = \frac{2}{1}\int_0^1 x^2 \sin((2n-1)\pi x)\,dx$$

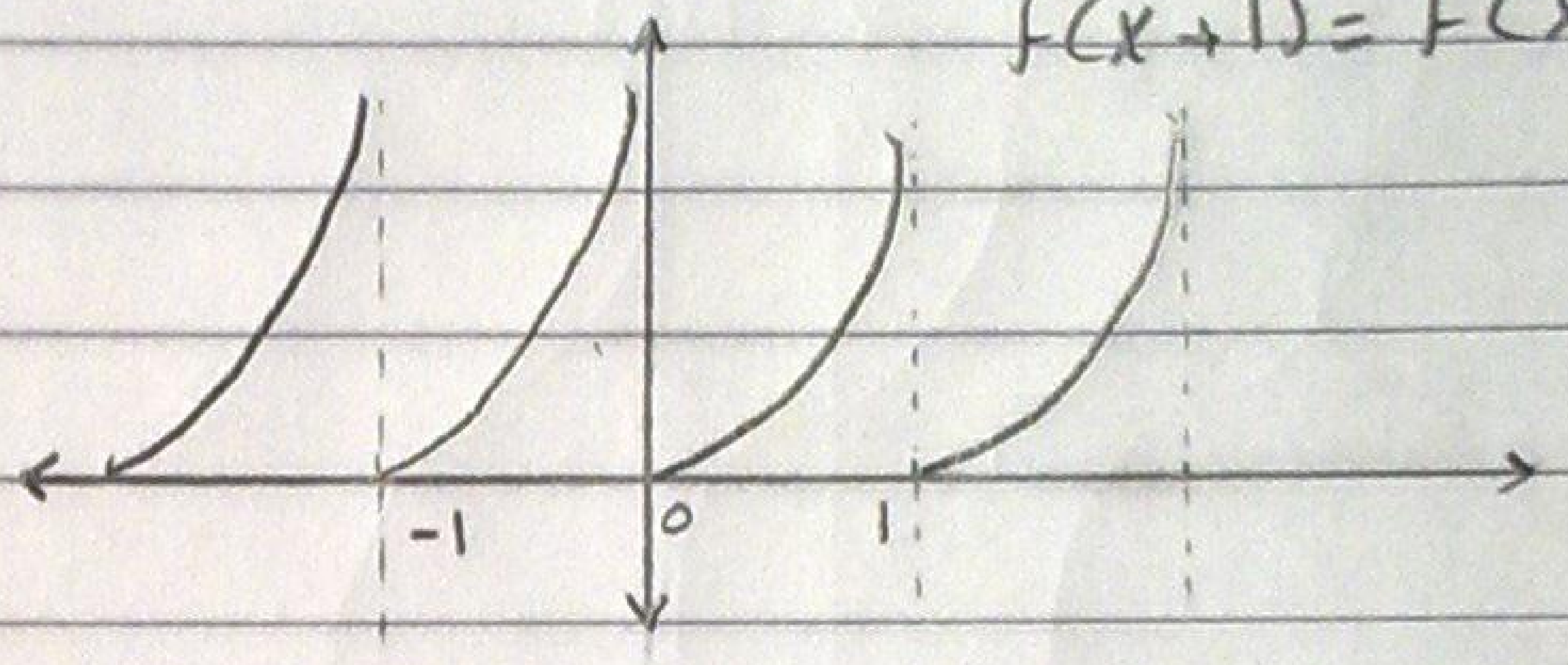
$$b_{2n-1} = 2\left[x^2 \cdot \frac{-\cos((2n-1)\pi x)}{(2n-1)\pi}\Big|_0^1 - \underbrace{2x \cdot \frac{-\sin(\cancel{(2n-1)\pi x})}{(2n-1)^2\pi^2}\Big|_0^1}_{\to 0} + 2\,\frac{\cos((2n-1)\pi x)}{(2n-1)^3\pi^3}\Big|_0^1 \right]$$

$$= 2\left[\frac{1}{(2n-1)\pi} - \frac{2}{(2n-1)^3\pi^3} \right] = \frac{2}{(2n-1)\pi} - \frac{4}{(2n-1)^3\pi^3}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{-4}{(2n-1)^2\pi^2}\cos((2n-1)\pi x) + \left(\frac{2}{(2n-1)\pi} - \frac{4}{(2n-1)^3\pi^3}\right)\sin((2n-1)\pi x)$$

$\boxed{iv}$ Even harmonic

$f(x+1) = f(x)$



$$a_0 = \frac{2}{1}\int_0^1 x^2\,dx = 2\left[\frac{x^3}{3}\right]_0^1$$

$$a_0 = \frac{2}{3}$$

$$a_{2n} = \frac{2}{1}\int_0^1 x^2 \cos(2n\pi x)\,dx$$

$$= 2\left[\underbrace{x^2\left(\frac{\sin(\cancel{2n\pi x})}{2n\pi}\right)\Big|_0^1}_{\to 0} - 2x\left(\frac{-\cos(2n\pi x)}{(2n\pi)^2}\right)\Big|_0^1 + \underbrace{2\left(\frac{-\sin(\cancel{2n\pi x})}{(2n\pi)^3}\right)\Big|_0^1}_{\to 0} \right]$$

$$a_{2n} = \frac{4}{(2n\pi)^2}\left(\cos(2n\pi)\right) = \frac{1}{n^2\pi^2}$$

$$b_{2n} = \frac{2}{1}\int_0^1 x^2 \sin(2n\pi x)\,dx$$

$$= 2\left[x^2 \cdot \frac{-\cos(2n\pi x)}{2n\pi}\Big|_0^1 - \underbrace{2x \cdot \frac{-\sin(\cancel{2n\pi x})}{(2n\pi)^2}\Big|_0^1}_{\to 0} + 2\,\frac{\cos(2n\pi x)}{(2n\pi)^3}\Big|_0^1 \right]$$

$$b_{2n} = 2\left[\frac{-1}{2n\pi} + \frac{2}{(2n\pi)^3}\right] \longrightarrow b_{2n} = \frac{-1}{n\pi} + \frac{1}{2n^3\pi^3}$$

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}\cos(2n\pi x) + \left(\frac{-1}{n\pi} + \frac{1}{2n^3\pi^3}\right)\sin(2n\pi x)$$